Introduction: Analytic Geometry and Calculus

Analytic Geometry and Calculus are closely related subjects. Analytic Geometry is the study of functions and relations as to how their graphs on the Cartesian Coordinate System relate to the algebra of their equations. Traditionally, Calculus has been the study of functions with a particular interest in tangent lines, maximum and minimum points, and area under the curve. (The Reform Calculus movement places an emphasis on how functions change, rather than the static graph, and consider Calculus to be a study of change and of motion.) Consequently, there is a great deal of overlap between the subjects. The advent of graphing calculators has blurred the distinctions between these fields and made subjects that had previously been strictly Calculus topics easily accessible at the lower level. The point of this course is to thoroughly discuss the subjects of Analytic Geometry that directly pertain to entry-level Calculus and to introduce the concepts and algebraic processes of first semester Calculus.

To this end, we have determined the following six Enduring Understandings are at the heart of this course:

- Understanding the relationship between a function and the derivative of that function.
- Understanding the similarities among and differences between the various members of the families of functions.
- Understanding the nature and purpose of the equations for a function and its derivative (plus, in Honors, the 2nd Derivative)
- Understanding word problems and mathematical modeling as a context for mathematical skills and thinking rather than as a separate topic.
- Understanding when it is appropriate to use a calculator and when it is not.
- Developing flexibility of thought by not getting locked into seeing things in a locked in, definite way.

A Note about the Text

This text is designed to study the various families of functions in light of the main characteristics—or traits—that the graphs of each family possess. Chapters 1 through 3 are really a single topic, Polynomial Functions. Each subsequent chapter will take a different family and a) review what is known about that family from Algebra 2, b) investigate the analytic traits, c) introduce the Calculus rule that most

applies to that family, d) put it all together in full sketches, and e) take one step beyond.

Basic Concepts and Definitions

A common vocabulary will be used throughout this course. Much of it comes directly from Algebra 1 and 2.

Domain – Defn: the set of values of the independent variable
Means: the set of *x*-values that can be substituted into the equation to get a real *y*-value (i.e., *no zero denominator* and *no negative under an even radical)*

Range – Defn: the set of values of the dependent variable
Means: the set of *y*-values that can come from the equation

Relation – Defn: a set of ordered pairs
Means: the equation that creates or defines the pairs

Function – Defn: a relation for which there is exactly one value of the dependent variable for each value of the independent variable
Means: an equation where every *x* gets only one *y*

Degree – Defn: the maximum number of variables that are multiplied together in any one term of the polynomial
Means: usually, the highest exponent

The Sets of Numbers

Natural Numbers – the counting numbers: 1, 2, 3, 4, …

Whole Numbers – the natural numbers, as well as 0

Integers – positive and negative whole numbers

Rational Numbers – any number that can be expressed as a fraction of two integers

Irrational Numbers – any number that cannot be expressed as a fraction of two integers (e.g., π, $\sqrt{2}$, or $e = 2.718281828459...$)

Real Numbers – all rational and irrational numbers

Imaginary Numbers – a real number times the square root of –1

Complex Numbers – a number that has a real part and an imaginary part (e.g., $2+3i$)

Transfinite Numbers – numbers involving infinity (∞)

The Cartesian Coordinate System

As noted before, Analytic Geometry is the study of functions and relations as to how their graphs on the Cartesian Coordinate System relate to the algebra of their equations. Rene Descartes created this system in the 17th century by putting two real number lines perpendicular to each other and defining any point on the plane by their horizontal and vertical distances form the origin. These numbers are called ordered pairs, because there is a specific order—the *x* is the first number and the *y* is the second. This allowed for writing equations that represented sets of points that were related in a specific way. *Because these functions are on the Cartesian Coordinate System, assume that they are real numbers, unless the directions state otherwise.*

NB. *This is not the only graphing system*. There are polar and rectangular coordinate systems, as well as others, but the Cartesian Coordinate System is the one that will be primarily used.

Families of Functions and Relations

On the Cartesian Coordinate System, *x*-values can be randomly assigned to *y*-values, but that is not particularly interesting. The equations mentioned above that related the ordered pairs, along with the sets of *x*-values and *y*-values, form what are called relations. There is one kind of relation that is of particular interest—the function. A function is a specific kind of relation, where every *x*-value gets exactly one *y*-value. There are several groups of functions, similar to the kinds of sets of numbers.

Algebraic Functions

Polynomial – Defn: an expression containing no other operations than addition, subtraction, and multiplication performed on the variable

Means: any equation of the form $y = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, where n is a non-negative integer

Rational – Defn: an expression that can be written as the ratio of one polynomial to another

Means: an equation with an x in the denominator

Irrational (radical) – Defn: an expression whose general equation contains a root of a variable and possibly addition, subtraction, multiplication, and/or division

Means: an equation with an x in a radical

Transcendental Functions

Exponential – Defn: a function whose general equation is of the form $y = a \cdot b^x$

Means: there is an x in the exponent

Logarithmic – Defn: the inverse of an exponential function

Means: there is a "log" or "ln" in the equation

Trigonometric – Defn: a function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure

Means: an equation with sine, cosine, tangent, secant, cosecant or cotangent in it

Other Kinds of Functions and Relations

Piece-wise Defined – a function that is defined by different equations for different parts of its domain

Inverse – two functions or relations that cancel/reverse each other (e.g., $y = x^2$ and $y = \sqrt{x}$)

Conic Sections – Defn: shapes formed by the intersection of a plane and a cone of two knaps

Means: circles, ellipses, parabolas, or hyperbolas (and occasionally lines or points)

Set Notation and Interval Notation

Interval Notation

Closed interval:	$x \in [a, b]$ means $a \le x \le b$
Open interval:	$x \in (a, b)$ means $a < x < b$
Half-open interval:	$x \in [a, b)$ means $a \le x < b$
or	$x \in (a, b]$ means $a < x \le b$