AP Calculus AB '20-21

Spring Final Part IIA v1

Calculator Allowed

Name:

Solution Key

1. The Peninsula Humane Society (PHS) is dedicated to the care and adoption of as many animals who they receive as possible. Since cats breed seasonally, the number of cats and kittens they receive into their facility in a given year varies roughly sinusoidally with time. The data available from 2019 shows the rate $R(t)$, measured in healthy cats per month, varies with time, measured in months after New Year's Day, according to the equation

$$R(t)=120-88\cos\left[\frac{\pi}{6}(t-2)\right].$$

The rate $A(t)$ at which adoption occur, measured in cats per month, varies with time, measured in months after New Year's Day, according to the equation

$$A(t)=125-85\cos\left[\frac{\pi}{6}(t-3)\right].$$

On New Years' Day ($t=0$), there were 131 cats in the PHS Nursery waiting to be adopted.

(a) How many cats and kittens were received at PHS in 2019? 1pt

$\int_0^{12} R(t)\,dt = 1440$

(b) Find $A'(10.3)$. Using the correct units, explain the meaning of $A'(10.3)$ in context of the problem. 2pts

$A'(10.3) = -28.008$

AT $t = 10.3$ MONTHS THE RATE AT WHICH CATS ARE BEING ADOPTED IS DECREASING BY 28.008 CATS/MONTH²

(c) Find the number of healthy cats and kittens predicted by the models to be in the PHS facility at the end of 2019. 3PTS

$$\text{TOTAL CATS} = 131 + \int_0^{12} R(t) - A(t)\, dt$$

$$= 71 \text{ CATS}$$

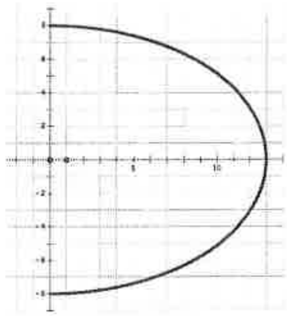
(d) Find the time when the number of healthy cats and kittens in the PHS facility during 2019 was at an absolute maximum. Include the units. 3PTS

$T' = R - A = 0 \rightarrow t = 2.8366968$ AND 8.4101621

t	TOTAL
0	131
2.837	48.443
8.410	190.897
12	71

THE MAXIMUM NUMBER OF CATS AT PHS OCCURS AT $t = 8.410$ MONTHS

2. The town of Northville, Michigan, purchased an elevated steel tank build by McDermott Fabrications. The shape of the tank is formed by rotating the right half of the ellipse $\frac{1}{169}x^2+\frac{1}{64}y^2=1$ on $-13\le x\le 13$ and $-8\le y\le 8$, about the y-axis to form a shape called an oblate spheroid, which looks something like a balloon which is being pressed down. Northville's tank is roughly 26 yards wide and 16 yards high.



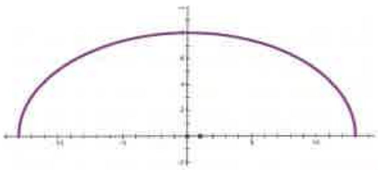
(a) Isolate x in the ellipse equation and find the volume of the oblate spheroid tank. Show the antiderivative. 4pts

$$\frac{1}{169}x^2+\frac{1}{64}y^2=1 \rightarrow \frac{1}{169}x^2=1-\frac{1}{64}y^2 \rightarrow x^2=169-\frac{169}{64}y^2$$

$$V=\pi\int_a^b R^2\,dy=\pi\int_{-8}^{8}\left(169-\frac{169}{64}y^2\right)dy$$

$$=\pi\left[169y-\frac{169}{192}y^3\right]_{-8}^{8}$$

$$=5663.244 \text{ yds}^3$$



b) If the top half of the ellise were rotated about the x-axis, the shape would be called a prolate spheroid and would look something like a football. Isolate y in the ellipse equation and find the volume of the resulting prolate spheroid tank. 2 PTS

$$y = \sqrt{64 - \frac{64}{169}x^2}$$

1 $$V = \pi \int_{-13}^{13} \left(64 - \frac{64}{169} y^2\right) dy$$

1 $$= 3485.073 \text{ yds}^3$$

(c) One cubic yard of water equals 200 gallons. How many gallons would each tank hold? 1 PTS

a) 1,132,648.871 GALLONS

b) 697014.6901 GALLONS

(d) One can estimate the volume of the water base on the depth h, using the partial volume formula for a spherical tank with a radius which is the average of the radii of the ellipse: $V = \frac{\pi}{3}h^2(32-h)$. The town of Northville uses 7000 gallons (or $35\,yd^3$) of water per month. How fast is the height h changing, in yards per month, when the water is 10 yards deep? 2pts

$$V = \frac{32\pi}{3}h^2 - \frac{\pi}{3}h^3$$

$$\frac{dV}{dt} = \frac{64\pi}{3}h\frac{dh}{dt} - \pi h^2\frac{dh}{dt}$$

$$-35 = \left(\frac{640\pi}{3} - 100\pi\right)\frac{dh}{dt}$$

$$\frac{dh}{dt} = .098 \frac{\text{yds}}{\text{month}}$$

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t	0	0.08	0.2	0.33	0.55	0.75	0.86	1
$H(t)$	60	5	63	25	75	28	70	6

3. Traffic flow on HWY 280 from San Francisco to San Jose during rush hour is notoriously variable because of bottlenecks caused by intersecting with HWYs 380, 92, and 85. The table above shows the average speed of traffic, in miles per hour, at time *t*, where *t* is measured in hours after entering the highway.

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a) Approximate $H'(0.6)$. Using the correct units, explain the meaning of $H'(0.6)$. 3 PTS

$$H'(.6) \approx \frac{28-75}{.75-.55} = -235 \frac{\text{MI}}{\text{HR}^2}$$

AT $t = 0.6$ HOURS THE VELOCITY OF THE CAR IS DECREASING BY -235 MILES PER HOUR PER HOUR

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b) Set up a left-hand Reimann Sum to approximate $\int_0^1 H(t)dt$. Using the correct units, explain the meaning of $\int_0^1 H(t)dt$.

$$\int_0^1 H(t)dt \approx .08(60) + .12(5) + .13(63) + .22(25) + .2(75) + .14(70)$$

$\int_0^1 H(t)\,dt$ IS THE APPROXIMATE NUMBER OF MILES TRAVELED BY THE CAR BETWEEN $t=0$ AND $t=1$ HOURS

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c) Assume that $S(t)=40-35\cos\left(\frac{\pi}{9}(t-5.5)\right)$ would accurqtely model the data on the table. Set up, but do not solve, an integral equation that would determine the time at which the car has been driving 50 miles. 2pt

$$\int_0^t S(\cancel{x})\,d\cancel{x} = 50$$

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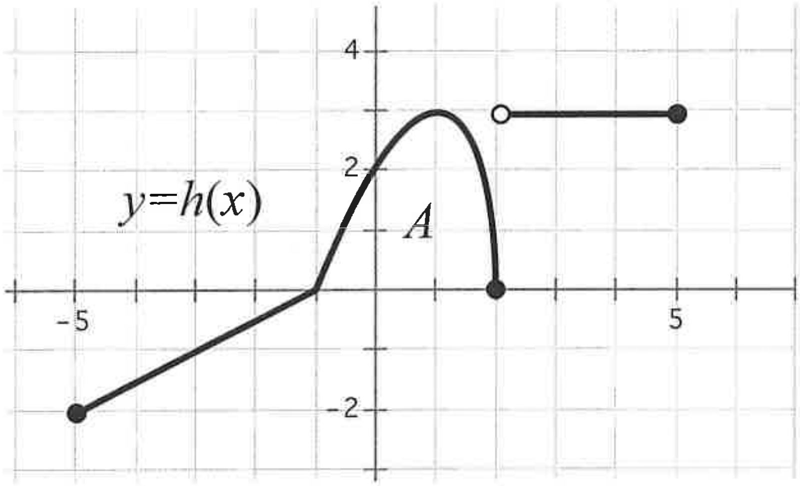
d) Set up, but do not solve, an integral equation that would determine the average $S(t)$ between $t=0.33$ and $t=0.55$. 2 pts

$$\frac{1}{.55-.33}\int_{.33}^{.55} S(t)\,dt$$



4. The graph above, $h(t)$ on $-5 \le x \le 5$, is comprised of two line segments and the graph of a radial function. Let $g(x)=4+\int_{-1}^{x} h(t)dt$. The area A is 5.

(a) Find $g(-5)$ and $g'(-5)$.

$g(-5) = 4 + \int_{-1}^{-5} h(t)\,dt = 4+4=8$

$g'(-5) = h(-5) = -2$

(b) Find $g(2)$.

$g(2) = 4 + \int_{-1}^{2} h(t)\,dt = 4+5=9$

(c) At what x-value, on $-5 \le x \le 5$, does $g(x)$ have the absolute minimum? Explain. 3PTS

$g' = h = 0$ AT $x = -1$ AND h SWITCHES FROM $-$ TO $+$

$x = -1$ IS THE ONLY MINIMUM, AS THE END POINTS ARE ~~MAX~~IMUMS $\therefore$ $x = -1$ IS AT THE ABSOLUTE MINIMUM

(d) On what interval(s) is $g(x)$ both increasing and concave down? Explain why.

$g'(x) = h(x)$

$g(x)$ IS INCREASING AND CONCAVE DOWN WHEN

$g'(x)$ IS POSITIVE AND DECREASING

$\therefore$ $x \in (1, 2)$

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5. The rate at which the population of a certain alien creature on another planet grows according to the differential equation $\frac{dA}{dt} = .005(100 - A)$, where A in creatures at time t days. The equation $y = A(t)$ is the particular solution to the differential equation wherein there are 10 creatures at time $t = 0$.

(a) Find the equation of the line tangent to $y = A(t)$ at $(0, 10)$. 1 PT

$\left.\frac{dA}{dt}\right|_{A=10} = .45$

$y - 10 = .45(t - 0)$

(b) Use the line tangent found in (a) to approximate the number of creatures at time $t = 12$ days. 1 PT

$A(12) \approx y(12) = .45(12 - 0) + 10$

$= 15.4$

(c) Find the particular solution to $\frac{dA}{dt} = .005(100 - A)$ with the initial condition $A(0) = 10$. 5PTS

$$\frac{1}{100-A}\, dA = .005\, dt$$

$$-\ln|100-A| = .005t + C$$

$$\ln|100-A| = -.005t + C \Rightarrow |100-A| = e^{-.005t + C}$$

$$100 - A = Ke^{-.005t}$$

$$A(0) = 10 \rightarrow K = 90$$

$$A = 100 - 90e^{-.005t}$$

(d) Determine whether the alien creature population is changing at an increasing or a decreasing rate at time $t = 12$ days. Explain your reasoning.

$$\frac{dA}{dt} = .005(100 - A) > 0 \therefore \text{INCREASING}$$

$$\frac{d^2A}{dt^2} = -.005 (-1)\frac{dA}{dt} = +.005(.005(100-A)) > 0$$

$\therefore$ THE POPULATION IS INCREASING AT AN INCREASING RATE

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6. $f(x)=\begin{cases} \ln(1-x), \text{ if } x\le 0 \\ \tan x, \text{ if } 0<x \end{cases}$

(a) Is $f(x)$ continuous at $x=0$? Justify your answer. 3ps

i) $f(0)$ EXISTS

ii) $\lim_{x\to 0} f(x)$ EXIST BECAUSE

$$\lim_{x\to 0^-} f(x) = \ln 1 = 0 = \tan 0 = \lim_{x\to 0^+} f(x)$$

iii) $f(0) = \lim_{x\to 0} f(x)$

(b) Find $f'(-1)$ and $f'\left(\frac{\pi}{4}\right)$. 2pts

$$f'(-1) = \frac{1}{1-x}(-1)\Big|_{x=-1} = \frac{-1}{2}$$

$$f'\left(\frac{\pi}{4}\right) = \sec^2 x\Big|_{x=\frac{\pi}{4}} = (\sqrt{2})^2 = 2$$

2PTS (c) Express $f'(x)$ as a piecewise-defined function. Explain why $f'(0)$ does not exist. 2PTS

$$f'(x) = \begin{cases} \dfrac{-1}{1-x} & \text{IF } x < 0 \\ \sec^2 x & \text{IF } x > 0 \end{cases}$$

$$\lim_{x \to 0^-} f'(x) \neq \lim_{x \to 0^+} f'(x)$$

2PT (d) Find $\lim_{x \to \pi} \dfrac{f'(x) - \cancel{2}\,1}{\pi - x}$. Justify your answer. CHANGE

$$x = \pi \rightarrow \frac{\sec^2 \pi - 1}{\pi - \pi} = \frac{0}{0}$$

$= ~~\text{E}~~$

$$\overset{L'H}{=} \lim_{x \to \pi} \frac{f''(x)}{-1} = \lim_{x \to \pi} \frac{2 \sec x \sec x \tan x}{-1}$$

$$= \frac{2(-1)(-1)(0)}{-1} = 0$$
