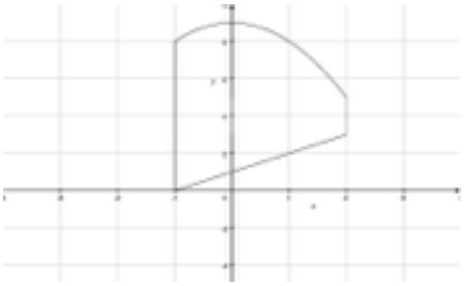
6.1 Free Response Solutions

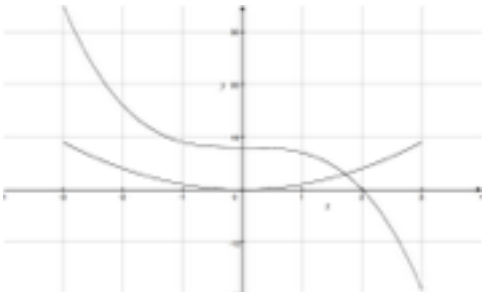
1. $y = x+1,\ y = 9-x^2,\ x = -1,\ x = 2$



$$A = \int_{-1}^{2}\left[\left(9-x^2\right)-\left(x+1\right)\right]dx = \int_{-1}^{2}\left(9-x^2-x-1\right)dx = \int_{-1}^{2}\left(8-x-x^2\right)dx$$

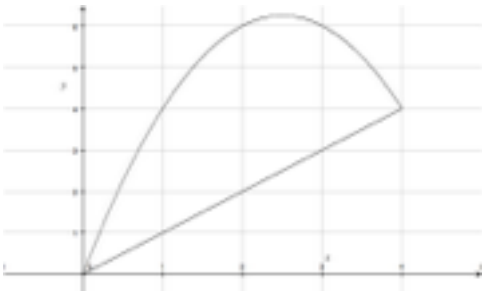
$$= \left[8x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{2} = \left[16-\frac{4}{2}-\frac{8}{3}\right]-\left[-8-\frac{1}{2}+\frac{1}{3}\right] = 19.5$$

3. $y = x^2,\ y = 8-x^3,\ x = -3,\ x = 3$



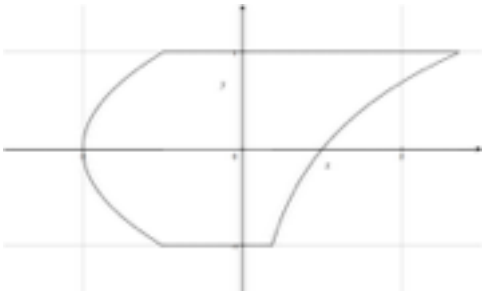
$$A = \int_{-3}^{1.716}\left[\left(8-x^3\right)-x^2\right]dx + \int_{1.716}^{3}\left[x^2-\left(8-x^3\right)\right]dx = 60.252$$

5. $y = 5x-x^2,\ y = x$



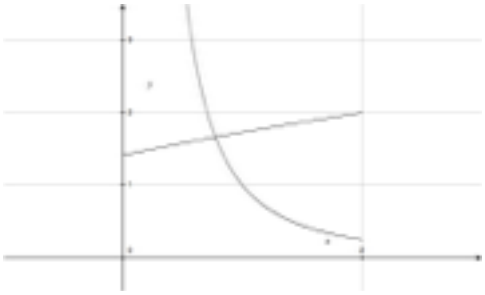
$$A = \int_{0}^{4}\left[\left(5x-x^2\right)-x\right]dx = \int_{0}^{4}\left[4x-x^2\right]dx = \left[2x^2-\frac{1}{3}x^3\right]_{0}^{4} = 32-\frac{8}{3} = 32/3$$

7. $x = e^y, x = y^2 - 2, y = -1, y = 1$



$$A = \int_{-1}^{1} \left[e^y - \left(y^2 - 2 \right) \right] dy = \left[e^y - \frac{1}{3} y^3 + 2y \right]_{-1}^{1} = \left[e^1 - \frac{1}{3} + 2 \right] - \left[\frac{1}{e} + \frac{1}{3} - 2 \right] = 5.684$$

9. $y = \sqrt{x+2}, y = \frac{1}{x^2}, x = 1, x = 2$

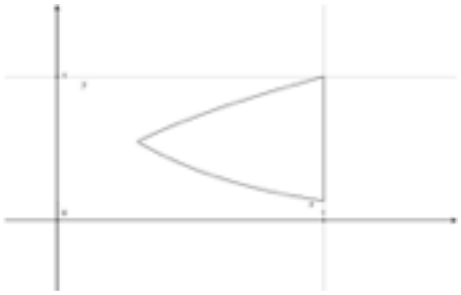


$$A = \int_{1}^{2} \left(\sqrt{x+2} - \frac{1}{x^2} \right) dx$$

$$= 1.369$$

$$\lim_{a \to 0^+} -\frac{1}{a} \text{ dne } \therefore \text{ the integral diverges}$$

11. $y=\sqrt{x},\ y=e^{-2x},\ x=1$



$$A=\int_{.301}^{1}\left(\sqrt{x}-e^{-2x}\right)dx=\left[\frac{2}{3}x^{3/2}+\frac{1}{2}e^{-2x}\right]_{.301}^{1}=.350$$

13. $y=x^2, y=2^x$

$$A=\int_{-.767}^{2}\left[2^x-x^2\right]dx=\left[\frac{2^x}{\ln 2}-\frac{1}{3}x^3\right]_{-.767}^{2}=2.106$$

6.1 Multiple Choice Solutions

1. $A=\int_{-9/8}^{9/8} e^{\left(\frac{-1}{1+x^2}\right)}dx=1.080$

The correct answer is B

---

3. $Area=\int_{0}^{1.456}\left(2\sin x-x\ln(2x+1)\right)dx=0.661$

The correct answer is B

---

5. $Area = \int_0^1 \left(e^{2x} - 1\right) dx = \left[\frac{1}{2}e^{2x} - x\right]_0^1 = \left(\frac{1}{2}e^2 - 2\right) - \left(\frac{1}{2}e^0 - 0\right) = \frac{1}{2}\left(e^2 - 3\right)$

The correct answer is B

6.2 Free Response Solutions

1. $y = 1 + 6x^{3/2}$ on $x \in [0, 1]$

$y' = 9x^{1/2} \rightarrow L = \int_0^1 \sqrt{1 + 81x}\, dx = 6.103$

3. $x = \frac{1}{3}\sqrt{y}\left(y - 3\right)$ on $y \in [1, 9]$

$x = \frac{1}{3}y^{3/2} - y^{1/2} \rightarrow x' = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$

$L = \int_1^9 \sqrt{1 + \left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2}\, dy = \int_1^9 \sqrt{1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1}}\, dy = \int_1^9 \sqrt{\frac{1}{2} + \frac{1}{4}y + \frac{1}{4}y^{-1}}\, dy = 10.667$

\

5. $y = \ln x^{3/2}$ on $x \in [1, \sqrt{3}]$

$y' = \frac{1}{x^{3/2}} \cdot \frac{3}{2}x^{1/2} = \frac{3}{2x} \rightarrow L = \int_1^{\sqrt{3}} \sqrt{1 + \frac{9}{4x^2}}\, dx = 1.106$

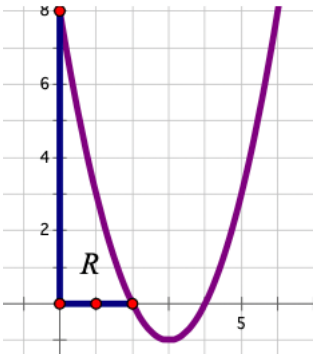
\

7. Find the length of the arc along $f(x) = \int_0^x \sqrt{\cos t}\, dt$ on $x \in \left[0, \frac{\pi}{2}\right]$.

$f'(x) = \sqrt{\cos x} \rightarrow L = \int_0^{\pi/2} \sqrt{1 + \cos x}\, dx = 2$

\

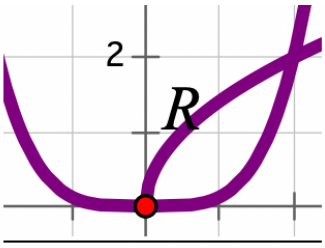
9. Find the perimeter of the region *R* bounded by $y = \left(x - 3\right)^2 - 1$, the *x*-axis, and the *y*-axis.



$$L=\int_0^2\sqrt{1+(2x-6)^2}\,dx+8+2=8.268+10=18.268$$

\

11. Find the perimeter of the region R bounded by $y=\sqrt{2x}$ and $y=\frac{1}{8}x^4$.



$$L=\int_0^2\sqrt{1+\left(0.5x^3\right)^2}\,dx+\int_0^2\sqrt{1+\left(\frac{1}{\sqrt{2x}}\right)^2}\,dx=3.200+2.958=6.158$$

\

6.2 Multiple Choice Solutions

1. $$\frac{dy}{dx}=\sqrt{x}\rightarrow L=\int_0^4\sqrt{1+\left(\sqrt{x}\right)^2}\,dx=\int_0^4\sqrt{1+x}\,dx=\left[\frac{2}{3}(1+x)^{3/2}\right]_0^4=\frac{2}{3}\left(5\sqrt{5}-1\right)$$

The correct answer is A

---

3. $\dfrac{dy}{dx}=\dfrac{1}{\cos x}(-\sin x)=-\tan x \rightarrow L=\int_0^{\pi/3}\sqrt{1+(-\tan x)^2}\,dx=\int_0^{\pi/3}\sqrt{\sec^2 x}\,dx=$

$\ln|\sec x+\tan x|_0^{\pi/3}=\ln\left|\sec\dfrac{\pi}{3}+\tan\dfrac{\pi}{3}\right|-\ln|\sec 0+\tan 0|=\ln|2+\sqrt{3}|-\ln 1=\ln|2+\sqrt{3}|$

The correct answer is D

---

5. $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}\rightarrow L=\int_a^b\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx=\int_a^b\sqrt{1+\left(\dfrac{1}{2\sqrt{x}}\right)^2}\,dx$

The correct answer is E

---

7. $\displaystyle\int_2^8\sqrt{1+\frac{1}{4-x^2}}\;dx=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\rightarrow\frac{dy}{dx}=\sqrt{\frac{1}{4-x^2}}=\frac{1}{\sqrt{4-x^2}}.$

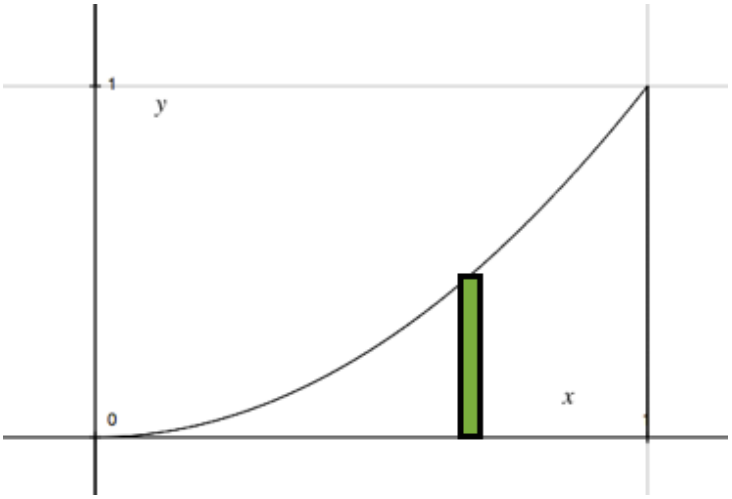
$\dfrac{dy}{dx}=\dfrac{1}{\sqrt{4-x^2}}\rightarrow y=\sin^{-1}\dfrac{x}{2}+c$

$(1,0)\rightarrow 0=\sin^{-1}\dfrac{1}{2}+c\rightarrow 0=\dfrac{\pi}{6}+c\rightarrow c=-\dfrac{\pi}{6}$

The correct answer is E

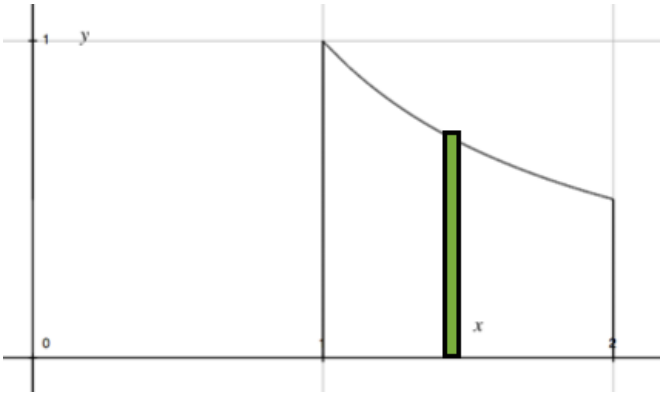
6.3 Free Response Solutions

1. $y = x^2, x = 1, y = 0;$ about the $x-$axis.



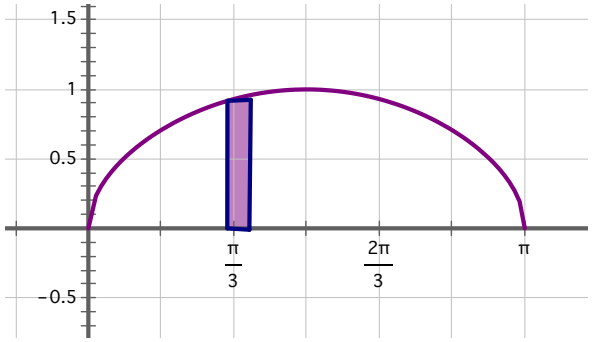
$$V = \pi \int_0^1 r^2\,dx = \pi \int_0^1 \left(x^2\right)^2\,dx = \pi \int_0^1 x^4\,dx = \pi \left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}$$

3. $y = \frac{1}{x}, x = 1, x = 2, y = 0;$ about the x-axis.



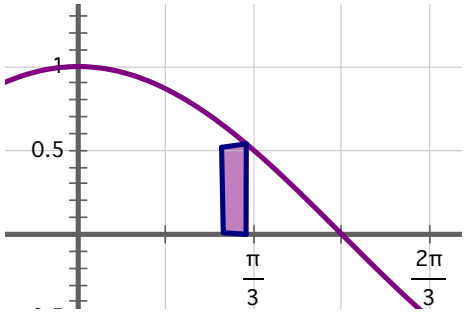
$$V = \pi \int_1^2 r^2\,dx = \pi \int_1^2 \left(\frac{1}{x}\right)^2\,dx = \pi \int_1^2 x^{-2}\,dx = \pi \left[\frac{x^{-1}}{-1}\right]_1^2 = \pi/2$$

5. The region bounded by $y=\sqrt{\sin x}$ and $y=0$.



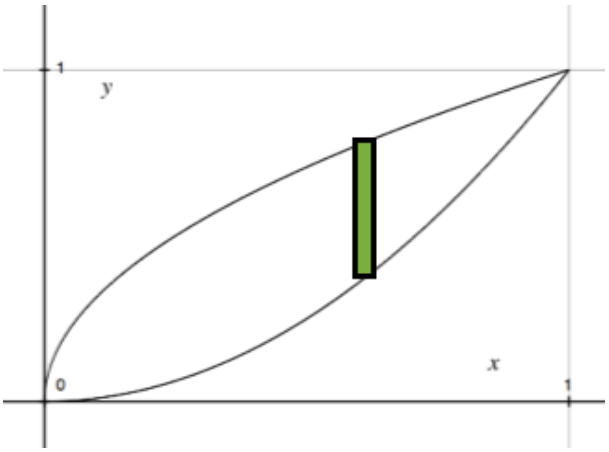
$$V=\pi\int_0^{\pi} r^2\,dx=\pi\int_0^{\pi}\left(\sqrt{\sin x}\right)^2 dx=\pi\int_0^{\pi}(\sin x)dx=\pi\left[-\cos x\right]_0^{\pi}=2\pi$$

7. The region bounded by $y=\cos x$, $x=0$, and $y=0$.



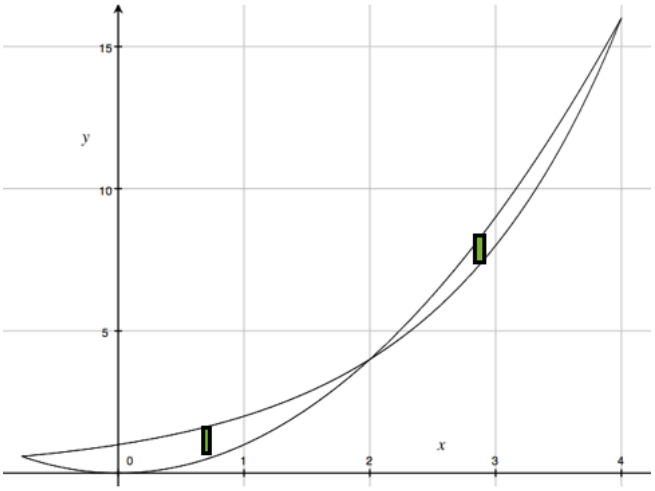
$$V=\pi\int_0^{\pi/2} r^2\,dx=\pi\int_0^{\pi/2}(\cos^2 x)dx=\pi\left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\pi}=\frac{\pi^2}{2}$$

9. $y = x^2, x = y^2;$ about the x-axis.



$$V = \pi\int_0^1 \left(R^2 - r^2\right)dx = \pi\int_0^1 \left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right]dx = \pi\int_0^1 \left(x - x^4\right)dx = \pi\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\Big|_0^1\right] = \frac{3\pi}{10}$$

11. $y = x^2, y = 2^x;$ about the x-axis.



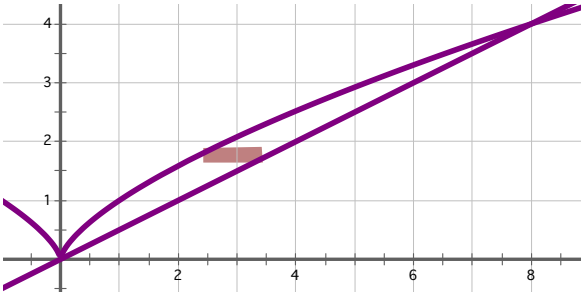
$$V = \pi\int_{-.767}^2 \left(R^2 - r^2\right)dx + \pi\int_2^4 \left(R^2 - r^2\right)dx$$

$$V = \pi\int_{-.767}^2 \left[\left(2^x\right)^2 - \left(x^2\right)^2\right]dx + \pi\int_2^4 \left[\left(x^2\right)^2 - \left(2^x\right)^2\right]dx = 94.612$$

$$V = \pi\int_{-.767}^2 \left(R^2 - r^2\right)dx + \pi\int_2^4 \left(R^2 - r^2\right)dx$$

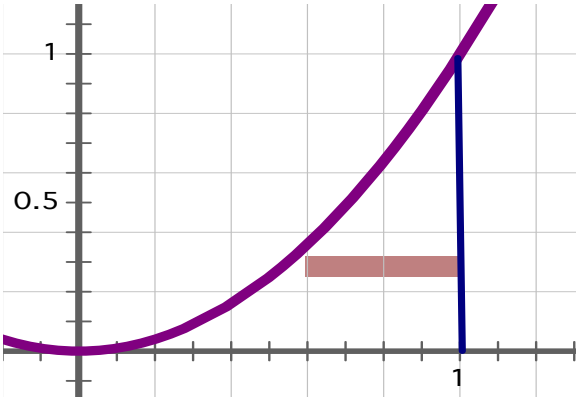
$$V = \pi\int_{-.767}^2 \left[\left(2^x\right)^2 - \left(x^2\right)^2\right]dx + \pi\int_2^4 \left[\left(x^2\right)^2 - \left(2^x\right)^2\right]dx = 94.612$$

13. $y^2 = x, x = 2y$; about the y-axis.



$$V = \pi\int_0^2 \left(R^2 - r^2\right)dy = \pi\int_0^2 \left[(2y)^2 - (y)^2\right]dy\pi\int_0^2 \left(4y^2 - y^4\right)dy = \pi\left[\frac{4}{3}y^3 - \frac{1}{5}y^5\Big|_0^2\right] = \frac{64\pi}{15}$$

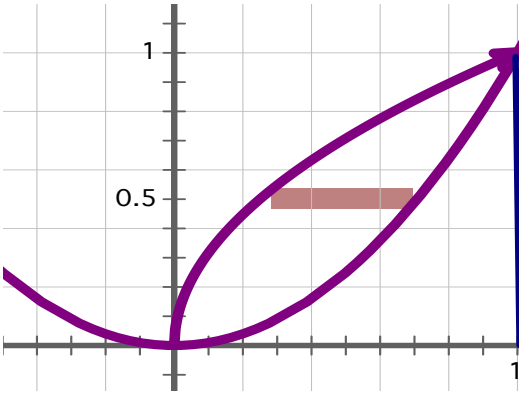
15. The region bounded by $y = x^2$, $x = 1$, and $y = 0$.



$y = x^2 \rightarrow x = \sqrt{x}$

$$V = \pi\int_0^1 \left(R^2 - r^2\right)dy = \pi\int_0^1 \left[(1)^2 - \left(y^{1/2}\right)^2\right]dy\pi\int_0^1 (1-y)dy = \pi\left[y - \frac{1}{2}y^2\Big|_0^1\right] = \frac{\pi}{2}$$

17. The region bounded by $y = x^2$ and $x = y^2$.



$y = x^2 \rightarrow x = \sqrt{y}$

$$V = \pi\int_0^1 \left(R^2 - r^2\right)dy = \pi\int_0^1 \left[\left(\sqrt{y}\right)^2 - \left(y^2\right)^2\right]dy = \pi\int_0^1 \left[y - y^4\right]dy = \pi\int_0^1 \left[\frac{1}{2}y^2 - \frac{1}{5}y^5\right]dy = \frac{3\pi}{10}$$

6.3 Multiple Choice Solutions

1. $$V = \pi\int_0^4 \left(\left(\sqrt{x}\right)^2 - \left(\frac{1}{2}x\right)^2\right)dx = \pi\int_0^4 \left(x - \frac{1}{4}x^2\right)dx =$$

$$\pi\left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^4 = \pi\left(8 - \frac{16}{3}\right) = \frac{8}{3}\pi$$

The correct answer is E

---

3. $$V = \pi\int_0^2 \left(\left(\sqrt{\frac{16x}{x^2+4}}\right)^2 - (x)^2\right)dx = \pi\int_0^2 \left(\frac{16x}{x^2+4} - x^2\right)dx = \pi\left[8\ln\left(x^2+4\right) - \frac{1}{3}x^3\right]_0^2 =$$

$$\pi\left[\left(8\ln 8 - \frac{8}{3}\right) - (8\ln 4 - 0)\right] = \pi\left(24\ln 2 - \frac{8}{3} - 16\ln 2\right) = \pi\left(8\ln 2 - \frac{8}{3}\right)$$

The correct answer is E

---

5. $$V = \pi\int_0^1 \left(3^2 - \left(\tan^{-1}x\right)^2\right)dx = 27.505$$

The correct answer is E

---

7. $y = \frac{\pi}{2} - \sin^{-1}(x-1) \rightarrow \sin^{-1}(x-1) = \frac{\pi}{2} - y \rightarrow x = 1 + \sin\left(\frac{\pi}{2} - y\right)$

$y = \frac{\pi}{2}\sqrt{x-2} \rightarrow \sqrt{x-2} = \frac{2y}{\pi} \rightarrow x = 2 + \left(\frac{2y}{\pi}\right)^2$

$Volume = \pi \int_0^{\pi} \left(2 + \left(\frac{2y}{\pi}\right)^2 - \left(1 + \sin\left(\frac{\pi}{2} - y\right)\right)\right)^2 dy = 108.895$

The correct answer is E

---

9. $Volume = \pi \int_0^2 \left(\pi^2 - \left(\frac{\pi}{2} - \sin^{-1}(x-1)\right)^2\right)dx + \pi \int_2^6 \left(\pi^2 - \left(\frac{\pi}{2}\sqrt{x-2}\right)^2\right)dx = 105.585$

The correct answer is D

6.4 Free Response Solutions

1. Washer Method

$V = \pi \int_{-4}^{4} \left(R^2 - r^2\right)dx = \pi \int_{-4}^{4} \left(\left(8 - (-1)\right)^2 - \left(\frac{1}{2}x^2 - (-1)\right)^2\right)dx =$

$2\pi \int_0^4 \left(81 - \left(\frac{1}{4}x^4 + x^2 + 1\right)\right)dx = 2\pi \int_0^4 \left(-\frac{1}{4}x^4 - x^2 + 80\right)dx$

$= 2\pi \left[-\frac{1}{20}x^5 - \frac{1}{3}x^3 + 80x\right]_0^4 = \frac{7424\pi}{15}$

3. Washer Method

$V = \pi \int_1^3 \left(R^2 - r^2\right)dx = \pi \int_1^3 \left(\left(1 + \frac{1}{x}\right)^2 - 1^2\right)dx = 8.997$

5. Washer Method $V = \pi \int_0^1 \left(R^2 - r^2\right)dx = \pi \int_0^1 \left(\left(1 - x^3\right)^2 - \left(1 - \sqrt{x}\right)^2\right)dx$

$$\pi\int_0^1\left(\left(1-2x^3+x^6\right)-\left(1-2\sqrt{x}+x\right)\right)dx=\pi\int_0^1\left(x^6-2x^3-x+2x^{1/2}\right)dx$$

$$=\pi\left(\frac{1}{7}x^7-\frac{1}{2}x^4-\frac{1}{2}x^2+\frac{4}{3}x^{3/2}\right)_0^1=\frac{10\pi}{21}$$

7. $y=\sqrt{x+1}\rightarrow x=y^2-1$

Disk Method: $V=\pi\int_0^2\left(\left[3-\left(y^2-1\right)\right]^2\right)dy=\pi\int_0^2\left[9-\left(y^4-2y^2+1\right)\right]dy$

$$=\pi\left[-\frac{1}{5}y^5+\frac{2}{3}y^3+8\right]_0^2=\pi\left[-\frac{32}{5}+\frac{16}{3}+8\right]=\frac{104\pi}{15}$$

9. Washer Method: $V=\pi\int_{-1}^1\left(R^2-r^2\right)dy=\pi\int_{-1}^1\left(\left(3-x_1\right)^2-\left(3-x_2\right)^2\right)dy$

$$=\pi\int_{-1}^1\left(\left(3-y^2\right)^2-(3-1)^2\right)dy=\pi\int_{-1}^1\left(y^4-6y^2+9-4\right)dy=2\pi\int_0^1\left(y^4-6y^2+5\right)dy$$

$$=2\pi\left(\frac{1}{5}y^4-2y^3+5y\right)_0^1=\frac{32\pi}{15}$$

11. Washer Method $V=\pi\int_{-.916}^{.916}\left(R^2-r^2\right)dx=\pi\int_{-.916}^{.916}\left(\left(1-y_1\right)^2-\left(1-y_2\right)^2\right)dx$

$$=\pi\int_{-.916}^{.916}\left(\left(1-\ln\left(x^2+1\right)\right)^2-(1-\cos x)^2\right)dx=3.447$$

13. Washer Method $y=\sqrt{6x-x^2}\rightarrow y^2=6x-x^2\rightarrow -y^2=x^2-6x$

$$-y^2+9=x^2-6x+9=\left(x-3\right)^2\rightarrow 9-y^2=\left(x-3\right)^2\rightarrow x=3+\sqrt{9-y^2}$$

$$g\left(x\right)=y=-1+e^{-0.5x}\rightarrow y+1=e^{-0.5x}\rightarrow\ln\left(y+1\right)=-0.5x\rightarrow -2\ln\left(y+1\right)$$

$$V=\pi\int_0^{2.9910233}\left[\left[\left(-2\ln\left(y+1\right)\right)-\left(-1\right)\right]^2-\left[\left(3+\sqrt{9-y^2}\right)-\left(-1\right)\right]^2\right]dy=374.525$$

6.4 Multiple Choice Solutions

1. This is a Washer Method problem. Both curves are below the line $y=3$, so $V=\pi\int_a^b\left(\left[k-f\left(x\right)\right]^2-\left[k-g\left(x\right)\right]^2\right)dx$. $y=2x$ is closer to the line, so

The correct answer is C

---

3. This is a Disk Method problem. $V = \pi \int_0^1 \left[3 - \tan^{-1} x\right]^2 dx = 20.773$ $y = \tan^{-1} x$

The correct answer is E

---

5. $Volume = \int_0^2 \left(\pi - \left(\frac{\pi}{2} - \sin^{-1}(x-1) \right) \right)^2 dx + \int_2^6 \left(\pi - \frac{\pi}{2}\sqrt{x-2} \right)^2 dx = 39.111$

The correct answer is C

6.5 Free Response Solutions

1. Rectangle length = $y = (x-3)^2 - 1 = x^2 - 6x + 8$

   Area = $y^2 = \left(x^2 - 6x + 8\right)^2$

   Volume $= \int_0^2 \left(x^2 - 6x + 8\right)^2 dx = 33.067$

3. $y = \sqrt{2x} \rightarrow x = \frac{y^2}{2}$; $y = \frac{1}{8}x^4 \rightarrow x = \sqrt[4]{8y}$; Rectangle length = $\sqrt[4]{8y} - \frac{y^2}{2}$

   Area = $3x^2 = 3\left(\sqrt[4]{8y} - \frac{y^2}{2} \right)^2$

   Volume $= \int_0^2 3\left(\sqrt[4]{8y} - \frac{y^2}{2} \right)^2 dx = 6.031$

5. The base is a circle of radius 4 and the cross-sections are squares.

$$V = \int_{-4}^{4} (side)^2 dx = \int_{-4}^{4} (2y)^2 dx = \int_{-4}^{4} 4y^2 dx = 4\int_{-4}^{4} \left(16 - x^2\right) dx = 341.333$$

7. The base is the region $x^2 \le y \le 1$ and the cross-sections perpendicular to the *y*-axis are squares.

$$V = \int_0^1 (side)^2\, dy = \int_0^1 (2x)^2\, dy = \int_0^1 4x^2\, dy = \int_0^1 4y\, dy = 2$$

9. The base is the region $x^2 \le y \le 1$ and the cross-sections perpendicular to the *y*-axis are equilateral triangles.

$$V = \int_0^1 \left(\frac{\sqrt{3}}{4} side\right)^2 dy = \int_0^1 \frac{\sqrt{3}}{4}(2x)^2\, dy = \sqrt{3}\int_0^1 x^2\, dy = \sqrt{3}\int_0^1 y\, dy = \frac{\sqrt{3}}{2}$$

11. $y = \ln\left(x^2+1\right)$ *and* $y = \cos x$; the cross-sections are squares.

$$V = \int_{-.916}^{.916} (side)^2\, dx = \int_{-.916}^{.916} \left(\cos x - \ln\left(x^2+1\right)\right)^2 dx = .916$$

13. The cross-sectional areas of a felled tree cut into 10-foot sections are given by in table below. Use the Midpoint Riemann Sum with n=5 to estimate the volume of the tree.

x	0	10	20	30	40	50	60	70	80	90	100
Area	.68	.65	.64	.61	.58	.59	.53	.55	.52	.50	.48

$$\begin{aligned} V &= \int_0^{100} A(x)\, dx \\ &\approx 20\cdot A(10) + 20\cdot A(30) + 20\cdot A(50) + 20\cdot A(70) + 20\cdot A(90) \\ &= 20\cdot(.65) + 20\cdot(.61) + 20\cdot(.59) + 20\cdot(.55) + 20\cdot(.5) \\ &= 58 \end{aligned}$$

6.5 Multiple Choice Solutions

1. $$V = \int_0^3 \left(e^{-x^2/2}\right)^2 dx = \int_0^3 \left(e^{-x^2}\right) dx = 0.886$$

The correct answer is A

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3. Perpendicular to the *y*-axis means the *x* must be isolated:

$x+3y=5\rightarrow x=5-3y.\quad V=\int_0^{5/3}(5-3y)^2\,dy=13.889$

The correct answer is E

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5. $Volume=\int_0^{\pi/4}\frac{\pi}{2}\left(\frac{\cos x-\sin x}{2}\right)^2 dx=0.112$ and $y=\cos x$

The correct answer is B

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7. $x^2+4y^2=4\rightarrow y=\sqrt{1-\frac{1}{4}x^2}\rightarrow$

$$V=\int_{-2}^{2}\left(\sqrt{1-\frac{1}{4}x^2}\right)^2 dx=2\int_0^2\left(1-\frac{1}{4}x^2\right)dx=2\left[x-\frac{1}{12}x^3\right]_0^2=2\left[2-\frac{2}{3}\right]=\frac{8}{3}$$

The correct answer is A

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9. $V=\int_0^2\frac{1}{2}\left(\pi-\left(\frac{\pi}{2}-\sin^{-1}(x-1)\right)\right)^2 dx+\int_2^6\frac{1}{2}\left(\pi-\left(\frac{\pi}{2}\sqrt{x-2}\right)\right)^2 dx=6.225$

The correct answer is A.

6.6 Free Response Solutions

1a. $Area = \int_0^2 \left(2x^2 - x^3\right) dx = \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16}{3} - 4 = \frac{4}{3}$

1b. $V = \frac{\pi}{8}\int_a^b r^2\,dx = \frac{\pi}{8}\int_0^2 \left(\frac{2x^2 - x^3}{2}\right)^2 dx$

1c. The rectangles need to be horizontal, so $y = 2x^2 \rightarrow x = \sqrt{\frac{y}{2}}$ and $y = x^3 \rightarrow x = y^{1/3}$.

Washer Method: $V = \pi\int_0^8 \left(\left[y^{2/3}\right]^2 - \left[\sqrt{\frac{y}{2}}\right]^2\right) dy = \pi\int_0^8 \left(y^{4/3} - \frac{1}{2}y\right) dy =$

$$= \pi\left[\frac{3}{7}y^{7/3} - \frac{1}{4}y^2\right]_0^8 = \pi\left(\frac{128}{7} - 16\right) = \frac{16\pi}{7}.$$

3a. $Area = \int_{-1}^3 (x+1)^{1/2}\,dx = \left[\frac{2}{3}(x+1)^{3/2}\right]_{-1}^3 = \frac{16}{3} - 0 = \frac{16}{3}$

3b. $Volume = \int_{-1}^3 (x+1)^{1/2}\left(2(x+1)^{1/2}\right) dx = \int_{-1}^3 (2x+2)\,dx =$

$\left[x^2 + 2x\right]_{-1}^3 = 15 - (-1) = 16$

3c. Washer Method with horizontal rectangles:

$y = \sqrt{x+1} \rightarrow x = y^2 - 1$

$V = \pi\int_0^2 \left(\left[4 - \left(y^2 - 1\right)\right]^2 - \left[4 - 3\right]^2\right) dy$

5a. $$Area = \int_0^2 \left(8\sqrt{2x} - x^4\right) dx = \left[\frac{8}{3}(2x)^{3/2} - \frac{1}{5}x^5\right]_0^2 = \frac{64}{3} - \frac{64}{5} = \frac{124}{15}$$

5b. Washer: $$Volume = \pi \int_0^2 \left(\left(8\sqrt{2x}\right)^2 - \left(x^4\right)^2\right) dx = \pi \int_0^2 \left(128x - x^8\right) dx =$$

$$\pi\left[64x^2 - \frac{1}{9}x^9\right]_0^2 = \pi\left(256 - \frac{512}{9}\right) = \frac{1792\pi}{9} \approx 625.526$$

5c. Washer Method with horizontal rectangles:

$$x = \frac{y^2}{128}$$

$$Volume = \pi \int_0^{16} \left(\left(y^{1/4}\right)^2 - \left(\frac{y^2}{128}\right)^2\right) dy = \pi \int_0^{16} \left(y^{1/2} - \frac{y^4}{16384}\right) dy$$

7a. Find the area of the regions R, S, and T.

$$Area_R = \int_0^{1.1418956} \left[\left(x^3 - 3x^2 + 2x + 4\right) - \left(2x\sqrt{4-x}\right)\right] dx = 2.463$$

$$Area_S = \int_{1.1418956}^{2.5444495} \left[\left(2x\sqrt{4-x}\right) - \left(x^3 - 3x^2 + 2x + 4\right)\right] dx = 1.551$$

$$Area_T = \int_0^4 \left[2x\sqrt{4-x}\right] dx - Area_S = 15.515$$

7b. Find the volume of the solid generated by rotating the curve $g(x) = 2x\sqrt{4-x}$ around the line $y = 8$ on the interval $x \in [1,3]$.

$$V = \pi \int_1^3 \left(8 - 2x\sqrt{4-x}\right)^2 dx = 48.152$$

7c. Find the volume of the solid generated by rotating the region S around the line $y = -1$.

$$V = \int_{1.1418956}^{2.5444495} \left[\left(2x\sqrt{4-x} - (-1)\right)^2 - \left(x^3 - 3x^2 + 2x + 4 - (-1)\right)^2\right] dx = 55.727$$

7d. Find the volume of the solid generated if *R* forms the base of a solid whose cross sections are squares whose bases are perpendicular to the *x*-axis

$$V=\int_{0}^{1.1418956}\left[\left(x^3-3x^2+2x+4\right)-\left(2x\sqrt{4-x}\right)\right]^2\,dx=6.962$$

9a. The volume of the solid generated when *h* is rotated around the line $y=2$ on the interval $x\in[0,2]$

$$V=\pi\int_{0}^{2}\left[2-\ln\left(x^2+1\right)\right]^2\,dx=12.004$$

9b. The volume of the solid formed if the region between the curve and the lines $y=-1$, $x=0$, and $x=2$ forms the base of a solid whose cross sections are rectangles with heights twice their base and whose bases are perpendicular to the *x*-axis.

$$V=\int_{0}^{2}2\left[\ln\left(x^2+1\right)+1\right]^2\,dx=12.840$$

11a. Find the area of R

$$A_R=\int_{0}^{.19436069}\left(e^{-x}-\left(\frac{1}{4}+\sin\left(\pi x\right)\right)\right)dx=.071$$

11b. Find the area of S

$$A_S=\int_{.19436069}^{.95723257}\left(\left(\frac{1}{4}+\sin\left(\pi x\right)\right)-e^{-x}\right)dx=.328$$

11c. Find the volume of the solid generated when S is revolved around the horizontal line $y=-2$

$$V_S=\pi\int_{.19436069}^{.95723257}\left(\left(\frac{1}{4}+\sin\left(\pi x\right)+2\right)^2-\left(e^{-x}+2\right)^2\right)dx=5.837$$

11d. The region R forms the base of a solid whose cross-sections are rectangles with bases perpendicular to the *x*-axis and heights equal to half the length of the base. Find the volume of the solid.

$$V_S = \pi \int_{.19436069}^{.95723257} \frac{1}{2}\left(\left(\frac{1}{4}+\sin(\pi x)\right)-\left(e^{-x}\right)\right)^2 dx = .007$$

13a) $ave = \frac{1}{27}\int_0^{27} k(x)\,dx$

13b) $V = \pi \int_0^{27} \left[\frac{k(x)}{2}\right]^2 dx = 4(1.4)^2 + 5(1.4)^2 + 6(1.55)^2 + 3(1.35)^2 + 4(1.45)^2 + 4(1.55)^2$

$= 55.5425 mm^3$

13c) Since $k(x)$ is differntiable, it is also continuous and the IVT applies. The range of $k(x)$ is $2.7 \le k(x) \le 3.1$. 2.83 is in that range, therefore, there must at least one *x*-value where $k(x) = 2.83$.

13d) Since $k(x)$ is differntiable, it is also continuous and the MVT applies. $k(4) = k(9)$ and $k(16) = k(27)$, therefore, there must be at least two *x*-values where $k'(x) = 0$.

15a) $-0.2 = \frac{1}{1.21}x^2 - 1 \rightarrow x = \pm\sqrt{.8 \cdot 1.21} = \pm 0.9838699$

$$Area = \int_{-0.9838699}^{0.9838699}\left[0-\left(\frac{1}{1.21}x^2-1\right)\right]dx = 1.443$$

15b) $0 = \frac{1}{1.21}x^2 - 1 \rightarrow x = \pm 1.1$

$$Area = \int_{-1.1}^{1.1}\left(\frac{1}{1.21}x^2-1\right)^2 dx = 1.173333...$$

$L = 4\ miles = 21{,}120\ ft$

$Volume = \int_0^{21120}(1.173333...)\,dx = 24{,}780.8\ ft^3$

15c) $Ave\ Flow = \frac{1}{60}\int_0^{60} 1.443 f(t)\,dx = 5.109\ {ft^3}/{min}$

15d) $Ave\ Flow = \frac{1}{20}\int_{20}^{40} 1.443 f(t)\,dx = 6.913 > 6.2$. Yes, the gates need to be closed.

17a) $\frac{1}{169}x^2+\frac{1}{64}y^2=1\rightarrow\frac{1}{169}x^2=1-\frac{1}{64}y^2\rightarrow x^2=169-\frac{169}{64}y^2\rightarrow x=\sqrt{169-\frac{169}{64}y^2}$

$V=\pi\int_{-8}^{8}\left(\sqrt{169-\frac{169}{64}y^2}\right)^2dy=\pi\int_{-8}^{8}\left(169-\frac{169}{64}y^2\right)dy=\pi\left[169y-\frac{169}{64}\frac{y^2}{3}\right]_{-8}^{8}=5663.244\,yd^3$

17b) $\frac{1}{169}x^2+\frac{1}{64}y^2=1\rightarrow\frac{1}{64}y^2=1-\frac{1}{169}x^2\rightarrow x^2=64-\frac{64}{169}x^2\rightarrow x=\sqrt{64-\frac{64}{169}x^2}$

$V=\pi\int_{-13}^{13}\left(\sqrt{64-\frac{64}{169}x^2}\right)^2dx=\pi\int_{-13}^{13}\left(64-\frac{64}{169}x^2\right)dx=\pi\left[169x-\frac{169}{64}\frac{x^2}{3}\right]_{-13}^{13}=3485.073\,yd^3$

17c) Oblate: $V=5663.244(200)=1{,}132{,}648.8721$ gallons

Prolate: $V=3485.073(200)=697{,}014.690$ gallons

17d) $V=\frac{\pi}{3}h^2(32-h)=\frac{32\pi}{3}h^2-\frac{\pi}{3}h^3$

$\frac{d}{dt}\left[V=\frac{32\pi}{3}h^2-\frac{\pi}{3}h^3\right]\rightarrow\frac{dV}{dt}=\left(\frac{64\pi}{3}h-\pi h^2\right)\frac{dh}{dt}$

$-35=\left(\frac{640\pi}{3}-100\pi\right)\frac{dh}{dt}\rightarrow\frac{dh}{dt}==0.098\,{yd^3}/{month}$

19b) $V=\pi\int_0^{12}\left(xe^{-.05x^2}\right)^2dx=\pi\int_0^{12}\left(xe^{-.1x^2}\right)dx=44.021$

19c) $V=\int_0^{12}\frac{3\sqrt{3}}{2}r^2\,dx=\frac{3\sqrt{3}}{2}\int_0^{12}\left(-xe^{-.05x^2}\right)^2dx=36.405$

19d) $\frac{d}{dt}\left[s=2xe^{-.05x^2}\right]\rightarrow\frac{ds}{dt}=\left[2xe^{-.05x^2}(-.1x)+e^{-.05x^2}(2)\right]\frac{dx}{dt}$

$\frac{ds}{dt}=\left[6e^{-.45}(-.3)+e^{-.45}(2)\right]=0.255$

6.7 Free Response Solutions

1. $y=\sec x, y=1, x=0, x=\frac{\pi}{6}$; about the y-axis.

$$V=\int_0^{\pi/6} 2\pi rl\,dx=\int_0^{\pi/6} 2\pi\cdot x\cdot(\sec x-1)\,dx=.064$$

3. $y=\frac{1}{x^{2/3}}, x=1, x=8, y=0$; about the y-axis.

$$V=\int_1^8 2\pi rl\,dx=\int_1^8 2\pi\cdot x\cdot\left(x^{-2/3}\right)dx=70.689$$

5. $y=x(x-1)^2$ and the x-axis; about the y-axis.

$$V=\int_0^1 2\pi rl\,dx=\int_0^1 2\pi\cdot x\cdot y\,dx=\int_0^1 2\pi x(x-1)^2\,dx=.209$$

7. $y=\sqrt{x}, y=e^{-2x}, x=1;$ about the line $x=1$.

$$V=\int_{.301}^1 2\pi rl\,dx=\int_{.301}^1 2\pi(1-x)\left(\sqrt{x}-e^{-2x}\right)dx=.554$$

9. $y=x^2,\ y=2^x$; about the line $x=-1$.

$$\begin{aligned}V&=\int_{-.767}^2 2\pi rl\,dx+\int_2^4 2\pi rl\,dx\\&=\int_{-.767}^2 2\pi(1+x)\left(2^x-x^2\right)dx+\int_2^4 2\pi(1+x)\left(x^2-2^x\right)dx\\&=55.428\end{aligned}$$

Volume Practice Test

1. $y=x-4$ is the top curve and $y=x^2-4$ is the bottom. The curves intersect at $x=0$ *and* 1.

The correct answer is A

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3. $$V=\pi\int_0^{\ln 3}\left[\left(e^{x/2}\right)^2-1^2\right]dx=2.83$$

The correct answer is B

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5. About the *y*-axis means the *x* is isolated: $x = \sin y$ and $y \in \left[0, \frac{\pi}{2}\right]$.

The correct answer is D

7a. $area = \int_0^6 \left(x\sqrt{12+4x-x^2} - \left(-3\sin\left(\frac{\pi}{3}x\right)\right)\right) = 54.295$

7b) Washer Method:

$$V = \pi \int_0^6 \left(\left(x\sqrt{12+4x-x^2} - (-4)\right)^2 - \left(-3\sin\left(\frac{\pi}{3}x\right) - (-4)\right)^2 \right) = 1012.159$$

c) $$V = \int_0^6 \left(\left(x\sqrt{12+4x-x^2} - \left(-3\sin\left(\frac{\pi}{3}x\right)\right)\right)^2 \right) = 713.902$$

9a. $area = \int_0^2 \left(\frac{8}{x^2+4}\right)dx = 8\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = 4\left(\tan^{-1}1 - \tan^{-1}0\right) = 4\left(\frac{\pi}{4}\right) = \pi$

9b. Disk Method: $V = \pi \int_0^2 \left(\frac{8}{x^2+4}\right)^2 dx$

9c. Disk Method with horizontal rectangles:

$$y = \frac{8}{x^2+4} \rightarrow x^2+4 = \frac{8}{y} \rightarrow x^2 = \frac{8}{y} - 4 \rightarrow x = \sqrt{\frac{8}{y}-4}$$

$$V = \pi \int_0^2 \left(\sqrt{\frac{8}{y}-4}\right)^2 dx$$