AP Calculus AB '21-22

Spring Final Part IIA v1

Calculator Allowed

Name:

Solution Key

1. The Pony Express achieved mythic status in the Old West despite only running for 19 months. The series of riders took mail from St. Joseph, MO, to Sacramento, CA—1966 miles in 10 days. One rider named Pony Bob Haslam impressed with both the fastest ride and the longest ride on record. The longest was 190 miles one way, from Friday's Station to Smith's Creek, Nevada. Because of active Paiute war parties, other relay riders refused to accept the mail and ride the next leg of the journey, so Bob continued on without rest. Estimates of his distances and average velocity one way at different times along the route are given in the table below.

Time t in hours	0	2.5	4.8	6.3	7.2	8.6	10
$v(t)$ in mi/hr	0	15.6	13.5	18.7	11.6	11.6	11.6

2

a) Approximate Bob's acceleration at $t = 5.1$ hours. Indicate units.

$$a(5.1) \approx \frac{18.7 - 13.5}{6.3 - 4.8} \quad 3.467 \ \mathrm{MI}/\mathrm{HR}^2$$

2

b) Use a Midpoint-hand Riemann Sum to approximate $\int_0^{10} v(t)dt$. Using the correct units, ~~explain why this is an over-estimate.~~

$$\int_0^{10} v(t)\,dt \approx 4.8(15.6) + 2.4(18.7) + 2.8(11.6)$$

$$152.24$$

$$= \cancel{170.94} \ \text{MILES}$$

2

c) Using your answer from part b), estimate Bob's average velocity during this ride.

$$\text{AVE VELOCITY} = \frac{1}{10-0}\int_0^{10} v(t)\,dt$$

$$\approx \frac{1}{10}(152.24) = 15.224\ \frac{\text{MI}}{\text{HR}}$$

3

d) Assume that $H(t) = 6.2te^{-0.21t}$ is an accurate model of Bob's velocity. Does the model indicate that his velocity is increasing at an increasing rate at $t = 3$ hours? Using the correct units, explain your reasoning.

$H(3) = 9.906$ MPH

$H'(3) = 1.221 > 0$

YES, BOB'S VELOCITY IN INCREASING AT AN INCREASING RATE BECAUSE $H(3)$ AND $H'(3) > 0$

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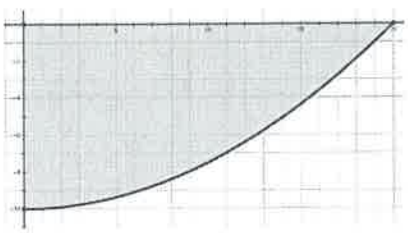
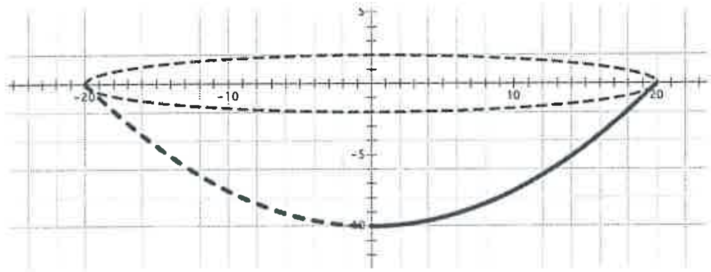
2. After the water from the Colorado River passes through the Yuma Desalting Plant to have its excess salts removed, the slurry (the densely salted byproduct water) is sent to evaporation pits in the desert where the salt sinks to the bottom and the water evaporates away. Assume the pit is in a shape formed by rotating the curve $y=\frac{x^2}{40}-10$ on $x\in[0,\ 20]$ about the ***y*-axis**.





4 a) How much slurry can this pit hold? Show the antiderivative steps.

$$y=\frac{x^2}{40}-10 \longrightarrow x=\sqrt{40(y+10)}$$

$$V=\pi\int_{-10}^{0}\left(\sqrt{40(y+10)}\right)^2 dy = 2000\pi\ ft^3$$

$$= 6\,283.185\ ft^3$$

$$=\pi\int_{-10}^{0}(40y+400)\,dy$$

$$=\pi\left[20y^2+400y\right]_{-10}^{0}$$

1 b) Write an equation that would give the volume of the slurry in the pit at any depth *h*, where is the number of feet below the rim of the pit.

$$V=\pi\int_{-10}^{h}\left(\sqrt{40(y+10)}\right)^2 dy$$

2

c) After a certain amount of time exposed to the desert heat, all the water evaporates, leaving just the salt. At that point, the level of salt is 3.9 feet below the rim of the pit. Use the equation in (b) to find the volume of the salt after all the water has evaporated.

$$V = \pi \int_{-10}^{-3.9} 40(y+10)\,dy$$

$$= 2337.973 \text{ Ft}^3/\text{ye}$$

2

d) The water in the slurry evaporates at a rate of $0.1 \, {}^{ft}\!/_{day}$. How fast is the volume of the slurry changing when $h = -1.9\,ft$?

$$\frac{d}{dt}\left[\pi \int_{-10}^{h} 40(y+10)\,dy\right] = \pi\left(40(h+10)\right)\frac{dh}{dt} = \frac{dV}{dt}$$

$$\left.\frac{dV}{dt}\right|_{t=-1.9} = \pi\left(40\left(-1.9+10\right)\right)(-.1) = -101.788 \ \frac{ft^3}{day}$$

End of

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AP Calculus AB '21-22

Spring Final Part IIB v1

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Name:

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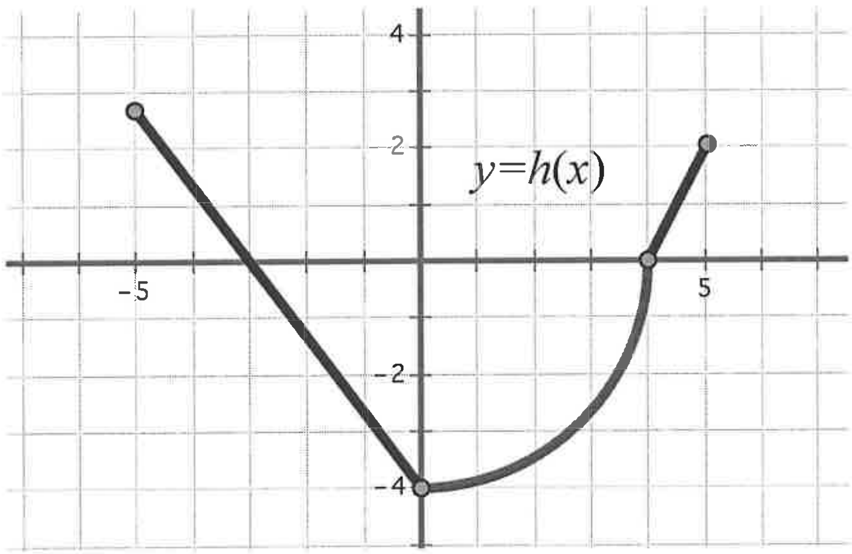

3. The graph above, $h(x)$ on $-5 \le x \le 5$, is comprised of two line segments and a quarter circle. Let $g(x) = 2 + \int_{-3}^{x} h(t)\,dt$.

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3 (a) Find $g(4)$, $g'(4)$, and $g''(4)$.

$g(4) = 2 + -6 + \left(-\frac{1}{4}(16\pi)\right) = -4 - 4\pi \approx -16.567$

$g'(4) = h(4) = 0$

$g''(4) = h'(4) = \text{DNE}$

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2 (b) At what x-value(s) on $-5 \le x \le 5$ does $g(x)$ have a relative minimum. Explain ~~n~~ your reasoning.

$x = -3$ AND $x = 5$, $g'(x) = h(x)$ AND $h(x)$ SWITCHES FROM − TO + AT $x = -3$ AND FROM ~~+ TO THE~~ THE LEFT ~~RIGHT~~ END POINT AT $x = 5$ TO A −

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2 (c) At what x-value(s) on $-5 \leq x \leq 5$ does $g(x)$ have a point of inflection. Explainn your reasoning.

$X = 0$ BECAUSE g' SWITCHES FROM DECREASING TO INCREASING

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2 (d) On what interval(s) is $g(x)$ both increasing and concave down? Explain why.

g IS INCREASING WHEN $g' = h$ IS POSITIVE

g IS CONCAVE DOWN WHEN h IS DECREASING

SO $g(x)$ IS INCREASING AND CONCAVE DOWN ON $x \in (-5, -3)$ BECAUSE $g' = h$ IS POSITIVE AND DECREASING

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4. Consider the curve given by $2x^2 - xy + y^2 = 28$.

2 a) Show that $\frac{dy}{dx} = \frac{4x-y}{x-2y}$.

$$\frac{d}{dx}\left[2x^2 - xy + y^2 = 28\right]$$

$$4x - x\frac{dy}{dx} - y(1) + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = -4x + y$$

$$\frac{dy}{dx} = \frac{-4x+y}{2y-x} = \frac{4x-y}{x-2y}$$

3 b) Find point(s) P where the tangent line is horizontal.

$$\frac{dy}{dx} = 0 = 4x - y \rightarrow y = 4x \quad ①$$

$$\cancel{2x^2} \quad 2x^2 - x(4x) + (4x)^2 = 28$$

$$14x^2 = 28$$

$$x^2 = 2$$

$$x = \pm\sqrt{2} \quad ①$$

$(\sqrt{2}, 4\sqrt{2})$ ①

$(-\sqrt{2}, -4\sqrt{2})$

2 c) Find $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left[\frac{dy}{dx} = \frac{4x-y}{x-2y}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(x-2y)\left(4-\frac{dy}{dx}\right) - (4x-y)\left(1-2\frac{dy}{dx}\right)}{(x-2y)^2}$$

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2 d) Determine if each value found in b) is at a maximum, a minimum, or neither. Justify your answer.

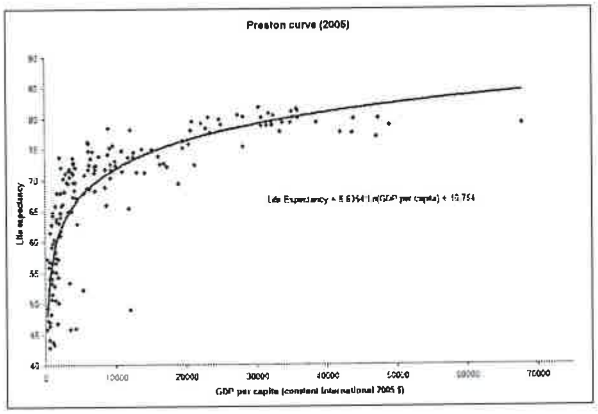
$\frac{dy}{dx} = 0$ AT $x = \pm\sqrt{2}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{2},4\sqrt{2}} = \frac{(\sqrt{2}-8\sqrt{2})(4-0)-(4\sqrt{2}-4\sqrt{2})(1-2(0))}{(\sqrt{2}-8\sqrt{2})^2} = \frac{-28\sqrt{2}}{+} < 0$$

$\therefore$ $\sqrt{2}, 4\sqrt{2}$ IS A MAX

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\sqrt{2},-4\sqrt{2}} = \frac{(-\sqrt{2}+8\sqrt{2})(4-0)-(-4\sqrt{2}+4\sqrt{2})(1-2(0))}{(-\sqrt{2}+8\sqrt{2})^2}$$

$= \frac{28\sqrt{2}}{+} > 0$ $\therefore$ $(-\sqrt{2}, -4\sqrt{2})$ IS AT A MIN



5. In 1975, Samuel Preston explored the correlation between life expectancy and real per capita income in the World over the previous 75 years. Show are the data and curve for 2005. The curve that best fits the data is a logarithmic growth curve, but another model might be a simple bounded exponential function. Assume that the 2022 data might be modeled by the differential equation $\frac{dL}{di} = 0.01(83 - L)$, where L is the average life expectancy in years and i is the per capita income in hundreds of doollars. Also assume that $L(0) = 5$.

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2 (a) If $L(200) = 72$, find the equation of the line tangent to the Preston curve.

$$m = \left.\frac{dL}{dt}\right|_{t=200} = .01(83 - 72) = .11$$

$$L - 72 = .11(t - 200)$$

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2 (b) Use the tangent line equation to approximate $L(350)$. Explain why this is an overestimate.

$$L - 72 = .11(350 - 200)$$

$$= .11(150)$$

$$L - 72 = 16.5$$

$$L = 88.5$$

B- 88.5 IS AN OVERESTIMATE BECAUSE $L(i)$ IS CONCAVE DOWN

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5 (c) Find the particular solution to $\frac{dL}{dt} = 0.01(83 - L)$ with the initial condition $L(0) = 5$.

$$\int \frac{1}{83-L}\, dL = \int .01\, dt$$

$$-\ln|83-L| = .01t + c$$

$$e^{\ln|83-L|} = e^{-.01t + c}$$

$$|83-L| = e^{-.01t+c} = Ke^{-.01t}$$

$$(0,5) \Rightarrow 78 = Ke^{0}$$

$$K = 78$$

$$83 - L = 78e^{-.01t}$$

$$L = 83 - 78e^{-.01t}$$

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