AP Calculus AB '22-23

Spring Final Part IIA v1

Calculator Allowed

Name:

Solution Key

1. The snowfall in Donner Summit is tracked by the US Weather Service. For the month of March, 2022, $S(t)$ represents the rate of snowfall in inches per day and its data is presented in the table below. $M(t) = 0.65 - 0.35\cos\left(\frac{5x^{0.95}}{6}\right)$ represents the rate at which the snow melts in inches per day, where t is measured in days.

t in days	1	3	4	7	11	15	21
$S(t)$ in inches per day	1.4	4.9	4.4	0.1	4.6	0.2	2.7

a) Find $\int_1^{21} M(t)\,dt$. Using the correct units, explain the meaning of $\frac{1}{21-1}\int_1^{21} M(t)\,dt$.

1 $\int_1^{21} M(t)dt = 13.023$

1 $\frac{1}{21-1}\int_1^{21} M(t)dt$ IS THE AVERAGE RATE, IN INCHES PER DAY, AT WHICH THE SNOW IS MELTING FROM $t=1$ TO $t=21$ DAYS

b) Using a Midpoint Reimann Sum, find $\int_1^{21} S(t)\,dt$. State the correct units.

$\int_1^{21} S(t)dt = 3(4.9) + 7(.1) + 10(.2) = 17.4$ INCHES.

c) Approximate $S'(5)$. Using the correct units, explain $S'(5)$ in context of the problem.

2

$$S'(5) = \frac{.1 - 4.4}{7-4} = -1.433$$

S'(5) ~~IS~~ IS THE RATE ~~AT~~ IN INCHES/DAY/DAY, AT WHICH THE SNOW IS DECREASING ON DAY 5.

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d) Assume $S(t) = 2.5 - 2.5\cos\left(\dfrac{10x^{.9}}{9}\right)$ would model the snow fall. If there were 9 inches of snow on the ground at the beginning of Day 1, find the minimum amount of snow on the ground between $t = 1$ and $t = 7$.

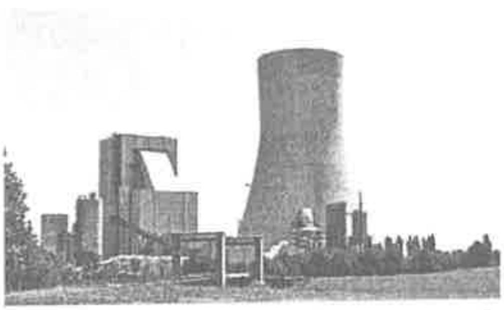
4

$$A = 9 + \int_1^t S(x) - M(x)\,dx \qquad A' = S - M$$

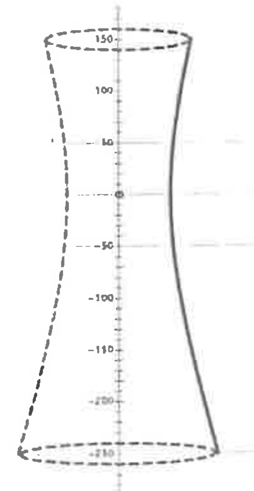
$$t = 5.918$$

t	A
1	9
5.918	21.729
7	21.325

THE MINIMUM AMOUNT OF SNOW ON THE GROUND IS 9 INCHES



2. Many power plants cool their reactions with convective air flow through a hyperboliod tower. The shape increases air flow while minimizing construction material. Consider the shape of a tower formed by revolving the hyperbola $f(y)=50\sqrt{\frac{1}{22500}y^2+1}$ on $y\in[-250,150]$ about the y-axis, where y is measured in feet.



a) Find the volume of the interior of the tower. Indicate the units. SHOW THE ANTI-DA

3

$$V=\pi\int_{-250}^{150}\left[50\sqrt{\frac{y^2}{22500}+1}\right]^2 dy$$

$$=\pi\int_{-250}^{150} 2500\left(\frac{y^2}{22500}+1\right)dy$$

$$=\pi\int_{-250}^{150}\left(2500+\frac{1}{9}y^2\right)dy$$

$$=\pi\left(2500y+\frac{1}{27}y^3\right)\Big|_{-250}^{150} = 5352343.039 \text{ ft}^3$$

b) Assume that the inner wall is of the shape formed by revolving $f(y)$ about the y-axis and the outer wall is of the the shape formed by revolving $g(y)=50.583\sqrt{\frac{1}{22500}y^2+1}$ about the y-axis. Find the volume of material needed to make the tower.

$$V = \pi \int^{150} g^2 - f^2 \, dy = \cancel{137399.554}$$

$125,544.321 \text{ ft}^3$

3

IN DEGREES FAHRENHEIT

c) The temperature S in the tower varies according to the function $S(y)=70e^{-0.001(y-150)}$, where y is measured in feet from the narrowest part of the tower. An object is dropped into the tower from the top. It falls at a rate of $R=-32t \; {}^{ft}\!/_{sec}$ and its height $y=-16t^2+150\,ft$. How fast is the temperature S changing when the object is at a level after falling for 3 seconds? Indicate the units.

3

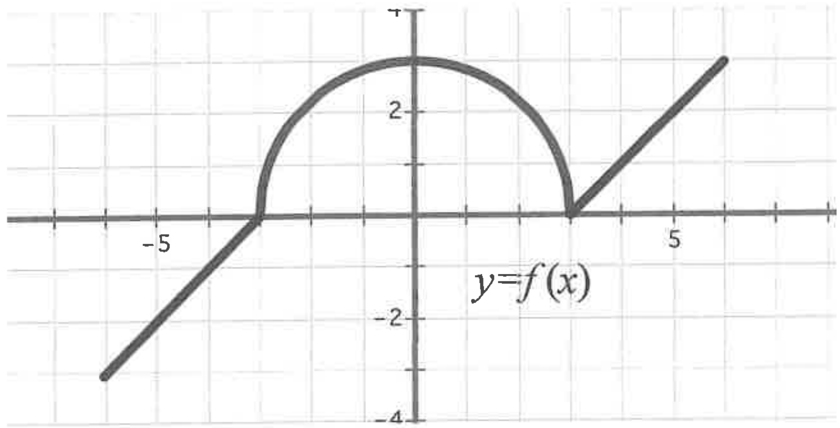
1

$y(3) = 6$

$$\frac{d}{dt}\left[S = 70e^{-.001(y-150)}\right]$$

$$\frac{dS}{dt} = 70e^{-.001(144-150)}(-.001)\Big(\overset{-32}{\frac{dy}{dt}}\Big) = 2.253 \ {}^{\circ}F/sec$$

1 1



3. The graph above, $h(x)$ on $-6 \le x \le 6$, is comprised of two line segments and a semi-circle. Let $g(x) = 2 + \int_{-3}^{x} f(t)\, dt$.

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3 (a) Find $g(3)$, $g'(3)$, and $g''(3)$.

$g(3) = 2 + \int_{-3}^{3} f(x) = 2 + \frac{9\pi}{2}$

$g'(3) = f(3) = 0$

$g''(3) = f'(3) = \text{DNE}$

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2 (b) At what x-value(s) on $-6 \le x \le 6$ does $g(x)$ have a relative minimum. Explain your reasoning.

MAX @ $x = -6$ & 6

MIN @ $x = -3$ BECAUSE $f = g'$ SWITCHES FROM ~~+~~ – TO +

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2 (c) At what x-value(s) on $-6 \le x \le 6$ does $g(x)$ have a point of inflection. Explain your reasoning.

$X=0$ BECAUSE f GOES FROM INCREASING TO DECREASING

$X=3$ " f " " DECREASING TO INCREASING

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2 (d) On what interval(s) is $g(x)$ both increasing and concave down? Explain why.

g IS INCREASING & CONCAVE DOWN

WHEN f IS POSITIVE AND DECREASING

$x \in (0,3)$

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4. Particle P moves along the x-axis such that, for time $t > 0$, its position is given by $x_P(t) = 8 - 2e^{-2t}$. Particle Q moves along the y-axis, for time $t > 0$, its velocity is given by $v_Q(t) = \frac{6}{t^3}$. At time $t = 1$, the position of particle Q is $y_Q(1) = -3.$ CHANGE IT TO -5

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1 a) Find $v_P(t)$, the velocity of particle P at time t.

$V_P(t) = 4e^{-2t}$

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3 b) Find $a_Q(t)$, the acceleration of particle Q at time t. Find all the times t, for $t > 0$, when the speed of particle Q is decreasing. Justify your answer.

$a_Q(t) = \frac{-18}{t^4}$

~~$t > 0$~~ THE SPEED IS ALWAYS DECREASING BECAUSE

~~$t0$~~ $a < 0$ FOR ALL $t \geq 0$

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c) Find $y_Q(t)$, the position of particle Q at time t.

3

$$y = \int \frac{6}{t^3}\,dt = \frac{6t^{-2}}{-2} + C$$

$$y(1) = -5 = -3t^{-2} + C \rightarrow C = -2$$

$$y = -3t^{-2} - 2$$

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d) As $t \to \infty$, which particle will eventually be farther from the origin. Give a reason for your answer.

2

$$\lim_{t \to \infty} x_p(t) = 8$$

$$\lim_{t \to \infty} y_Q(t) = -2$$

PARTICLE P IS FURTHER FROM THE ORIGIN

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5. Consider the function $y^2 - y + e^x = \cos x$.

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2 a) Prove that $\dfrac{dy}{dx} = \dfrac{e^x + \sin x}{1-2y}$.

$$\frac{d}{dx}\left[y^2 - y + e^x = \cos x\right]$$

$$2y\frac{dy}{dx} - 1\frac{dy}{dx} + e^x = -\sin x$$

$$(2y-1)\frac{dy}{dx} = -e^x - \sin x$$

$$\frac{dy}{dx} = \frac{-(e^x + \sin x)}{2y-1} = \frac{e^x + \sin x}{1-2y}$$

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2 b) Find the equation of the tangent line at (0, 1).

$$\cancel{m = \frac{1-1}{\quad} + e^0 =}$$

$$m = \frac{e^0 + \sin 0}{1-2(1)} = \frac{1}{-1} = -1$$

$$y - \cancel{0} = -1(x-0)$$

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c) Find the value of $\frac{d^2y}{dx^2}$ at (0, 1). Does the curve have a relative maximum, a relative minimum, or neither at (0, 1)? Justify your answer.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{e^x + \sin x}{1-2y}\right]$$

$$= \frac{(1-2y)(e^x + \cos x) - (e^x + \sin x)\left(-2\frac{dy}{dx}\right)}{(1-2y)^2}$$

$$= \frac{(1-2y)(e^x + \cos x) + 2(e^x + \sin x)^2}{(1-2y)^{\cancel{2}\,3}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{(-1)(2)+2}{1} = 0$$

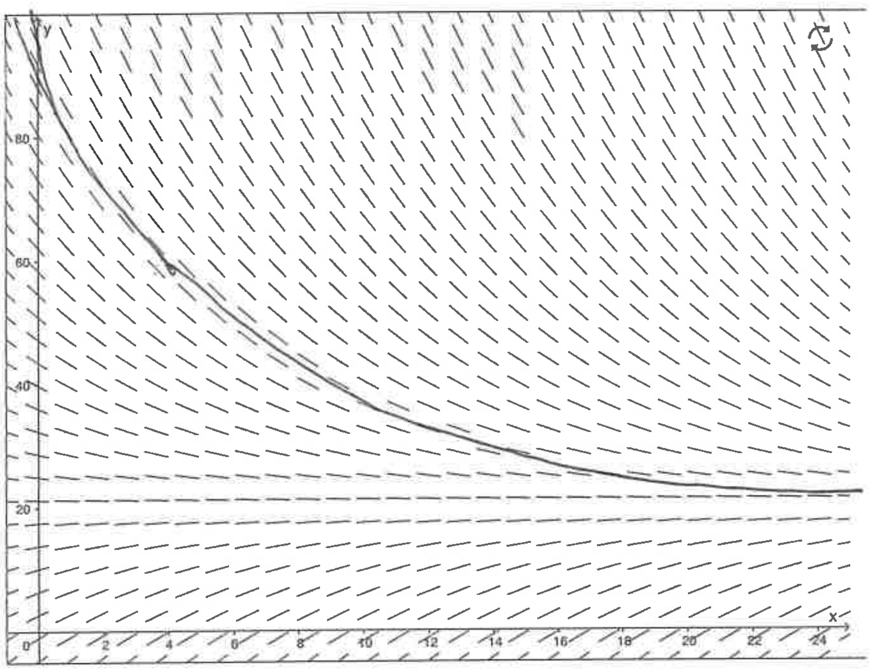
NEITHER BECAUSE $\left.\frac{dy}{dx}\right|_{(0,1)} \neq 0$

6. A cup of coffee is made with boiling water at a temperature of 100 C°, in a room at temperature 20 C°. After two minutes, it has cooled to 80 C°. According to Newton's Law of Cooling, the temperature of the coffee follows the differential equation

$$\frac{dy}{dt} = -0.14(y-20),$$

where y is the temperature of the coffee at time t minutes.

1



(a) Above is given a partial slope field for the temperature differential equation. Draw the solution to the differential equation at (5, 60).

(b) If $y(0) = 100$, find the equation of the line tangent to the temperature curve and use the tangent line equation to approximate $y(\cancel{20})$. Explain what this estimate means.
5

3

$m = \frac{dy}{dt} = -.14(100 - 20) = -11.2$

$y - 100 = -11.2t$

$y(\cancel{20}) = -\cancel{224}. + 100 = \cancel{97.76}\ C^\circ$

5, −56, 44

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5 (c) Find the particular solution to $\frac{dy}{dt} = -0.14(y - 20)$ with the initial condition $y(0) = 100$.

$\frac{1}{y-20}\, dy = -.14\, dt$

$\ln|y - 20| = -.14t + C$

$y - 20 = ke^{-1.4t}$

$80 = ke^{0} \Rightarrow k = 80$

$y = 20 + 80e^{-1.4t}$
