

5.1 Free Response Solutions

- 1a) The particle is moving right when the velocity is positive.

$$v(t) = 3t^2 - 8t + 4 = (3t - 2)(t - 2)$$

v	+	0	-	0	+
t	$\xleftarrow[2/3]{}$				2

$$t \in [0, 2/3] \cup [2, 5]$$

1b) $a(t) = v'(t) = 6t - 8 \rightarrow a(2) = 6(2) - 8 = 4$

1c) $y(t) = \int (3t^2 - 8t + 4) dt = t^3 - 4t^2 + 4t + c .$
 $y(1) = 3 \rightarrow 3 = (1)^3 - 4(1)^2 + 4(1) + c \rightarrow c = 2$
 $y(t) = t^3 - 4t^2 + 4t + 2$

- 3a) The particle is moving left when the velocity is negative.

$$v(t) = 12te^{-3t^2} = 0 \rightarrow x = 0$$

v	-	0	+
t	$\xleftarrow[0]{}$		

$$t \in (-\infty, 0]$$

3b) $a(t) = v'(t) = 12t(e^{-3t^2}(-6t)) + e^{-3t^2}(12) = 12e^{-3t^2}(-6t^2 + 1) \rightarrow a(0.6) = -12.323$

3c) $a(t) = v'(t) = 12t(e^{-3t^2}(-6t)) + e^{-3t^2}(12) = 12e^{-3t^2}(-6t^2 + 1) = 0 \rightarrow t = \pm \frac{1}{\sqrt{6}} = \pm 0.408$

v	-	0	+	0	-
t	$\xleftarrow[-1/\sqrt{6}]{}$				

v	-	0	+
t	$\xleftarrow[0]{}$		

The particle is slowing down when the velocity and acceleration have opposite signs. The velocity and acceleration have opposite signs on $t \in [-0.408, 0] \cup [0.408, 1.5]$

3d) $x(t) = \int (12te^{-3t^2}) dt = -2 \int (e^{-3t^2})(-6tdt) = -2e^{-3t^2} + c$
 $x(0) = -1 \rightarrow -1 = -2e^0 + c \rightarrow c = 1 \rightarrow x(t) = -2e^{-3t^2} + 1$

5a) Graph the function and find the zeros by calculator: $t = 2.292$ and 6.752

5b) Use Math 8: $a(7.3) = 3.536$

5c) $\text{Displacement} = \int_1^6 v(t) dt = -18.957$

5d) $\text{Dist} = \int_2^9 |v(t)| dt = 31.831$

7a) Graph the function and find the zeros by calculator: $t = 5.160$ and 7.718

7b) Based on the graph, we can see that the displacement will be most negative (left) at $t = 7.718$. $\text{Position} = -1.6 + \int_0^{7.817} v(t) dt = -1.817$

7c) $\text{Dist} = \int_3^6 |v(t)| dt = 8.455$

7d) $v(t) = \ln(t+3) - e^{\frac{t-1}{2}} \cos t$

$$\begin{aligned} a(t) &= \frac{d}{dt} \left[\ln(t+3) - e^{\frac{t-1}{2}} \cos t \right] = \frac{1}{x+3} - \frac{d}{dt} \left[e^{\frac{t-1}{2}} \cos t \right] \\ &= \frac{1}{t+3} - e^{\frac{t-1}{2}} (-\sin t) - \cos t \left(e^{\frac{t-1}{2}} \left(\frac{1}{2} \right) \right) \\ &= \frac{1}{t+3} + e^{\frac{t-1}{2}} \left(\sin t - \frac{1}{2} \cos e^{\frac{t-1}{2}} \right) \end{aligned}$$

Graph the acceleration equation to find the zeros. The acceleration is negative on the interval $x \in [3.664, 6.738]$

9a) $a(7) \approx \frac{40-55}{9-6} = \frac{-15}{3} = -5 \text{ ft/min}^2$.

9b) $\int_0^{18} v(t) dt \approx 1(55) + 3(70) + 2(68) + 3(55) + 1(40) + 3(38) + 2(46) + 3(50) = 962$.

The meaning is that the car traveled 962 feet in these 18 minutes.

9c) Since $v(t)$ is continuous and differentiable, the Mean Value Theorem applies. Since $v(0) = v(6)$ and $v(1) = v(18)$, $a(t) = 0$ at least twice.

9d) $\text{average} = \frac{1}{18} \int_0^{18} v(t) dt = \frac{1}{18} \int_0^{18} P(t) dt = 54.35 \text{ ft/min}$

11a) $\int_0^{60} v(t) dt \approx 30(0.132) + 30(0.123) = 7.65$. The team ran approximately 7.65 miles in these 60 minutes.

11b) $\frac{1}{60} \int_0^{60} v(t) dt$ would be the average pace in minutes per mile. That is, the result would equal the time, on average, it would take to complete one mile.

$$11c) a(37) \approx \frac{0.123 - 0.121}{45 - 30} = \frac{2}{15000} \text{ mi/min}^2$$

11d) Because $v(t)$ is continuous and differentiable, the MVT applies. Since $p(0) = p(45)$, there must be at least one place where $p'(t) = 0$. Since $p(0) < p(30)$, there must be a maximum.

$$13a) a(100) = v'(100) \approx \frac{38 - 43}{120 - 90} = \frac{-5}{30} = -\frac{1}{6} \text{ m/sec}^2$$

13b) The car is slowing down at $t = 100$ because $v(100) > 0$ but $a(100) < 0$.

13c) $t = 4$ and 7

This means the car traveled approximately 13,190 meters between $t = 0$ and $t = 360$.

15a) $t = 4$ and 10 because the graph of $v(t)$ switches from above the x -axis to below, or vice versa.

15b) $t = 4$ and 7 . Speed is the absolute value of the velocity. The speed at $t = 4$ and 7 is 2.

$$15c) \text{Total distance} = \int_0^{10} |v(t)| dt = \int_0^2 v(t) dt - \int_2^6 v(t) dt + \int_6^9 v(t) dt - \int_9^{10} v(t) dt \\ = \frac{\pi}{2} - (-2\pi) + 3 - \left(-\frac{1}{2}\right) = \frac{5\pi}{2} + \frac{7}{2}$$

$$15d) x(8) = x(2) + \int_2^8 v(t) dt = 6 + (-2\pi) + \frac{5}{2} = \frac{17}{2} - 2\pi$$

17a) According to the graph, the highest acceleration is 20 mi/hr^2 . This occurs at $t \in [3, 5]$.

$$17b) a'(6.3) = \frac{0 - 20}{8 - 5} = -\frac{20}{3} \text{ mi/hr}^3$$

17c) $v(3) = v(0) + \int_0^3 a(t) dt = 25 + (-5) + 20 = 40 \text{ mi/hr}$

17d) Maximum velocity occurs when the acceleration switches from positive to negative or at the left endpoint. This happens at $t = 0$ and 8.

$$v(0) = 25 \text{ mi/hr}; \quad v(8) = v(0) + \int_0^8 a(t) dt = 25 + (-5) + 20 + 40 + 30 = 110 \text{ mi/hr}$$

The maximum velocity is 110 mi/hr .

19a) The acceleration is equal to the slope of the line segment at $t = 3.5$, which is $-1 \frac{\text{mi}}{\text{min}^2}$

19b) The particle switches direction when the velocity switches sign, namely at $t = 4$ and 7.

19c) distance = $4 + 3 + 1 = 8 \text{ miles}$

19d) $\text{Ave} = \frac{2-1}{8-3} = \frac{1}{5} \frac{\text{mi}}{\text{min}}$. One can draw a horizontal line at $v(t) = \frac{1}{5}$ and see that the velocity graph intersects that line twice.

17EC. The equations of the line segments on $t \in [3, 6]$ and $t \in [6, 8]$ are $v - 0 = -1(t - 4)$ and $v - 0 = 2(t - 6)$, respectively, because of the slopes and x -intercepts.

$$v = \frac{1}{5} \rightarrow 0.2 = -1(t - 4) \rightarrow t = 3.8; \quad v = \frac{1}{5} \rightarrow 0.2 = 2(t - 6) \rightarrow t = 6.1$$

21. See AP Central

5.1 Multiple Choice Solutions

1. $s = t^3 + t^2 \rightarrow v = 3t^2 + 2t \rightarrow a = 6t + 2 = 0 \rightarrow t = -\frac{1}{3}$

The correct answer is B

3. $v(t) = 2t - 6 = 0 \rightarrow t = 3$

The correct answer is C

5. $s(t) = 6t^3 - 7t^2 - 9t + 2 \rightarrow v(t) = 18t^2 - 14t - 9$
 $a(t) = 36t - 14 \rightarrow a(9) = 36(9) - 14 = 310$

The correct answer is A

7. $v(t) = \frac{d}{dt} \int_0^t (x^3 - 2x^2 + x) dx = t^3 - 2t^2 + t$. Maximum velocity is attained when $t = 0$. $v'(t) = 3t^2 - 4t + 1 = (3t^2 - 1)(t - 1)$.

$$\begin{array}{c} v' \\ t \end{array} \leftarrow \begin{array}{ccccc} + & 0 & - & 0 & + \\ \frac{1}{3} & & & & 1 \end{array}$$

The correct answer is A

9. $v = 3t^2 + 2t = t(3t + 2)$; $a(t) = 6t + 2$

I. True: $v\left(\frac{2}{3}\right) > 0$

II. False: $v\left(\frac{1}{3}\right) > 0$

III. True: $v(1)$ and $a(1)$ are both positive

The correct answer is D

5.2 Free Response Solutions

1. $\frac{dy}{dt} = ky \rightarrow y = y_0 e^{kt} \rightarrow 2y_0 = y_0 e^{k(30)} \rightarrow k = \frac{\ln 2}{30} = 0.023 \rightarrow y = y_0 e^{0.023t}$
 $3y_0 = y_0 e^{0.023t} \rightarrow 3 = e^{0.023t} \rightarrow \ln 3 = 0.023t \rightarrow t = 47.549 \text{ min}$

3. $\frac{dP}{dt} = kP \rightarrow P = P_0 e^{kt} \rightarrow 44,000 = 50,000 e^{k(10)} \rightarrow k = \frac{\ln 0.88}{10} = -0.0128$
 $P = 50,000 e^{-0.0128t}$. In 2035, $P = 50,000 e^{-0.0128(25)} \approx 36,322 \text{ or } 36,323$

5a. $\frac{dF}{(100-F)} = .4 dt \rightarrow -\int \frac{-dF}{(100-F)} = \int .4 dt$
 $-\ln|100-F| = .4t$
 $\ln|100-F| = -.4t$
 $|100-F| = e^{-0.4t+c}$
 $100-F = ke^{-0.4t}$
 $F(0) = 10 \rightarrow 100-10 = ke^0 \rightarrow k = 90$
 $100-F = 90e^{-0.4t}$
 $-F = -100 + 90e^{-0.4t}$
 $F = 100 - 90e^{-0.4t}$

5b. $F(1.2) = 100 - 90e^{-0.4(32)} \approx 70$

5c. $\lim_{t \rightarrow \infty} F(t) = \lim_{t \rightarrow \infty} (100 - 90e^{-0.4t}) = 100 \lim_{t \rightarrow \infty} (1000 - 400e^{0.256t}) = 1000 - 0 = 1000$

7a. $\left. \frac{dM}{dt} \right|_{(0, 600)} = .256(1000 - 600) = 102.4 \Rightarrow M - 600 = 102.4t$

7b. $M(10) \approx f(10) = 102.4(10) + 600 = 1624 \text{ Kellertons}$

7c. $\int \frac{1}{1000-M} dM = \int 0.256 dt \rightarrow -\ln|1000-M| = 0.256t + c \rightarrow \ln|1000-M| = -0.256t + c$
 $|1000-M| = e^{-0.256t+c} \rightarrow 1000-M = ke^{-0.256t}$ $(0, 600) \rightarrow 400 = ke^0 \rightarrow 400 = k$

$$1000 - M = 400e^{-0.256t} \rightarrow M = 1000 - 400e^{-0.256t}$$

7d. $\lim_{t \rightarrow \infty} (1000 - 400e^{-0.256t}) = 1000 - 0 = 1000$

9a. $\frac{dy}{dt} = k(y - 20) \rightarrow \frac{1}{y-20} dy = k dt$

$$\int \frac{1}{y-20} dy = \int k dt$$

$$\ln|y-20| = kt + c$$

$$|y-20| = e^{kt+c} \rightarrow y-20 = Be^{-kt}$$

$$(0, 100) \rightarrow 100 - 20 = Be^0 \rightarrow B = 80$$

$$(2, 80) \rightarrow 80 - 20 = 80e^{k(2)} \rightarrow k = -0.144$$

$$y = 20 + 80e^{-0.144t}$$

9b. $y = 20 + 80e^{-0.144(5)} = 58.940 \text{ } C^\circ$

9c. $40 = 20 + 80e^{-0.144t} \rightarrow 20 = 80e^{-0.144t} \rightarrow 0.25 = e^{-0.144t} \rightarrow t = 9.627 \text{ min}$

11. See AP Central

5.2 Multiple Choice Solutions

1. $x^2 + y^2 = 1$

<u>Implicit</u> $2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$	<u>Explicit</u> $x^2 + y^2 = 1 \rightarrow y^2 = 1 - x^2 \rightarrow y = \pm\sqrt{1-x^2}$ $y = \pm\sqrt{1-x^2} = \pm(1-x^2)^{\frac{1}{2}}$
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$$\begin{aligned}
y &= \pm \frac{1}{2} (1-x^2)^{-1/2} (-2x) \\
&= \frac{-x}{\pm (1-x^2)^{1/2}} \\
&= \frac{-x}{y}
\end{aligned}$$

3. $\frac{1}{x} + \frac{1}{y} = 1$

Implicit

$$-x^{-2} - y^{-2} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x^{-2}}{-y^{-2}} = -\frac{y^2}{x^2}$$

Explicit

$$y+x = xy \Rightarrow y-xy = -x \Rightarrow y = \frac{-x}{1-x}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1-x) \cdot -1 - (-x) \cdot (-1)}{(1-x)^2} \\
&= \frac{-1+x-x}{(1-x)^2} = -\frac{1}{(1-x)^2}
\end{aligned}$$

$$\frac{dy}{dx} = -\frac{y^2}{x^2} = -\frac{\left(\frac{-x}{1-x}\right)^2}{x^2} = -\frac{x^2}{(1-x)^2} \cdot \frac{1}{x^2} = -\frac{1}{(1-x)^2}$$

5. $x^2 + xy - 4y - 1 = 0$

$$2x + x \frac{dy}{dx} + y(1) - 4 \frac{dy}{dx} = 0$$

$$(x-4) \frac{dy}{dx} = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x-4}$$

$$7. \quad x^2 + 4xy - 5y^2 = 4$$

$$2x + 4x \frac{dy}{dx} + y(4) - 10 \frac{dy}{dx} = 0$$

$$\begin{aligned}(4x - 10) \frac{dy}{dx} &= -2x - 4y \\ \frac{dy}{dx} &= \frac{-2x - 4y}{4x - 10} \\ &= -\frac{x + 2y}{2x - 5}\end{aligned}$$

$$9. \quad x^3 + 10x^2y + 7y^2 = 60$$

$$3x^2 + 10 \left(x^2 \frac{dy}{dx} + y \cdot 2x \right) + 14y \frac{dy}{dx} = 0$$

$$3x^2 + 10x^2 \frac{dy}{dx} + 20xy + 14y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3x^2 - 20xy}{10x^2 + 14y}$$

$$11. \quad 4\cos(x)\sin(y) = 1$$

$$4 \left[\cos x \cdot \cos y \frac{dy}{dx} + \sin y \cdot -\sin x \right] = 0$$

$$4\cos x \cos y \frac{dy}{dx} = 4\sin x \sin y$$

$$\frac{dy}{dx} = \tan x \tan y$$

$$13. \quad \tan(x - y) = \frac{y}{1+x^2}$$

$$\sec^2(x-y) \left(1 - \frac{dy}{dx} \right) = \frac{(1+x^2) - y \cdot 2x}{(1+x^2)^2}$$

$$\left[\sec^2(x-y) - \sec^2(x-y) \frac{dy}{dx} \right] \cdot (1+x^2)^2 = \frac{dy}{dx} + x^2 \frac{dy}{dx} - 2xy$$

$$\sec^2(x-y)(1+x^2)^2 - \sec^2(x-y)(1+x^2)^2 \frac{dy}{dx} = \frac{dy}{dx} + x^2 \frac{dy}{dx} - 2xy$$

$$\sec^2(x-y)(1+x^2)^2 + 2xy = \frac{dy}{dx} \left[1 + x^2 + \sec^2(x-y)(1+x^2)^2 \right]$$

$$\frac{dy}{dx} = \frac{\sec^2(x-y)(1+x^2)^2 + 2xy}{1 + x^2 + \sec^2(x-y)(1+x^2)^2}$$

$$15. \quad \frac{d}{dx} \left[x^2 = \frac{x-y}{x+y} \right] \rightarrow 2x = \frac{(x+y) \left(1 - \frac{dy}{dx} \right) - (x-y) \left(1 + \frac{dy}{dx} \right)}{(x+y)}$$

$$2x = \frac{2y - 2x \frac{dy}{dx}}{(x+y)^2}$$

$$2x(x+y)^2 = 2y - 2x \frac{dy}{dx}$$

$$2x(x+y)^2 - 2y = 2x \frac{dy}{dx}$$

$$(x+y)^2 - \frac{y}{x} = \frac{dy}{dx}$$

$$\begin{aligned}
17. \quad & \frac{d}{dx} \left[y^2 = \frac{x-y}{x+y} \right] \rightarrow 2y \frac{dy}{dx} = \frac{(x+y) \left(1 + \frac{dy}{dx} \right) - (x-y) \left(1 + \frac{dy}{dx} \right)}{(x+y)^2} \\
& 2y(x+y)^2 \frac{dy}{dx} = \left(x+y + x \frac{dy}{dx} + y \frac{dy}{dx} \right) - \left(x-y + x \frac{dy}{dx} - y \frac{dy}{dx} \right) \\
& 2y(x+y)^2 \frac{dy}{dx} = 2y + 2y \frac{dy}{dx} \\
& \left(2y(x+y)^2 - 2y \right) \frac{dy}{dx} = 2y \\
& \frac{dy}{dx} = \frac{2y}{2y(x+y)^2 - 2y} = \frac{1}{(x+y)^2 - 1}
\end{aligned}$$

$$19. \quad x^2 - y^2 - 6y - 3 = 0 \text{ at } (\sqrt{3}, 0)$$

$$2x - 2y \frac{dy}{dx} - 6 \frac{dy}{dx} = 0$$

$$2\sqrt{3} - 6 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\sqrt{3}}{3}$$

$$y = \frac{\sqrt{3}}{3} (x - \sqrt{3})$$

$$21. \quad 12x^2 - 4y^2 + 72x + 16y + 44 = 0 \text{ at } (-1, -3)$$

$$24x - 8y \frac{dy}{dx} + 72 + 16 \frac{dy}{dx} = 0$$

$$-24 + 24 \frac{dy}{dx} + 72 + 16 \frac{dy}{dx} = 0$$

$$40 \frac{dy}{dx} = -48 \rightarrow \frac{dy}{dx} = -\frac{6}{5}$$

$$y + 3 = -\frac{6}{5}(x + 1)$$

23. Find the equation of the line tangent to $x^2 + 3xy + y^2 = 11$, through the point (1,2).

$$2x + 3\left[x \frac{dy}{dx} + y\right] + 2y \frac{dy}{dx} = 0$$

$$2 + 3\left[\frac{dy}{dx} + 2\right] + 4\frac{dy}{dx} = 0 \Rightarrow 2 + 3\frac{dy}{dx} + 6 + 4\frac{dy}{dx} = 0 \Rightarrow 7\frac{dy}{dx} = -8 \Rightarrow \frac{dy}{dx} = -\frac{8}{7}$$

$$y - 2 = -\frac{8}{7}(x - 1)$$

25. Find the equation of the lines tangent and normal to $y - \frac{4}{\pi^2}x^2 = 2e^{y \sin x} + y^3 - 3$ through the point $\left(\frac{\pi}{2}, 0\right)$.

$$\frac{dy}{dx} - \frac{8}{\pi^2}x = 2e^{y \sin x} \left[y \cos x + \sin x \frac{dy}{dx} \right] + 3y^2 \frac{dy}{dx}$$

$$\frac{dy}{dx} - \frac{8}{\pi^2}\left(\frac{\pi}{2}\right) = 2e^0 \left[0 + \sin \frac{\pi}{2} \frac{dy}{dx} \right] + 0$$

$$\frac{dy}{dx} - \frac{4}{\pi} = 2 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{4}{\pi}$$

Tangent line: $y - 0 = -\frac{4}{\pi}(x - \frac{\pi}{2})$,

Normal line: $y - 0 = \frac{\pi}{4}(x - \frac{\pi}{2})$

27. Find $\frac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$

$$8x + 18y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-8x}{18y} = \frac{-4x}{9y}$$

$$\frac{d^2y}{dx^2} = \frac{9y \cdot -4 - (-4x) \cdot 9 \frac{dy}{dx}}{81y^2} = \frac{-36y + 36x \cdot \frac{-4x}{9y}}{81y^2} = \frac{-36y - \frac{144x^2}{9y}}{81y^2}$$

$$= \frac{-324y^2 - 144x^2}{729y^3} = \frac{-36y^2 - 16x^2}{81y^2}$$

29. Find $\frac{d^2y}{dx^2}$ if $x^3 + 4y^2 = 16$

$$\frac{d}{dx} [x^3 + 4y^2 = 16] \rightarrow 3x^2 + 8y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-3x^2}{8y}$$

$$\frac{d}{dx} \left[\frac{-3x^2}{8y} \right] \rightarrow \frac{d^2y}{dx^2} = \frac{8y(-6x) - (-3x^2)\left(8\frac{dy}{dx}\right)}{(8y)^2}$$

$$= \frac{8y(-6x) + 3x^2 \left(8\left(\frac{-3x^2}{8y}\right)\right)}{64y^2}$$

$$= \frac{-48xy - \frac{9x^4}{y}}{64y^2}$$

$$= \frac{-48xy^2 - 9x^4}{64y^3}$$

$$= \frac{-3x(16y^2 - 3x^3)}{64y^3}$$

5.3 Multiple Choice Solutions

1. $\frac{d}{dx} [x^3 - y^2 + x^2 = 0] \rightarrow 3x^2 - 2y \frac{dy}{dx} + 2x = 0 \rightarrow \frac{dy}{dx} = \frac{3x^2 + 2x}{2y}$

Vertical tangent line have $\frac{dy}{dx} = dne$;

$$2y = 0 \rightarrow y = 0 \rightarrow x^3 + x^2 = 0 \rightarrow x = 0 \text{ or } -1$$

The correct answer is D

3. $\frac{d}{dx} [y^2 + x = -2xy - 5] \rightarrow 2y \frac{dy}{dx} + 1 = -2x \frac{dy}{dx} + y(-2)$

$$2(1) \frac{dy}{dx} + 1 = -2(2) \frac{dy}{dx} + (1)(-2) \rightarrow \frac{dy}{dx} = -\frac{3}{6} = -\frac{1}{2}$$

The correct answer is A

$$5. \quad \frac{d}{dx} [x + xy + 2y^2 = 6] \rightarrow 1 + x \frac{dy}{dx} + y(1) + 4y \frac{dy}{dx} = 0$$

$$\left. \frac{dy}{dx} \right|_{(2,1)} = 1 + (2) \frac{dy}{dx} + (1)(1) + 4(1) \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{1}{3}$$

The correct answer is C

$$7. \quad \frac{d}{dx} [xy - y^3 + 6 = 0] \rightarrow x \frac{dy}{dx} + y - 3y^2 \frac{dy}{dx} = 0$$

$$\left. \frac{dy}{dx} \right|_{(1,2)} = 1 \frac{dy}{dx} + 2 - 3(2)^2 \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{2}{11}.$$

The correct answer is C

$$9. \quad \frac{d}{dx} [\sin^{-1} x = \ln y] \rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{1}{y} \frac{dy}{dx}$$

The correct answer is A

$$11. \quad \frac{dy}{dx} [y = x + \sin(xy)] \rightarrow \frac{dy}{dx} = 1 + [\cos(xy)] \left[x \frac{dy}{dx} + y(1) \right] \rightarrow$$

$$[\cos(xy)] \frac{dy}{dx} = 1 + y \cos(xy)$$

$$\frac{dy}{dx} = \frac{1 + y \cos(xy)}{\cos(xy)}$$

The correct answer is E

$$13. \quad y = \ln(x^2 + y^2) \rightarrow \frac{dy}{dx} = \frac{1}{x^2 + y^2} \left(2x + 2y \frac{dy}{dx} \right)$$

$$\left. \frac{dy}{dx} \right|_{(1,0)} = \frac{1}{1} \left(2(1) + 2(0) \frac{dy}{dx} \right) = 2$$

The answer is D.

5.4 Free Response Key

1a. $\frac{d}{dx} [3x^2 - 4xy + 5y^2 = 25] \rightarrow 6x - 4x\frac{dy}{dx} + y(-4) + 10y\frac{dy}{dx} = 0$

$$6x - 4y = 4x\frac{dy}{dx} - 10y\frac{dy}{dx} = (4x - 10y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x - 4y}{4x - 10y} = \frac{3x - 2y}{2x - 5y}$$

b) $3(2)^2 - 4(2)y + 5y^2 = 25$

$$5y^2 - 8y - 13 = 0$$

$$(5y - 13)(y + 1) = 0$$

$$y = 1, \frac{13}{5} \rightarrow (2, -1) \left(2, \frac{13}{5} \right)$$

c) $\frac{dy}{dx} \Big|_{(2, -1)} = \frac{3(2) - 2(-1)}{2(2) - 5(-1)} = \frac{8}{9} \rightarrow y + 1 = \frac{8}{9}(x - 2)$

$$\frac{dy}{dx} \Big|_{\left(2, \frac{13}{5}\right)} = \frac{3(2) - 2\left(\frac{13}{5}\right)}{2(2) - 5\left(\frac{13}{5}\right)} = \frac{\frac{4}{5}}{-9} = -\frac{4}{45} \rightarrow y - \frac{13}{5} = -\frac{4}{45}(x - 2)$$

d) $\frac{dy}{dx} = \frac{3x - 2y}{2x - 5y} = 0 \rightarrow 3x - 2y = 0 \rightarrow y = \frac{3}{2}x$

$$3x^2 - 4x\left(\frac{3}{2}x\right) + 5\left(\frac{3}{2}x\right)^2 = 25$$

$$3x^2 - 6x^2 + \frac{45}{4}x^2 = 25$$

$$33x^2 = 100 \rightarrow x = \pm \frac{10}{\sqrt{33}} \rightarrow \left(\frac{10}{\sqrt{33}}, \frac{5}{\sqrt{33}} \right) \left(-\frac{10}{\sqrt{33}}, -\frac{5}{\sqrt{33}} \right)$$

3a) $\frac{d}{dx} [2x^2 - xy + y^2 = 44] \rightarrow 4x - x\frac{dy}{dx} + y(-1) + 2y\frac{dy}{dx} = 0$

$$(2y - x)\frac{dy}{dx} = y - 4x \rightarrow \frac{dy}{dx} = \frac{y - 4x}{2y - x} = \frac{4x - y}{x - 2y}$$

b) $\frac{d}{dx} [2(5)^2 - 5y + y^2 = 44] \rightarrow y^2 - 5y + 6 = 0 \rightarrow (y-3)(y-2) = 0 \rightarrow y = 2, 3$

$$(5, 2), (5, 3)$$

c) $\left. \frac{dy}{dx} \right|_{(5, 2)} = \frac{4(5) - 2}{5 - 2(2)} = 18 \rightarrow y - 2 = 18(x - 5)$

$$\left. \frac{dy}{dx} \right|_{(5, 3)} = \frac{4(5) - 3}{5 - 2(3)} = -17 \rightarrow y - 3 = -17(x - 5)$$

d) Vertical tangent line means $x - 2y = 0 \rightarrow x = 2y$

$$2(2y)^2 - (2y)y + y^2 = 44 \rightarrow 7y^2 = 44 \rightarrow y = \pm \frac{2\sqrt{11}}{\sqrt{7}} \rightarrow \left(\frac{4\sqrt{11}}{\sqrt{7}}, \frac{2\sqrt{11}}{\sqrt{7}} \right), \left(-\frac{4\sqrt{11}}{\sqrt{7}}, -\frac{2\sqrt{11}}{\sqrt{7}} \right)$$

5a) $\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} = xy + y \right] = \frac{d}{dx} \left[\frac{dy}{dx} = y(x+1) \right]$

$$\frac{d^2y}{dx^2} = y(1) + (x+1)\frac{dy}{dx} = y + (x+1)y(x+1) = y[1 + (x+1)^2] = y[x^2 + 2x + 2]$$

b) $\left. \frac{dy}{dx} \right|_{(-1, 2)} = y(x+1)|_{(-1, 2)} = 0; \quad \left. \frac{d^2y}{dx^2} \right|_{(-1, 2)} = y[1 + (x+1)^2]|_{(-1, 2)} = 2$

$\left. \frac{dy}{dx} \right|_{(-1, 2)} = 0$ and $\left. \frac{d^2y}{dx^2} \right|_{(-1, 2)} > 0$, therefore, $(-1, 2)$ is a minimum.

c) $\frac{1}{y} dy = (x+1) dx \rightarrow \int y^{-1} dy = \int (x+1) dx \rightarrow \ln y = x^2 + x + c$

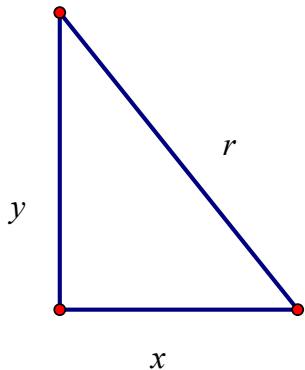
$$y = e^{x^2+x+c} = ke^{x^2+x}$$

$$(-1, 2) \rightarrow 2 = ke^0 \rightarrow k = 2$$

$$y = 2e^{x^2+x}$$

5.5 Free Response Solutions

1a.



b) $x, y, r, \frac{dx}{dt}, \frac{dy}{dt}$, and $\frac{dr}{dt}$

c) $x = 1, y = 1.4, \frac{dx}{dt} = 5, \frac{dy}{dt} = 12,$
 $1^2 + 1.4^2 = r^2 \rightarrow r = 1.720.$

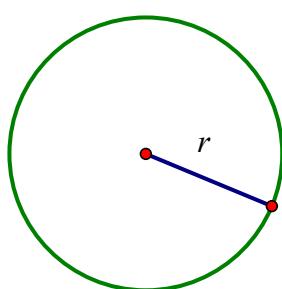
Find $\frac{dr}{dt}$

d) $x^2 + y^2 = r^2$

e) $\frac{d}{dt}[x^2 + y^2 = r^2] \rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt} \rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$

$$(1)(5) + (1.4)(12) = (1.720)\frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{21.8}{1.720} = 12.674 \text{ mph}$$

3a.



b) $A, r, \frac{dA}{dt}$, and $\frac{dr}{dt}$

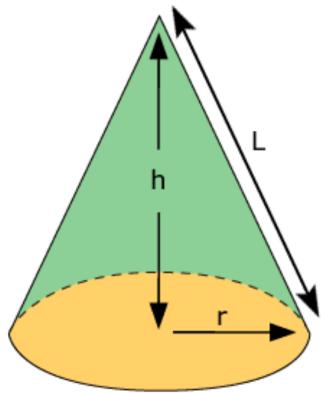
c) $\frac{dr}{dt} = \frac{1}{2} \text{ in/min}, d = 1 \text{ in} (\text{ } r = \frac{1}{2} \text{ in})$

Find $\frac{dA}{dt}$

d) $A = \pi r^2$

e) $A = \pi r^2 \rightarrow \frac{dA}{dt} = 2\pi r \left(\frac{dr}{dt} \right) \rightarrow \frac{dA}{dt} = 2\pi \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) = \frac{\pi}{2} \text{ in}^2/\text{min}$

5a.

b) Volume, diameter, height, radius, $\frac{dV}{dt}$, and $\frac{dh}{dt}$ 

c)

$$\frac{dV}{dt} = 30\pi \text{ ft}^3 \text{ min}^{-1}, \quad h = 5$$

$$h = 5 \rightarrow r = 2.5$$

Find $\frac{dh}{dt}$

d)

$$\frac{1}{2}h = r; \quad V = \frac{\pi}{3}r^2h = \frac{\pi}{12}h^3;$$

$$V = \frac{\pi}{3}r^2h = \frac{\pi}{12}h^3$$

e)

$$\frac{d}{dt} \left[V = \frac{\pi}{12}h^3 \right] \rightarrow \frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

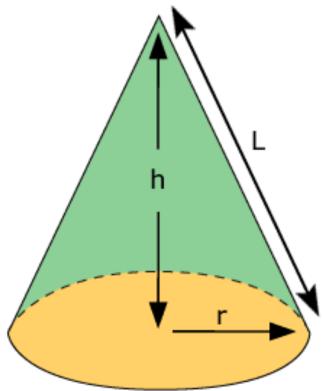
$$\frac{dV}{dt} = 30 = \frac{\pi}{4}10^2 \frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{6}{5\pi}$$

$$-300 = 300\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{\pi} \text{ ft min}^{-1}$$

7a)

b) $h, d, r, V, \frac{dh}{dt}, \frac{dr}{dt}, \frac{dV}{dt}$



c)

$$\frac{dV}{dt} = -5000 \text{ cm}^3/\text{min}; h = 8; d = 4$$

$$d = 4 \rightarrow r = 2$$

Find $\frac{dh}{dt}$ and $\frac{dr}{dt}$

d)

$$V = \frac{\pi}{3} r^2 h, d = 2r$$

e) $\frac{r}{h} = \frac{2}{8} = \frac{1}{4} \rightarrow r = \frac{1}{4} h \rightarrow V = \frac{\pi}{3} \left(\frac{1}{4} h \right)^2 h = \frac{\pi}{48} h^3$

$$\frac{d}{dt} \left[V = \frac{\pi}{48} h^3 \right] \rightarrow \frac{dV}{dt} = \frac{\pi}{16} h^2 \frac{dh}{dt}$$

$$-5000 = \frac{\pi}{16} (8^2) \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1250}{\pi} \text{ m}^3/\text{min}$$

f) $\frac{r}{h} = \frac{1}{4} \rightarrow 4r = h \rightarrow V = \frac{\pi}{3} r^2 (4r) = \frac{4\pi}{3} r^3$

$$\frac{d}{dt} \left[V = \frac{4\pi}{3} r^3 \right] \rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$-5000 = 4\pi (2^2) \frac{dr}{dt}$$

$$\frac{dr}{dt} = -\frac{625}{2\pi} \text{ m}^3/\text{min}$$

9. $\frac{dV}{dt} = 6 \text{ ft}^3/\text{min}$; $r = 10 \text{ ft}$; $\frac{dr}{dt} = ?$

$$V = \frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt} \right)$$

$$6 = 4\pi(10)^2 \left(\frac{dr}{dt} \right)$$

$$\frac{6}{400\pi} = \frac{dr}{dt}$$

$$\frac{3}{200\pi} \text{ ft}/\text{min} = \frac{dr}{dt}$$

11. $\frac{dx}{dt} = 5$; $y = 7$; $x^2 + y^2 = 25^2 \rightarrow x = 24$

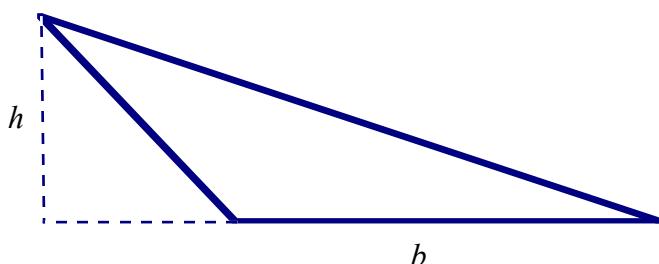
$$\frac{d}{dt}(x^2 + y^2 = 25)$$

$$2x \left(\frac{dx}{dt} \right) + 2y \left(\frac{dy}{dt} \right) = 0$$

$$2(24)(5) + 2(7) \left(\frac{dy}{dt} \right) = 0$$

$$\frac{dy}{dt} = -\frac{120}{7} \text{ ft/sec}$$

13. $\frac{dh}{dt} = 2$; $\frac{dA}{dt} = 5$; $A = \frac{1}{2}bh$; find $\frac{db}{dt}$.



$$A = \frac{1}{2}bh \rightarrow 144 = \frac{1}{2}b(12) \rightarrow 24 = b$$

$$\frac{d}{dt} \left[A = \frac{1}{2} b h \right] \rightarrow \frac{dA}{dt} = \frac{1}{2} b \frac{dh}{dt} + \frac{1}{2} h \frac{db}{dt}$$

$$5 = \frac{1}{2}(24)(5) + \frac{1}{2}(12) \frac{db}{dt} \rightarrow \frac{db}{dt} = -\frac{55}{6}$$

15. $P = \frac{kT}{V}$

$$P = kTV^{-1}$$

T is constant, so $\frac{dT}{dt} = 0$

$$\frac{dP}{dt} = 20 \text{ kPa} \cancel{/ \text{min}}; \frac{dV}{dt} = ? \text{ when } V = 600, P = 150$$

$$\frac{dP}{dt} = -1kTV^{-2} \left(\frac{dV}{dt} \right) \rightarrow 20 = \frac{-kt}{600^2} \left(\frac{dV}{dt} \right) \rightarrow \frac{7200000}{-kT} = \frac{dV}{dt}$$

Use the original equation to find kT :

$$150 = \frac{kT}{600} \rightarrow kT = 90000$$

$$\frac{dV}{dt} = \frac{7200000}{-90000} = -80 \text{ in}^3 \cancel{/ \text{min}}$$

17a. Person A: $Dist \ a = 220 - 10(10) = 120 \text{ ft}$

Person B: $Dist \ b = 10(5) = 50 \text{ ft}$

17b. $Dist \ c = \sqrt{50^2 + 120^2} = 130 \text{ ft}$

17c. $\frac{d}{dt} [a^2 + b^2 = c^2] \rightarrow 2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}$

$$2(120)(-10) + 2(50)(5) = 2(130) \frac{dc}{dt} \rightarrow \frac{dc}{dt} = -7.308 \text{ ft/sec}$$

17d. $\tan \theta = \frac{b}{a} \rightarrow \frac{d}{dt} \left[\tan \theta = \frac{b}{a} \right] \rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{a \frac{db}{dt} - b \frac{da}{dt}}{a^2} \rightarrow \frac{c^2}{a^2} \frac{d\theta}{dt} = \frac{a \frac{db}{dt} - b \frac{da}{dt}}{a^2}$

$$\frac{(130)^2}{(120)^2} \frac{d\theta}{dt} = \frac{(120)(5) - (50)(-10)}{(120)^2}$$

$$\frac{d\theta}{dt} = 0.065 \text{ rad/sec}$$

19a. $t = 15 \rightarrow x = 20(15) = 300 \text{ ft}$
 $r(15) = \sqrt{20^2 + 300^2} = 300.666 \text{ ft}$

19b. $r^2 = 20^2 + x^2 \rightarrow 2r \frac{dr}{dt} = 2x \frac{dx}{dt} \rightarrow 2\sqrt{20^2 + 150^2} \frac{dr}{dt} = 2(150)(20)$
 $\frac{dr}{dt} = 19.825 \text{ ft/sec}$

19c. $\tan \theta = \frac{20}{x} \rightarrow \theta = \tan^{-1} \frac{20}{140} = 0.142$

19d. $\sec^2 \theta \frac{d\theta}{dt} = -20x^{-2} \frac{dx}{dt} \rightarrow \frac{d\theta}{dt} = -20 \left(\frac{20}{140} \right)^2 (15) \cos^2 0.142 = -0.015 \text{ rad/sec}$

21a. $A = \frac{1}{2}(730)(730)\sin 0.79 = 189,273.629 \text{ sq.ft.}$

21b. $c^2 = 370^2 + 370^2 - 2(370)(370)\cos 0.79 \rightarrow c = 284.758 \text{ ft}$

21c. $c^2 = a^2 + a^2 - 2(a)(a)\cos 0.79 \rightarrow c^2 = 0.593a^2$
 $\frac{d}{dt} [c^2 = a^2 + a^2 - 2(a)(a)\cos 0.79] = \frac{d}{dt} [c^2 = 0.593a^2]$
 $2c \frac{dc}{dt} = 1.186a \frac{da}{dt} \rightarrow \frac{dc}{dt} = \frac{1.186(370)}{2(284.758)}(-15.2) = -5.856 \text{ ft/hr}$

21d. $\frac{d}{dt} \left[A = \frac{1}{2}(\sin 0.79)a^2 \right] \rightarrow \frac{dA}{dt} = (\sin 0.79)a \frac{da}{dt} = 0.710(370)(-15.2) = -3995.027 \text{ sq.ft.}$

$$23. \quad \frac{dC}{dt} = 2\pi \frac{dr}{dt} = 6\pi \rightarrow \frac{dr}{dt} = 3$$

Because the right triangle inside the equilateral triangle is a 30-60-90 triangle, $a = \frac{1}{2}r$

$$\text{and } P = 3\sqrt{3}a = \frac{3\sqrt{3}}{2}r$$

$$A = \pi r^2 - \frac{3\sqrt{3}}{2}r \rightarrow \frac{d}{dt} \left[A = \pi r^2 - \frac{3\sqrt{3}}{2}r \right] \rightarrow \frac{dA}{dr} = 2\pi r \frac{dr}{dt} - \frac{3\sqrt{3}}{2} \frac{dr}{dt}$$

$$\pi r^2 = 64\pi \rightarrow r = 8$$

$$\frac{dA}{dr} = 2\pi(8)(3) - \frac{3\sqrt{3}}{2}(3) = 143.002 \frac{\text{in}^2}{\text{sec}}$$

5.5 Multiple Choice Solutions

1. $V = s^3 \rightarrow \frac{dV}{dt} = 3s^2 \frac{ds}{dt}; A = 6s^2 = 54 \rightarrow s = 3$
 $\frac{dV}{dt} = 3(3)^2(6) = 162$

The correct answer is D

3. $x = 5(70) = 350 \text{ miles}; y = 5(28) = 140 \text{ miles}$
 $x^2 + y^2 = r^2 \rightarrow r = \sqrt{350^2 + 140^2} = 376.962$
 $x^2 + y^2 = r^2 \rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt} \rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} = r \frac{dr}{dt}$
 $(350)70 + (140)28 = 376.962 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = 75.39$

The correct answer is D

5. $V = \frac{\pi h^2}{3}(18 - h) = 6\pi h^2 - \frac{\pi}{3}h^3 \rightarrow \frac{dV}{dt} = 12\pi h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt}.$
 $30\pi = [12\pi(2) - \pi(2)^2] \frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{3}{2}$

The correct answer is C

7. $\frac{dx}{dt} = x; \frac{d}{dt}\left(\frac{1}{x}\right) = \frac{-1}{x^2} \frac{dx}{dt} = \frac{-1}{x^2}(x) = -\frac{1}{x} \therefore \text{At } x = -\frac{1}{4}, \frac{d}{dt}\left(\frac{1}{x}\right) = 4?$

The correct answer is E

Derivative Applications II Practice Chapter Test: PART 1

1. $v(t) = 9(t-4)^2 \rightarrow v'(t) = 18(t-4) = 0 \rightarrow t = 4$
 $v(4) = 0$
 $v(0) = 324$
 $v(10) = 324$

The correct answer is B

3. $v_{ave} = \frac{v(b)-v(a)}{b-a} = \frac{v(\pi/4)-v(0)}{\pi/4-0} = \frac{3-2}{\pi/4} = \frac{4}{\pi}$

The correct answer is E

5. $\frac{d}{dx}[3x^3 - 4xy - 4y^2 = 1] \rightarrow 9x^2 - 4x\frac{dy}{dx} + y(-4) - 8y\frac{dy}{dx} = 0$
 $(-4x - 8y)\frac{dy}{dx} = 4y - 9x^2 \rightarrow \frac{dy}{dx} = \frac{4y - 9x^2}{(-4x - 8y)}$

The correct answer is B

7. $\frac{dB}{dt} = 0.2B\left(1 - \frac{B}{900}\right) \rightarrow A = 900 \rightarrow \frac{A}{2} = 450$

The correct answer is B

9. $\frac{dy}{dx} = \frac{1}{x^2 + y^2} \left(2x + 2y\frac{dy}{dx}\right); (0, 1) \rightarrow \frac{dy}{dx} = \frac{1}{1+0} \left(2 + 0\frac{dy}{dx}\right) = 2.$

The correct answer is D

11. $\frac{dI}{dt} = 4I \left(1 - \frac{I}{12,000}\right) \rightarrow A = 12,000$, so only C or E could be correct. E has the variables in the wrong places.

The correct answer is C

Derivative Applications II Practice Chapter Test: PART 2

CALCULATOR ALLOWED

Directions: Show all work.

$$1a) \quad \frac{v_A(t)}{10} = \frac{0.182}{8} \rightarrow v_A(t) = 0.2275 \text{ miles/min}$$

$$v_B(10) = \frac{12(10) - (10)^2}{10(10)^2 + 11} = 0.020 \text{ miles/min}$$

$$b) \quad a_A(10) = v'_A(10) = \frac{0.182 - 0}{8 - 16} = -0.023 \text{ miles/min}^2$$

$$a_B(10) = v'_B(10) = -0.012 \text{ miles/min}^2$$

$$c) \quad D_A(16) = \int_0^{16} v_A(t) dt = \frac{1}{2}(2)(0.182) + 6(0.182) + \frac{1}{2}(8)(0.182) = 2.002 \text{ miles}$$

$$D_B(12) = \int_0^{12} v_B(t) dt = 1.885 \text{ miles}$$

$D_A(16) > 2 \text{ miles}$, so the lineman made his time.

$D_B(12) < 2 \text{ miles}$, so the receiver did not make his time.

$$3a. \quad \frac{d}{dx}[x^2 + 4xy + y^2 = -12] \rightarrow 2x + 4x\frac{dy}{dx} + y(4) + 2y\frac{dy}{dx} = 0$$

$$(4x + 2y)\frac{dy}{dx} = -2x - 4y \rightarrow \frac{dy}{dx} = -\frac{x + 2y}{2x + y}$$

$$3b. \quad x + 2y = 0 \rightarrow x = -2y$$

$$x^2 + 4xy + y^2 = (-2y)^2 + 4(-2y)y + y^2 = -3y^2 = -12 \rightarrow y = \pm 2$$

$$(4, -2) \text{ and } (-4, 2)$$

$$\begin{aligned}
 3c. \quad \frac{d^2y}{dx^2} &= \frac{d}{dx} \left[-\frac{x+2y}{2x+y} \right] = \frac{(2x+y)\left(-1-2\frac{dy}{dx}\right) - (-x-2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2} \\
 \frac{d^2y}{dx^2} \Big|_{(4,-2)} &= \frac{(2(4)+(-2))(-1-2(0)) - (-(-4)-2(-2))(2+(0))}{(2(4)+(-2))^2} = \frac{6}{36} = \frac{1}{6} \\
 \frac{d^2y}{dx^2} \Big|_{(-4,2)} &= \frac{(2(-4)+(2))(-1+2(0)) - (-(-4)-2(2))(2+(0))}{(2(4)+(-2))^2} = -\frac{6}{36} = -\frac{1}{6}
 \end{aligned}$$

$(4, -2)$ is at a minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

$(-4, 2)$ is at a minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$