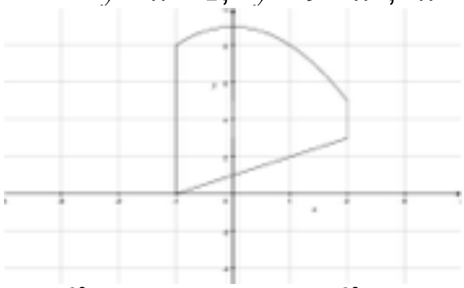
6.1 Free Response Solutions

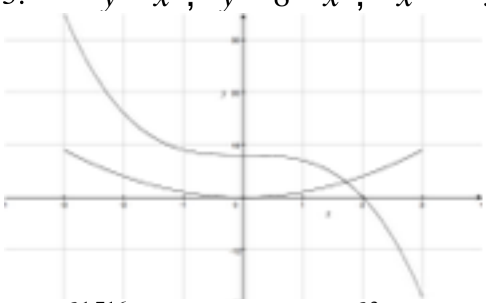
1. $y = x + 1, \quad y = 9 - x^2, \quad x = -1, \quad x = 2$



$$A = \int_{-1}^{2}\left[\left(9 - x^2\right) - (x+1)\right]dx = \int_{-1}^{2}\left(9 - x^2 - x - 1\right)dx = \int_{-1}^{2}\left(8 - x - x^2\right)dx$$

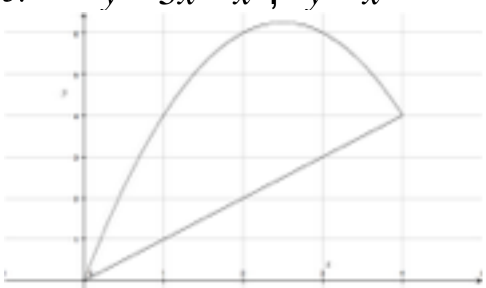
$$= \left[8x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \left[16 - \frac{4}{2} - \frac{8}{3}\right] - \left[-8 - \frac{1}{2} + \frac{1}{3}\right] = 19.5$$

3. $y = x^2, \quad y = 8 - x^3, \quad x = -3, \quad x = 3$



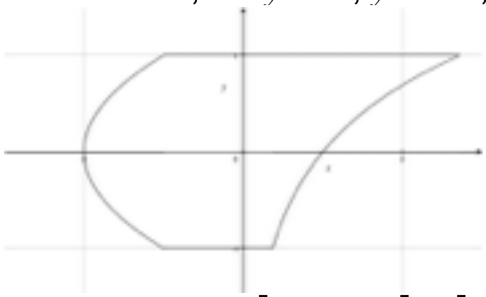
$$A = \int_{-3}^{1.716}\left[\left(8 - x^3\right) - x^2\right]dx + \int_{1.716}^{3}\left[x^2 - \left(8 - x^3\right)\right]dx = 60.252$$

5. $y = 5x - x^2, \quad y = x$



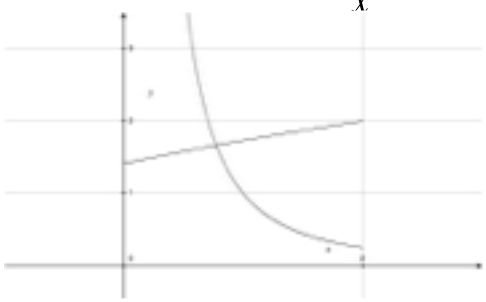
$$A = \int_{0}^{4}\left[\left(5x - x^2\right) - x\right]dx = \int_{0}^{4}\left[4x - x^2\right]dx = \left[2x^2 - \frac{1}{3}x^3\right]_{0}^{4} = 32 - \frac{8}{3} = 32/3$$

7. $x = e^y,\ x = y^2 - 2,\ y = -1,\ y = 1$



$$A = \int_{-1}^{1}\left[e^y - \left(y^2 - 2\right)\right]dy = \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1} = \left[e^1 - \frac{1}{3} + 2\right] - \left[\frac{1}{e} + \frac{1}{3} - 2\right] = 5.684$$

9. $y = \sqrt{x+2},\ y = \dfrac{1}{x^2},\ x = 1,\ x = 2$

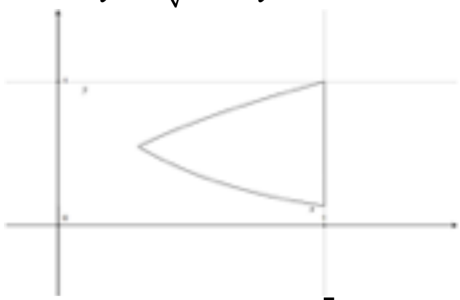


$$A = \int_{1}^{2}\left(\sqrt{x+2} - \frac{1}{x^2}\right)dx$$

$$= 1.369$$

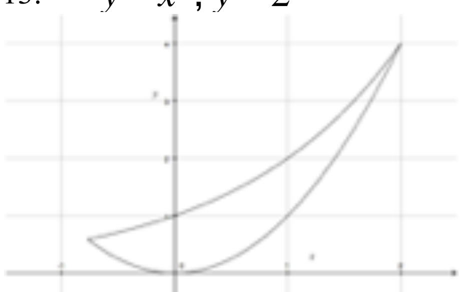
$\lim_{a \to 0^+} -\dfrac{1}{a}$ dne $\therefore$ the integral diverges

11. $y = \sqrt{x}, \quad y = e^{-2x}, \quad x = 1$



$$A = \int_{.301}^{1} \left(\sqrt{x} - e^{-2x}\right)dx = \left[\frac{2}{3}x^{3/2} + \frac{1}{2}e^{-2x}\right]_{.301}^{1} = .350$$

13. $y = x^2, y = 2^x$



$$A = \int_{-.767}^{2} \left[2^x - x^2\right]dx = \left[\frac{2^x}{\ln 2} - \frac{1}{3}x^3\right]_{-.767}^{2} = 2.106$$

<u>6.1 Multiple Choice Solutions</u>

1. $$A = \int_{-9/8}^{9/8} e^{\left(\frac{-1}{1+x^2}\right)}dx = 1.080$$

The correct answer is B

---

3. $$Area = \int_{0}^{1.456} (2\sin x - x\ln(2x+1))dx = 0.661$$

The correct answer is B

---

5. $Area = \int_0^1 (e^{2x} - 1)dx = \left[\frac{1}{2}e^{2x} - x\right]_0^1 = \left(\frac{1}{2}e^2 - 2\right) - \left(\frac{1}{2}e^0 - 0\right) = \frac{1}{2}(e^2 - 3)$

The correct answer is B

6.2 Free Response Solutions

1. $y = 1 + 6x^{3/2}$ on $x \in [0, 1]$

$y' = 9x^{1/2} \rightarrow L = \int_0^1 \sqrt{1 + 81x}\,dx = 6.103$

3. $x = \frac{1}{3}\sqrt{y}(y - 3)$ on $y \in [1, 9]$

$x = \frac{1}{3}y^{3/2} - y^{1/2} \rightarrow x' = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$

$L = \int_1^9 \sqrt{1 + \left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2}\,dy = \int_1^9 \sqrt{1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1}}\,dy = \int_1^9 \sqrt{\frac{1}{2} + \frac{1}{4}y + \frac{1}{4}y^{-1}}\,dy = 10.667$

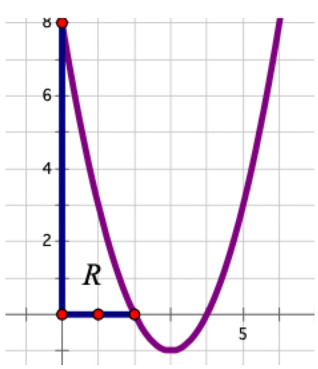
5. $y = \ln x^{3/2}$ on $x \in \left[1, \sqrt{3}\right]$

$y' = \frac{1}{x^{3/2}} \cdot \frac{3}{2}x^{1/2} = \frac{3}{2x} \rightarrow L = \int_1^{\sqrt{3}} \sqrt{1 + \frac{9}{4x^2}}\,dx = 1.106$

7. Find the length of the arc along $f(x) = \int_0^x \sqrt{\cos t}\,dt$ on $x \in \left[0, \frac{\pi}{2}\right]$.

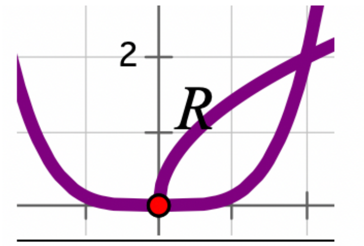
$f'(x) = \sqrt{\cos x} \rightarrow L = \int_0^{\pi/2} \sqrt{1 + \cos x}\,dx = 2$

9. Find the perimeter of the region R bounded by $y = (x - 3)^2 - 1$, the x-axis, and the y-axis.



$$L = \int_0^2 \sqrt{1+(2x-6)^2}\,dx + 8 + 2 = 8.268 + 10 = 18.268$$

11. Find the perimeter of the region R bounded by $y = \sqrt{2x}$ and $y = \frac{1}{8}x^4$.



$$L = \int_0^2 \sqrt{1+(0.5x^3)^2}\,dx + \int_0^2 \sqrt{1+\left(\frac{1}{\sqrt{2x}}\right)^2}\,dx = 3.200 + 2.958 = 6.158$$

6.2 Multiple Choice Solutions

1. $\frac{dy}{dx} = \sqrt{x} \to L = \int_0^4 \sqrt{1+(\sqrt{x})^2}\,dx = \int_0^4 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^4 = \frac{2}{3}(5\sqrt{5}-1)$

The correct answer is A

---

3. $\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x \to L = \int_0^{\pi/3} \sqrt{1+(-\tan x)^2}\,dx = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx =$

$\ln|\sec x + \tan x|\Big|_0^{\pi/3} = \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| = \ln\left|2+\sqrt{3}\right| - \ln 1 = \ln\left|2+\sqrt{3}\right|$

The correct answer is D

---

5. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \to L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx$

The correct answer is E

---

7. $\int_2^8 \sqrt{1+\frac{1}{4-x^2}}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \to \frac{dy}{dx} = \sqrt{\frac{1}{4-x^2}} = \frac{1}{\sqrt{4-x^2}}.$

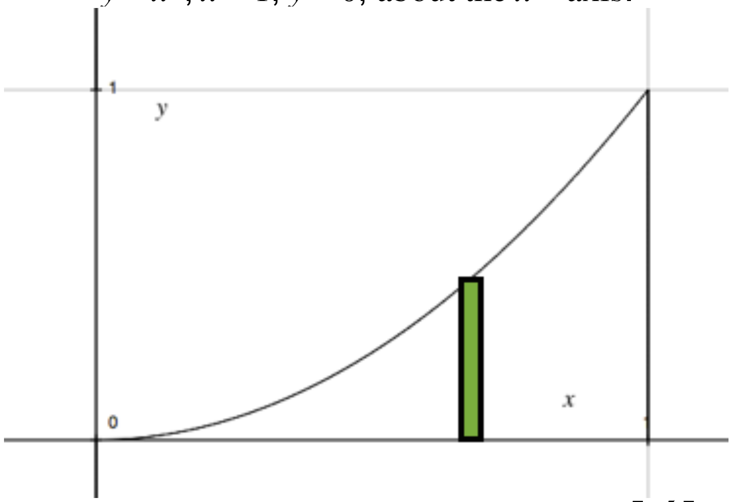
$$\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}} \to y = \sin^{-1}\frac{x}{2} + c$$

$$(1, 0) \to 0 = \sin^{-1}\frac{1}{2} + c \to 0 = \frac{\pi}{6} + c \to c = -\frac{\pi}{6}$$

The correct answer is E

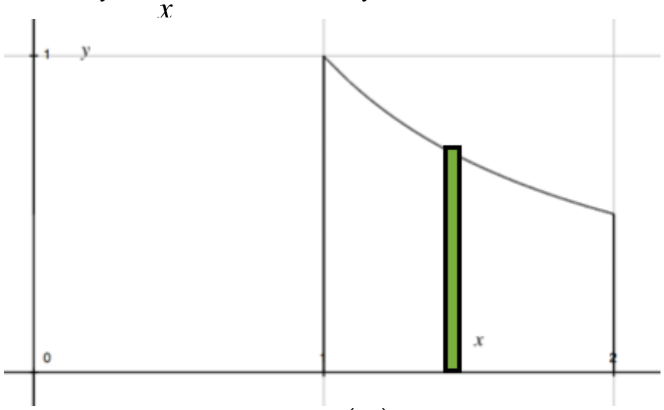
6.3 Free Response Solutions

1. $y = x^2,\ x = 1,\ y = 0$; about the $x-$axis.



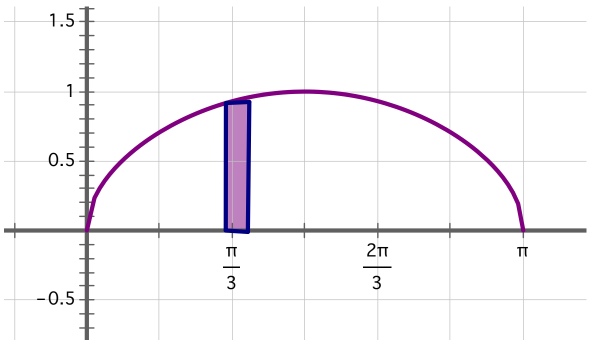
$$V = \pi\int_0^1 r^2 dx = \pi\int_0^1 (x^2)^2 dx = \pi\int_0^1 x^4 dx = \pi\left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}$$

3. $y = \dfrac{1}{x},\ x = 1,\ x = 2,\ y = 0$; about the $x-$axis.



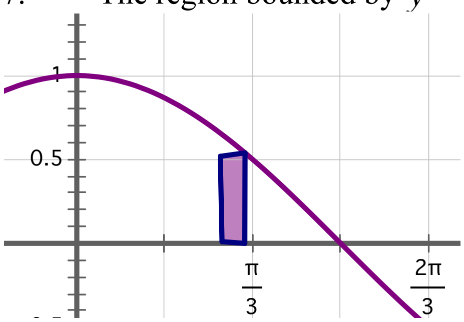
$$V = \pi\int_1^2 r^2 dx = \pi\int_1^2 \left(\frac{1}{x}\right)^2 dx = \pi\int_1^2 x^{-2} dx = \pi\left[\frac{x^{-1}}{-1}\right]_1^2 = \pi/2$$

5. The region bounded by $y=\sqrt{\sin x}$ and $y=0$.



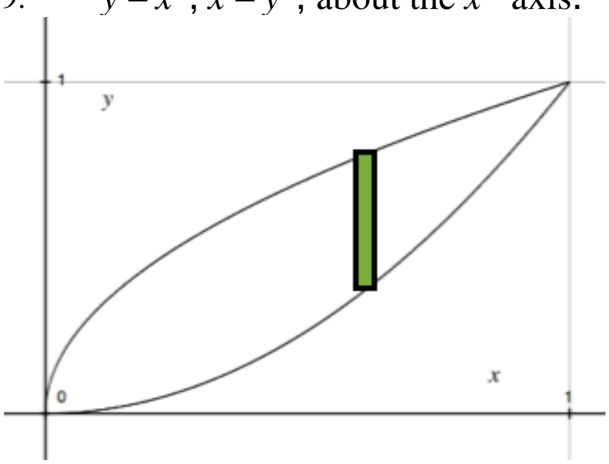
$$V=\pi\int_0^{\pi} r^2 dx=\pi\int_0^{\pi}\left(\sqrt{\sin x}\right)^2 dx=\pi\int_0^{\pi}(\sin x)dx=\pi[-\cos x]_0^{\pi}=2\pi$$

7. The region bounded by $y=\cos x$, $x=0$, and $y=0$.



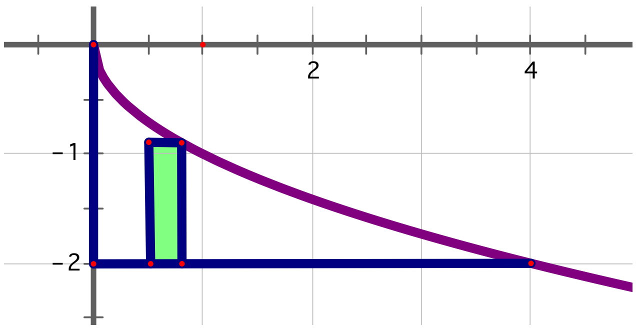
$$V=\pi\int_0^{\pi/2} r^2 dx=\pi\int_0^{\pi/2}\left(\cos^2 x\right)dx=\pi\left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\pi}=\frac{\pi^2}{2}$$

9. $y=x^2$, $x=y^2$; about the $x-$axis.



$$V=\pi\int_0^1\left(R^2-r^2\right)dx=\pi\int_0^1\left[\left(\sqrt{x}\right)^2-\left(x^2\right)^2\right]dx=\pi\int_0^1\left(x-x^4\right)dx=\pi\left[\frac{1}{2}x^2-\frac{1}{5}x^5\Big|_0^1\right]=\frac{3\pi}{10}$$

11. The region bounded by $y = -\sqrt{x}$, $y = -2$ and the y-axis.



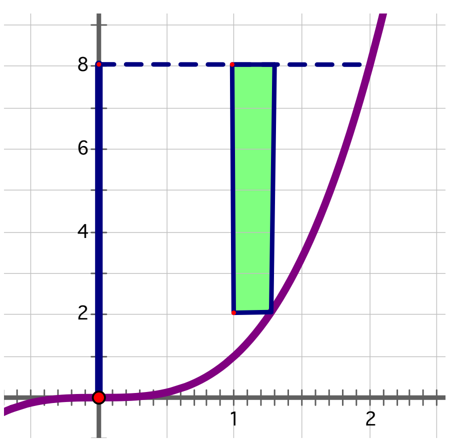
$$V = \pi \int_0^4 \left(R^2 - r^2\right) dx$$

$$= \pi \int_0^4 \left[(1)^2 - \left(-\sqrt{x}\right)^2\right] dx$$

$$= \pi \int_{.301}^1 (1 - x) dx$$

$$= \pi \left[x - \frac{1}{2} x^2 \Big|_0^4 \right] = 8\pi$$

13. The region bounded by $y = x^3$, $y = 8$, and the y-axis.



$$V=\pi\int_0^2 (R^2-r^2)dx = 2\pi\int_0^2 [(8)^2-(x^3)^2]dx$$

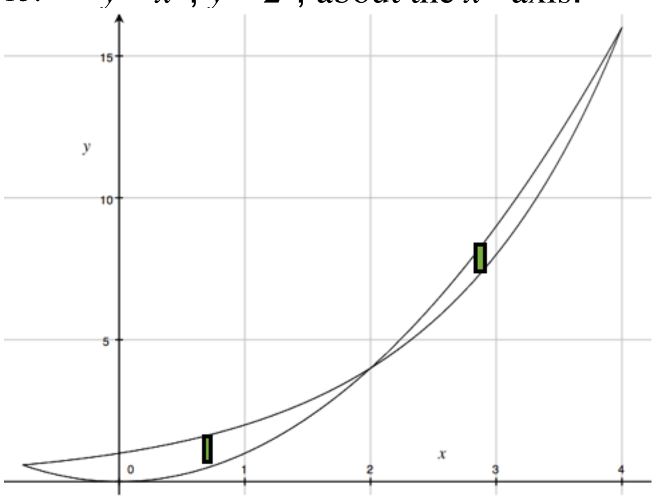
$$=\pi\int_0^2 [64-x^6]dx$$

$$=\pi\left[64x-\frac{1}{7}x^7\Big|_0^2\right]$$

$$\pi\left[128-\frac{128}{7}\right]$$

$$=\frac{768\pi}{7}$$

15. $y=x^2$, $y=2^x$; about the $x-$axis.



$$V=\pi\int_{-.767}^{2}(R^2-r^2)dx+\pi\int_2^4(R^2-r^2)dx$$

$$V=\pi\int_{-.767}^{2}[(2^x)^2-(x^2)^2]dx+\pi\int_2^4[(x^2)^2-(2^x)^2]dx=94.612$$

$$V=\pi\int_{-.767}^{2}(R^2-r^2)dx+\pi\int_2^4(R^2-r^2)dx$$

$$V=\pi\int_{-.767}^{2}[(2^x)^2-(x^2)^2]dx+\pi\int_2^4[(x^2)^2-(2^x)^2]dx=94.612$$

6.3 Multiple Choice Solutions

1. $$V = \pi \int_0^4 \left((\sqrt{x})^2 - \left(\frac{1}{2}x \right)^2 \right) dx = \pi \int_0^4 \left(x - \frac{1}{4}x^2 \right) dx =$$
$$\pi \left[\frac{1}{2}x^2 - \frac{1}{12}x^3 \right]_0^4 = \pi \left(8 - \frac{16}{3} \right) = \frac{8}{3}\pi$$

The correct answer is E

---

3. $$V = \pi \int_1^e (\sqrt{\ln x})^2 \, dx = \pi \int_1^e (\ln x) \, dx = \pi$$

The correct answer is C

---

5. $$V = \pi \int_0^1 \left(3^2 - \left(\tan^{-1} x\right)^2\right) dx = 27.505$$

The correct answer is E

---

7. $$x^2 + 4y^2 = 1 \rightarrow y = \sqrt{1 - \frac{1}{4}x^2} \rightarrow V = \pi \int_{-2}^{2} \left(\sqrt{1 - \frac{1}{4}x^2} \right)^2 dx$$
$$V = 2\pi \int_0^2 \left(1 - \frac{1}{4}x^2 \right) dx = 2\pi \left[x - \frac{1}{12}x^3 \right]_0^2 = \frac{8\pi}{3}$$

The correct answer is B

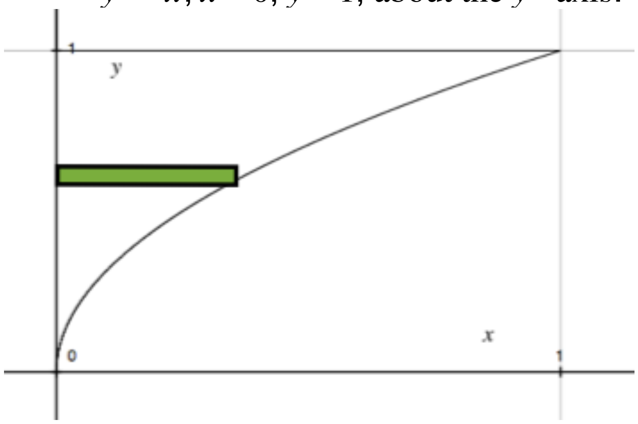
---

9. $$Volume = \pi \int_0^2 \left(\pi^2 - \left(\frac{\pi}{2} - \sin^{-1}(x-1) \right)^2 \right) dx + \pi \int_2^6 \left(\pi^2 - \left(\frac{\pi}{2} \sqrt{x-2} \right)^2 \right) dx = 105.585$$

The correct answer is D

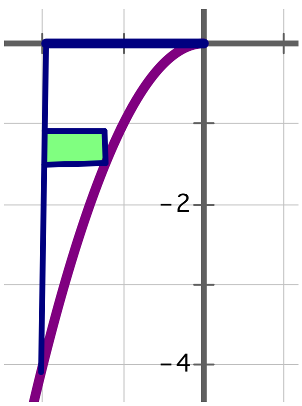
6.4 Free Response Solutions

1. $y^2 = x,\ x = 0,\ y = 1$; about the $y-$axis.



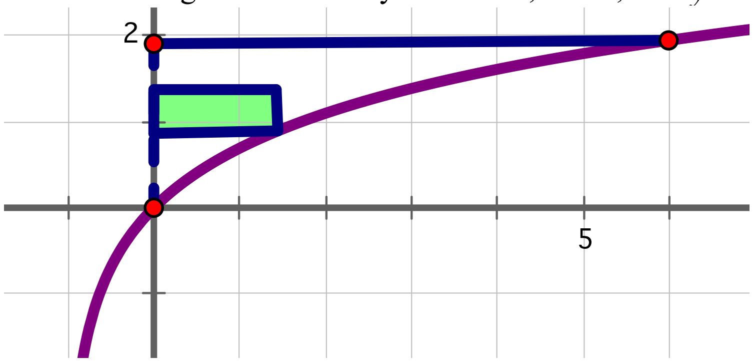
$$V = \pi\int_0^1 r^2 dy = \pi\int_0^1 x^2 dy = \pi\int_0^1 (y^2)^2 dy = \pi\int_0^1 y^4 dy = \pi\left[\frac{y^5}{5}\right]_0^1 = \frac{\pi}{5}$$

3. The region bounded by by $x = -\sqrt{-y}$ on $y \in [-4, 0]$.



$$V = \pi\int_{-4}^0 r^2 dy = \pi\int_{-4}^0 x^2 dy = \pi\int_{-4}^0 (-\sqrt{-y})^2 dy = \pi\int_{-4}^0 (-y) dy = \pi\left[\frac{-y^2}{2}\right]_{-4}^0 = 8\pi$$

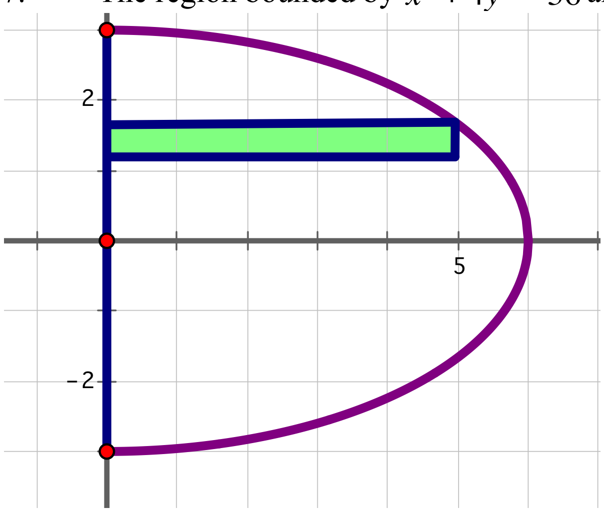
5. The region bounded by $x = e^y - 1$, $x = 0$, and $y = \ln 3$.



$$V = \pi \int_0^{\ln 3} r^2 dy = \pi \int_0^{\ln 3} (e^y - 1)^2 dy = \pi \int_0^{\ln 3} (e^{2y} - 2e^y + 1) dy = \pi \left[\frac{1}{2} e^{2y} - 2e^y + y \right]_0^{\ln 3} =$$

$$= \pi[(4.5 - 6 + \ln 3) - (0.5 - 3 + 0)] = 2.986$$

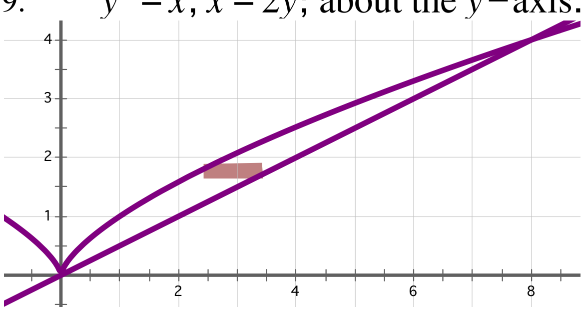
7. The region bounded by $x^2 + 4y^2 = 36$ and the y-axis on $y \in [-3, 3]$.



$$x^2 + 4y^2 = 36 \to x = \sqrt{36 - 4y^2}$$

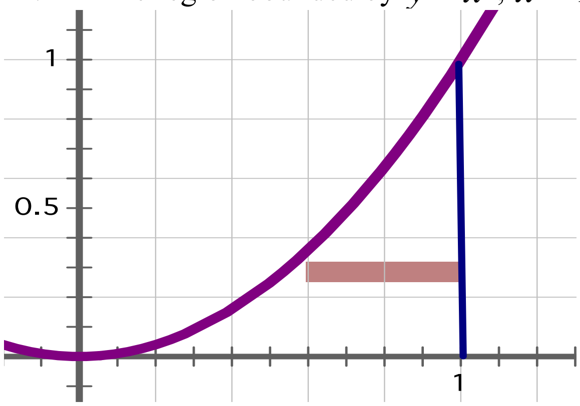
$$V = \pi \int_{-3}^{3} r^2 dy = \pi \int_{-3}^{3} \left(\sqrt{36 - 4y^2}\right)^2 dy = 2\pi \int_0^3 (36 - 4y^2) dy = 2\pi \left[36y - \frac{4y^3}{3} \right]_0^3 = 144\pi$$

9. $y^2 = x$, $x = 2y$; about the y-axis.



$$V=\pi\int_0^2\left(R^2-r^2\right)dy=\pi\int_0^2\left[(2y)^2-(y)^2\right]dy\,\pi\int_0^2\left(4y^2-y^4\right)dy=\pi\left[\frac{4}{3}y^3-\frac{1}{5}y^5\Big|_0^2\right]=\frac{64\pi}{15}$$

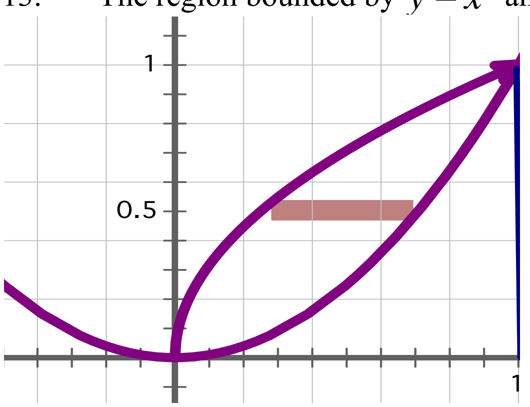
11. The region bounded by $y=x^2$, $x=1$, and $y=0$.



$$y=x^2\rightarrow x=\sqrt{x}$$

$$V=\pi\int_0^1\left(R^2-r^2\right)dy=\pi\int_0^1\left[(1)^2-\left(y^{1/2}\right)^2\right]dy\,\pi\int_0^1(1-y)dy=\pi\left[y-\frac{1}{2}y^2\Big|_0^1\right]=\frac{\pi}{2}$$

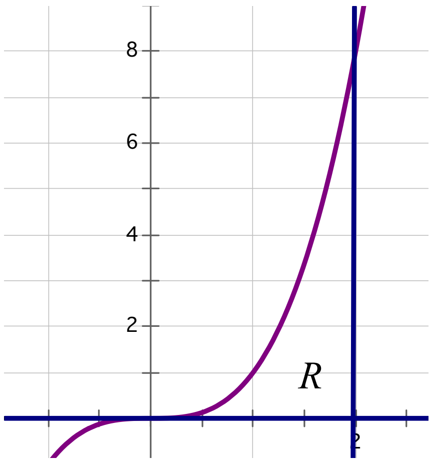
13. The region bounded by $y=x^2$ and $x=y^2$.



$$y=x^2\rightarrow x=\sqrt{y}$$

$$V=\pi\int_0^1\left(R^2-r^2\right)dy=\pi\int_0^1\left[\left(\sqrt{y}\right)^2-\left(y^2\right)^2\right]dy=\pi\int_0^1\left[y-y^4\right]dy=\pi\int_0^1\left[\frac{1}{2}y^2-\frac{1}{5}y^5\right]dy=\frac{3\pi}{10}$$

15. Let R be the region bounded by $y = x^3$, $x = 2$, and $y = 0$.



$y = x^3 \to x = y^{1/3} y = x^3,\ x = 2$

$$V = \pi\int_0^8 (R^2 - r^2)dy = \pi\int_0^8 \left[(2)^2 - \left(y^{1/3}\right)^2\right]dy = \pi\int_0^8 \left[4 - y^{2/3}\right]dy = \pi\int_0^8 \left[4y - \frac{3}{5}y^{5/3}\right]dy = \frac{128\pi}{5}$$

6.4 Multiple Choice Solutions

1. $$V = \pi\int_0^2 \left(\left(\sqrt{\frac{16x}{x^2+4}}\right)^2 - (x)^2\right)dx = \pi\int_0^2 \left(\frac{16x}{x^2+4} - x^2\right)dx = \pi\left[8\ln\left(x^2+4\right) - \frac{1}{3}x^3\right]_0^2 =$$

$$\pi\left[\left(8\ln 8 - \frac{8}{3}\right) - \left(8\ln 4 - 0\right)\right] = \pi\left(24\ln 2 - \frac{8}{3} - 16\ln 2\right) = \pi\left(8\ln 2 - \frac{8}{3}\right)$$

The correct answer is E

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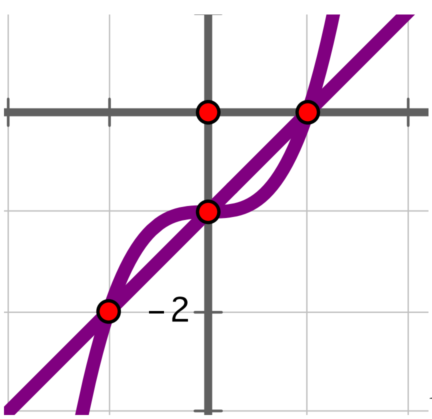
3. 

$y = x^3 - 1 \to x = (y+1)^{1/3}$; $y = x - 1 \to x = y + 1$

Boundaries: $y + 1 = (y+1)^{1/3} \rightarrow y = -2,\ -1,\ and\ 0$

$$Volume = \pi \int_{-2}^{0} \left((x_1)^2 - (x_2)^2 \right) dy = 2\pi \int_{-1}^{0} \left((y+1)^2 - (y+1)^{2/3} \right) dy = 6.462$$

The correct answer is E

---

5. The solid is generated when R is rotated about the y-axis, so B and C are incorrect. $y = \dfrac{\pi}{2}$ is a boundary, so E is incorrect. The formula for the Disk Method about the y-axis is $\pi \int_{0}^{\pi/2} x^2 dy$, so A is incorrect.

The correct answer is D

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7. The solid is formed when the region is revolved about the x-axis, so only A or E can be correct. The graph shows the problem uses the Washer Method, but A uses the Disk Method.

The correct answer is E

6.5 Free Response Solutions

1. Washer Method

$$V = \pi \int_{-4}^{4} \left(R^2 - r^2\right) dx = \pi \int_{-4}^{4} \left((8-(-1))^2 - \left(\frac{1}{2}x^2 - (-1)\right)^2 \right) dx =$$

$$2\pi \int_0^4 \left(81 - \left(\frac{1}{4}x^4 + x^2 + 1 \right) \right) dx = 2\pi \int_0^4 \left(-\frac{1}{4}x^4 - x^2 + 80 \right) dx$$

$$= 2\pi \left[-\frac{1}{20}x^5 - \frac{1}{3}x^3 + 80x \right]_0^4 = \frac{7424\pi}{15}$$

3. Washer Method

$$V = \pi \int_1^3 \left(R^2 - r^2\right) dx = \pi \int_1^3 \left(\left(1 + \frac{1}{x}\right)^2 - 1^2 \right) dx = 8.997$$

5. Washer Method $\quad V = \pi \int_0^1 \left(R^2 - r^2\right) dx = \pi \int_0^1 \left((1 - x^3)^2 - (1 - \sqrt{x})^2 \right) dx$

$$\pi \int_0^1 \left((1 - 2x^3 + x^6) - (1 - 2\sqrt{x} + x) \right) dx = \pi \int_0^1 \left(x^6 - 2x^3 - x + 2x^{1/2} \right) dx$$

$$= \pi \left(\frac{1}{7}x^7 - \frac{1}{2}x^4 - \frac{1}{2}x^2 + \frac{4}{3}x^{3/2} \right)_0^1 = \frac{10\pi}{21}$$

7. $y = \sqrt{x+1} \rightarrow x = y^2 - 1$

Disk Method: $\quad V = \pi \int_0^2 \left(\left[3 - (y^2 - 1)\right]^2 \right) dy = \pi \int_0^2 \left[9 - (y^4 - 2y^2 + 1) \right] dy$

$$= \pi \left[-\frac{1}{5}y^5 + \frac{2}{3}y^3 + 8 \right]_0^2 = \pi \left[-\frac{32}{5} + \frac{16}{3} + 8 \right] = \frac{104\pi}{15}$$

9. Washer Method: $\quad V = \pi \int_{-1}^{1} \left(R^2 - r^2\right) dy = \pi \int_{-1}^{1} \left((3 - x_1)^2 - (3 - x_2)^2 \right) dy$

$$= \pi \int_{-1}^{1} \left((3 - y^2)^2 - (3-1)^2 \right) dy = \pi \int_{-1}^{1} \left(y^4 - 6y^2 + 9 - 4 \right) dy = 2\pi \int_0^1 \left(y^4 - 6y^2 + 5 \right) dy$$

$$= 2\pi \left(\frac{1}{5}y^4 - 2y^3 + 5y \right)_0^1 = \frac{32\pi}{15}$$

11. Washer Method $\quad V = \pi \int_{-.916}^{.916} \left(R^2 - r^2\right) dx = \pi \int_{-.916}^{.916} \left((1 - y_1)^2 - (1 - y_2)^2 \right) dx$

$$= \pi \int_{-.916}^{.916} \left(\left(1 - \ln(x^2 + 1)\right)^2 - (1 - \cos x)^2 \right) dx = 3.447$$

13. Washer Method $y=\sqrt{6x-x^2} \to y^2=6x-x^2 \to -y^2=x^2-6x$

$-y^2+9=x^2-6x+9=(x-3)^2 \to 9-y^2=(x-3)^2 \to x=3+\sqrt{9-y^2}$

$g(x)=y=-1+e^{-0.5x} \to y+1=e^{-0.5x} \to \ln(y+1)=-0.5x \to -2\ln(y+1)$

$V=\pi\int_0^{2.9910233}\left[\left[(-2\ln(y+1))-(-1)\right]^2-\left[\left(3+\sqrt{9-y^2}\right)-(-1)\right]^2\right]dy=374.525$

6.5 Multiple Choice Solutions

1. This is a Washer Method problem. Both curves are below the line $y=3$, so $V=\pi\int_a^b\left(\left[k-f(x)\right]^2-\left[k-g(x)\right]^2\right)dx$. $y=2x$ is closer to the line, so

The correct answer is C

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3. This is a Disk Method problem. $V=\pi\int_0^1\left[3-\tan^{-1}x\right]^2dx=20.773y=\tan^{-1}x$

The correct answer is E

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5. $Volume=\int_0^2\left(\pi-\left(\frac{\pi}{2}-\sin^{-1}(x-1)\right)\right)^2dx+\int_2^6\left(\pi-\frac{\pi}{2}\sqrt{x-2}\right)^2dx=39.111$

The correct answer is C

6.6 Free Response Solutions

1. Area = $y = \left(\sqrt{x}\right)^2 = x$

Volume $= \int_0^4 x^2 dx = \left[\frac{1}{3}x^3\right]_0^4 = \frac{64}{3}$

3. $y = \sqrt{2x} \to x = \frac{y^2}{2};\ y = \frac{1}{8}x^4 \to x = \sqrt[4]{8y}$; Rectangle length = $\sqrt[4]{8y} - \frac{y^2}{2}$

Area = $3s^2 = 3\left(\sqrt[4]{8y} - \frac{y^2}{2}\right)^2$

Volume $= \int_0^2 3\left(\sqrt[4]{8y} - \frac{y^2}{2}\right)^2 dy = 6.031$

5. Rectangle length = $y = (x-3)^2 - 1 = x^2 - 6x + 8$

Area = $y^2 = \left(x^2 - 6x + 8\right)^2$

Volume $= \int_0^2 \left(x^2 - 6x + 8\right)^2 dx = 33.067$

7. Rectangle length = $e^{-.5x} - x^2 = w$; Area = $\frac{1}{2}w^2 = \frac{1}{2}\left(e^{-.5x} - x^2\right)^2$

Volume $= \int_{-1.429612}^{.81555342} \frac{1}{2}\left(e^{-.5x} - x^2\right)^2 dx = 0.687$

9. The base is the ellipse $9x^2 + 4y^2 = 36$ and the cross-sections perpendicular to the *x*-axis are isosceles right triangles with the hypotenuse in the base.

$$V = \int_{-2}^{2} \frac{1}{2} bh dx = \int_{-2}^{2} \frac{1}{2} \cdot \frac{2}{\sqrt{2}} y \cdot \frac{2}{\sqrt{2}} y dx = \int_{-2}^{2} y^2 dx = \int_{-2}^{2} \left(9 - \frac{9}{4}x^2\right) dx = 24$$

11. The base is the region $x^2 \le y \le 1$ and the cross-sections perpendicular to the *x*-axis are squares.

$$V = \int_{-1}^{1} (side)^2 dx$$
$$= \int_{-1}^{1} (1 - x^2)^2 dx = 2\int_{0}^{1} (1 - 2x^2 + x^4) dx = 2\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = \frac{16}{15}$$

13. $y = \sqrt{x}$, $y = e^{-2x}$, $x = 1$; the cross-sections are semi-circles.

$$V = \int_{.301}^{1} \frac{1}{2}\pi r^2 dx = \frac{\pi}{2}\int_{.301}^{1} \left(\frac{\sqrt{x} - e^{-2x}}{2}\right)^2 dx = .085$$

15. The region bounded by $y = \dfrac{1}{x}$, $x = 1$ and $x = 4$ in Quadrant I where the cross-sections are squares.

$$V = \int_{1}^{4} \frac{1}{x^2} dx = \left[\frac{x^{-1}}{-1}\right]_1^4 = -\frac{1}{4} - (-1) = \frac{3}{4}$$

17a) $V \approx (36\pi) \cdot 3 + (25\pi) \cdot 3 + (9\pi) \cdot 3 + (4\pi) \cdot 3 + (9\pi) \cdot 3 + (25\pi) \cdot 3 + (36\pi) \cdot 3$
$= 582\pi$

17b) $V = \pi \int_{0}^{27} \left(4 + 2\cos\left(\frac{\pi}{27}h\right)\right)^2 dh = 13186.456$

6.6 Multiple Choice Solutions

1. $$V = \int_{0}^{3} \left(e^{-x^2/2}\right)^2 dx = \int_{0}^{3} \left(e^{-x^2}\right) dx = 0.886$$

The correct answer is A

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3. Perpendicular to the *y*-axis means the *x* must be isolated:

$$x + 3y = 5 \rightarrow x = 5 - 3y. \quad V = \int_{0}^{5/3} (5 - 3y)^2 \, dy = 13.889$$

The correct answer is E

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5. $Volume = \int_0^{\pi/4} \frac{\pi}{2}\left(\frac{\cos x - \sin x}{2}\right)^2 dx = 0.112$ and $y = \cos x$

The correct answer is B

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7. $x^2 + 4y^2 = 4 \to y = \sqrt{1 - \frac{1}{4}x^2} \to$

$$V = \int_{-2}^{2}\left(\sqrt{1-\frac{1}{4}x^2}\right)^2 dx = 2\int_0^2\left(1-\frac{1}{4}x^2\right)dx = 2\left[x - \frac{1}{12}x^3\right]_0^2 = 2\left[2-\frac{2}{3}\right] = \frac{8}{3}$$

The correct answer is A

---

9. $V = \int_0^2 \frac{1}{2}\left(\pi - \left(\frac{\pi}{2} - \sin^{-1}(x-1)\right)\right)^2 dx + \int_2^6 \frac{1}{2}\left(\pi - \left(\frac{\pi}{2}\sqrt{x-2}\right)\right)^2 dx = 6.225$

The correct answer is A.

6.7 Free Response Solutions

1a. $Area = \int_0^2 (2x^2 - x^3)\, dx = \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 = \frac{16}{3} - 4 = \frac{4}{3}$

1b. $V = \frac{\pi}{8}\int_a^b r^2 dx = \frac{\pi}{8}\int_0^2 \left(\frac{2x^2 - x^3}{2}\right)^2 dx$

1c. The rectangles need to be horizontal, so $y = 2x^2 \to x = \sqrt{\frac{y}{2}}$ and $y = x^3 \to x = y^{1/3}$.

Washer Method: $V = \pi\int_0^8 \left(\left[y^{2/3}\right]^2 - \left[\sqrt{\frac{y}{2}}\right]^2\right) dy = \pi\int_0^8 \left(y^{4/3} - \frac{1}{2}y\right) dy =$

$= \pi\left[\frac{3}{7}y^{7/3} - \frac{1}{4}y^2\right]_0^8 = \pi\left(\frac{128}{7} - 16\right) = \frac{16\pi}{7}.$

3a. $Area = \int_{-1}^3 (x+1)^{1/2}\, dx = \left[\frac{2}{3}(x+1)^{3/2}\right]_{-1}^3 = \frac{16}{3} - 0 = \frac{16}{3}$

3b. $Volume = \int_{-1}^3 (x+1)^{1/2}\left(2(x+1)^{1/2}\right) dx = \int_{-1}^3 (2x+2)\, dx =$

$\left[x^2 + 2x\right]_{-1}^3 = 15 - (-1) = 16$

3c. Washer Method with horizontal rectangles:

$y = \sqrt{x+1} \to x = y^2 - 1$

$V = \pi\int_0^2 \left(\left[4 - (y^2 - 1)\right]^2 - [4-3]^2\right) dy$

5a. $A = \int_0^4 \left[-6(x^2 - 4x)e^{-x} - (x^2 - 4x)\right] dx = 23.326$

5b. $V = \pi\int_0^4 \left[\left[-6(x^2 - 4x)e^{-x} + 4\right]^2 - (x^2 - 4x + 4)^2\right] dx = 676.640$

5c. $V = \int_0^4 \left[-6(x^2 - 4x)e^{-x} - (x^2 - 4x)\right]^2 dx = 170.182$

7a. $A=\int_0^3\left[\frac{9}{2x+3}\right]dx=\frac{9}{2}\int_0^3\left[\frac{1}{2x+3}\right]2dx=\frac{9}{2}\int_3^9\frac{1}{u}\,du=\frac{9}{2}[\ln u]_3^9=\frac{9}{2}[\ln 9-\ln 3]=\frac{9}{2}\ln 3$

7b. $y=\frac{9}{2x+3}\rightarrow\frac{dy}{dx}=-9(2x+3)^{-2}(2)=\frac{-18}{(2x+3)^2}$

$$P=3+3+1+\int_0^3\sqrt{1+\left(\frac{-18}{(2x+3)^2}\right)^2}\,dx=10.747$$

7c. $V=\pi\int_0^3\left(\frac{9}{2x+3}\right)^2dx=\frac{81\pi}{2}\int_0^3(2x+3)^{-2}\,2dx=\frac{81\pi}{2}\left[\frac{(2x+3)^{-1}}{-1}\right]_0^3=\frac{81\pi}{2}\left[-\frac{1}{9}-\left(-\frac{1}{3}\right)\right]$

$V=9\pi=28.274$

9a. $A=\int_0^1\left[4x(1-x)-\sqrt[4]{2x}(x-1)\right]\,dx=1.089$

b. $V=\int_0^1\left[[3-4x(1-x)]^2-\left[3-\sqrt[4]{2x}(x-1)\right]^2\right]\,dx=6.630.$

c. $V=\int_0^1\left[[4x(1-x)+2]^2-\left[\sqrt[4]{2x}(x-1)+2\right]^2\right]\,dx=17.665$

d. $V=\int_0^1\left[[k-x(1-x)]^2-\left[k-\sqrt[4]{2x}(x-1)\right]^2\right]\,dx$

11a. Find the area of the regions R, S, and T.

$$Area_R=\int_0^{1.1418956}\left[(x^3-3x^2+2x+4)-(2x\sqrt{4-x})\right]\,dx=2.463$$

$$Area_S=\int_{1.1418956}^{2.5444495}\left[(2x\sqrt{4-x})-(x^3-3x^2+2x+4)\right]\,dx=1.551$$

$$Area_T=\int_0^4\left[2x\sqrt{4-x}\right]\,dx-Area_S=15.515$$

11b. Find the volume of the solid generated by rotating the curve $g(x)=2x\sqrt{4-x}$ around the line $y=8$ on the interval $x\in[1,3]$.

$$V=\pi\int_1^3\left(8-2x\sqrt{4-x}\right)^2\,dx=48.152$$

11c. Find the volume of the solid generated by rotating the region S around the line $y=-1$.

$$V=\int_{1.1418956}^{2.5444495}\left[(2x\sqrt{4-x}-(-1))^2-(x^3-3x^2+2x+4-(-1))^2\right]\,dx=55.727$$

11d. Find the volume of the solid generated if *R* forms the base of a solid whose cross sections are squares whose bases are perpendicular to the *x*-axis

$$V=\int_0^{1.1418956}\left[\left(x^3-3x^2+2x+4\right)-\left(2x\sqrt{4-x}\right)\right]^2dx=6.962$$

13a) $A=\int_0^{30}w(x)\ dx\approx\left(\frac{0+14.6}{2}\right)(5)+\left(\frac{14.6+13.5}{2}\right)(5)+\left(\frac{13.5+14.5}{2}\right)(5)+$
$\left(\frac{14.5+17.4}{2}\right)(5)+\left(\frac{17.4+11.6}{2}\right)(5)+\left(\frac{11.6+0}{2}\right)(5)=359\text{ft}^2$

13b) $V=\int_0^{30}w(x)\cdot h(x)\ dx\approx(3\cdot 14.6)(10)+(6\cdot 14.5)(10)+(8\cdot 11.6)(10)=2236\text{ft}^3$

13c) $w'(15)\approx\frac{17.4-13.5}{20-10}=.39$. Tangent Line: $w-15.5=.39(x-15)$

13d) $x=2t$ $f(2t)=-.0001565(2t+1)^4+0.007969(2t+1)^3-0.12954(2t+1)^2+1.06724(2t+1)-0.931776$
$f'(2t)=-.000626(2t+1)^3(2)+0.023907(2t+1)^2(2)-0.25908(2t+1)(2)+1.06724(2)$
$f'(2(8))=0.9929\,\text{ft}/\text{sec}$

15a) Disk Method with horizontal rectangles: $y=\frac{1}{8}x^5\rightarrow x=(8y)^{1/5}$

$$V=\pi\int_0^4\left[(8y)^{1/5}\right]^2dy=\pi\int_0^4\left[(8y)^{2/5}\right]dy=\frac{\pi}{8}\int_0^4(8y)^{2/5}8\ dy=\frac{\pi}{8}\left[\frac{5}{7}(8y)^{7/5}\right]_0^4=\frac{20\pi}{7}\text{ft}^3$$

15b) $V=\pi\int_0^h(8y)^{2/5}dy=\frac{\pi}{8}\left[\frac{5}{7}(8y)^{7/5}\right]_0^h=\frac{5\pi}{56}(8h)^{7/5}$

15c) $\frac{d}{dt}\left[V=\frac{5\pi}{56}(8h)^{7/5}\right]\rightarrow\frac{dV}{dt}=\frac{5\pi}{56}\left[\frac{7}{5}(8h)^{2/5}(8)\right]\frac{dh}{dt}=\pi(8h)^{2/5}\frac{dh}{dt}$

$$\left.\frac{dV}{dt}\right|_{h=3}=\pi(8(3))^{2/5}\left(-\frac{1}{5}\right)=-7.800\,\text{ft}^2/\text{hr}$$

15d) $A=\pi r^2$ and $r=x=(8y)^{1/5}$, so $\frac{dV}{dt}=\pi(8h)^{2/5}\frac{dh}{dt}=\pi\left[(8h)^{1/5}\right]^2\frac{dh}{dt}=A\frac{dh}{dt}$

17a. $V=\pi\int_0^4\left(5x^{1/2}\right)^2dx=25\pi\int_0^4x\ dx=25\pi\left[\frac{x^2}{2}\right]_0^4=200\pi=628.319in^3$

b) $y=5\sqrt{x}\rightarrow\frac{dy}{dx}=\frac{5}{2}x^{-1/2}$. $L=\int_{-4}^4\sqrt{1+\left(\frac{5}{2}x^{-1/2}\right)^2}dx=21.965in$

c) $\tan\theta = \dfrac{y}{4-x}$

d) $\dfrac{d}{dt}\left[\tan\theta = \dfrac{y}{4-x}\right] \to \sec^2\theta\dfrac{d\theta}{dt} = \dfrac{(4-x)\frac{dy}{dt} - y\left(-\frac{dx}{dt}\right)}{(4-x)^2}$

$\theta = \tan^{-1}\dfrac{5}{3} = 1.030$

$\left(\sec^2 1.030\right)\dfrac{d\theta}{dt} = \dfrac{3(-.1)-5(-.03)}{3^2} \to \dfrac{d\theta}{dt} = -0.013\,{}^{rad}\!/_{\min}$

19a) $V = \pi\displaystyle\int_{-18}^{18}\left(\frac{1}{108}x^2 - 14\right)^2 dx = \pi\int_{-18}^{18}\left(\frac{1}{11664}x^4 - \frac{7}{27}x^2 - 14\right)\,dy$

$V = \pi\left[\dfrac{1}{11664}\dfrac{x^5}{5} - \dfrac{7}{27}\dfrac{x^3}{3} - 14x\right]_{-18}^{18} = 19{,}203.928\text{in}^3$

19b) $A = 2\pi(11)^2 + 2\pi\displaystyle\int_{-18}^{18}\left(\frac{1}{108}x^2 - 14\right)\sqrt{1+\left[\frac{1}{54}x\right]^2}\,dx = 760.265 + 2990.835 = 3751.100\ \text{in}^2$

19c) $V_{cyl} = \pi(12.5)^2 h \to \dfrac{d}{dt}\left[V_{cyl} = \pi(12.5)^2 h\right] \to \dfrac{dV}{dt} = (12.5)^2\pi\dfrac{dh}{dt}$

$-4800 = (12.5)^2\pi\dfrac{dh}{dt} \to \dfrac{dh}{dt} = -9.778\ {}^{in}\!/_{hr}$

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21. See AP Central

6.8 Free Response Solutions

1. $y = \sec x$, $y = 1$, $x = 0$, $x = \dfrac{\pi}{6}$; about the y-axis.

$V = \displaystyle\int_0^{\pi/6} 2\pi r l\,dx = \int_0^{\pi/6} 2\pi\cdot x\cdot(\sec x - 1)\,dx = .064$

3. $y = \dfrac{1}{x^{2/3}}$, $x = 1$, $x = 8$, $y = 0$; about the y-axis.

$V = \displaystyle\int_1^8 2\pi r l\,dx = \int_1^8 2\pi\cdot x\cdot\left(x^{-2/3}\right)dx = 70.689$

5. $y = x(x-1)^2$ and the $x-$axis; about the $y-$axis.

$$V = \int_0^1 2\pi r l dx = \int_0^1 2\pi \cdot x \cdot y dx = \int_0^1 2\pi x(x-1)^2 dx = .209$$

7. $y = \sqrt{x}$, $y = e^{-2x}$, $x = 1$;about the line $x = 1$.

$$V = \int_{.301}^{1} 2\pi r l dx = \int_{.301}^{1} 2\pi(1-x)\left(\sqrt{x} - e^{-2x}\right)dx = .554$$

9. $y = x^2$, $y = 2^x$; about the line $x = -1$.

$$V = \int_{-.767}^{2} 2\pi r l dx + \int_{2}^{4} 2\pi r l dx$$

$$= \int_{-.767}^{2} 2\pi(1+x)\left(2^x - x^2\right)dx + \int_{2}^{4} 2\pi(1+x)\left(x^2 - 2^x\right)dx$$

$$= 55.428$$

Volume Practice Test

1. $y = x - 4$ is the top curve and $y = x^2 - 4$ is the bottom. The curves intersect at $x = 0$ *and* 1.

The correct answer is A

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3. $V = \pi \int_0^{\ln 3} \left[\left(e^{x/2}\right)^2 - 1^2\right]dx = 2.83$

The correct answer is B

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5. About the *y*-axis means the *x* is isolated: $x = \sin y$ and $y \in \left[0, \frac{\pi}{2}\right]$.

The correct answer is D

7a. $area = \int_0^6 \left(x\sqrt{12 + 4x - x^2} - \left(-3\sin\left(\frac{\pi}{3}x\right)\right)\right) = 54.295$

7b) Washer Method: $V = \pi \int_0^6 \left(\left(x\sqrt{12 + 4x - x^2} - (-4)\right)^2 - \left(-3\sin\left(\frac{\pi}{3}x\right) - (-4)\right)^2\right) = 1012.159$

c) $V = \int_0^6 \left(\left(x\sqrt{12+4x-x^2} - \left(-3\sin\left(\frac{\pi}{3}x\right)\right)\right)^2\right) = 713.902$

9a. $area = \int_0^2 \left(\frac{8}{x^2+4}\right)dx = 8\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = 4\left(\tan^{-1}1 - \tan^{-1}0\right) = 4\left(\frac{\pi}{4}\right) = \pi$

9b. Disk Method: $$V = \pi \int_0^2 \left(\frac{8}{x^2+4}\right)^2 dx$$

9c. Disk Method with horizontal rectangles:

$$y = \frac{8}{x^2+4} \to x^2+4 = \frac{8}{y} \to x^2 = \frac{8}{y} - 4 \to x = \sqrt{\frac{8}{y} - 4}$$

$$V = \pi \int_0^2 \left(\sqrt{\frac{8}{y} - 4}\right)^2 dx$$