AP Calculus BC '15-16

Fall Final
Part IIa

Calculator Required

Name:

Solution Key

1. At an electric substation, readings of the rate at which power is being used were recorded every three hours over a 24-hour period. The readings are listed in the table below. The rate is in kilowatts per hour.

t	0	3	6	9	12	15	18	21	24
P(t)	1245	1268	1321	1316	1393	1369	1369	1451	1428

2

a. Using midpoint Rheiman rectangles with four equal subintervals, approximate $\int_0^{24} P(t)\,dt$.

$= 6(1268) + 6(1316) + 6(1369) + 6(1451)$

$= 32{,}424$

2

b. Using the correct units, explain the meaning of $\int_0^{24} P(t)\,dt$ in terms of power usage.

$\int_0^{24} P(t)\,dt =$ THE TOTAL AMOUNT OF POWER IN KILOWATTS USED IN THESE 24 HOURS.

2

c. Estimate how fast the rate of change of the power usage is increasing at t = 12. Show the computations and indicate units of measure.

$$P'(12) \approx \frac{1369 - 1316}{15 - 9} = \frac{53}{6} \; \frac{KW/HR}{HR}$$

3

d) Assume the function $F(t) = 1245 + 10te^{.25\cos t}$ is an accurate model of the rate of power usage at time t, where $F(t)$ is measured in kilowatts per hour and t is measured in hours. Use $F(t)$ to approximate the average rate of power usage during the 24-hour period. Indicate units of measure.

$$\frac{1}{24}\int_0^{24} F(t)\,dt$$

$$= \cancel{1246.279} \; \frac{KW}{HR}$$

$$1364.477$$

2. A 12,000-liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where r is given by the piecewise-defined function

$$r(t)=\begin{cases} \dfrac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-.2t} & \text{for } 5 < t \end{cases}$$

a. Is $r(t)$ continuous at $t = 5$? Show the work that leads to your answer.

$$\frac{600(5)}{5+3} = 375$$

$$1000e^{-.2(5)} = 367.879$$

$$\lim_{t \to 5^-} f(t) \neq \lim_{t \to 5^+} f(t)$$

$\therefore$ IT IS <u>NOT</u> CONTINUOUS

b. Find the average rate at which water is draining from the tank between time $t = 0$ and time $t = 8$ hours.

$$\frac{1}{8}\int_0^8 r(t)\,dt$$

$$= \frac{1}{8}\left[\int_0^5 \frac{600t}{t+3}\,dt + \int_5^8 1000e^{-.2t}\,dt\right]$$

$$= 258.052$$

2

c. Find $r'(3)$. Using correct units, explain the meaning of that value in the context of this problem

$$\frac{d}{dt}\left(\frac{600t}{t+3}\right)\Bigg|_{t=3} = 50 \ \frac{\text{LT}}{\text{HR}^2}$$

~~T'(3)~~ THE RATE AT WHICH THE TANK IS EMPTYING IS INCREASING AT 50 LT/HR/HR

3

d. Write, but do not solve, an equation involving an integral to find the time A when the amount of water in the tank is 9000 liters.

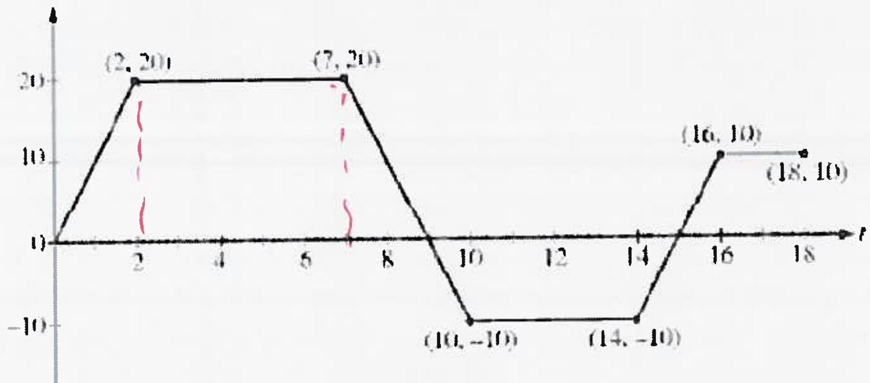
$$\int_0^A r(t)\,dt = 3000$$

OR $$3000 = \int_0^5 \frac{600t}{t+3}\,dt + \int_5^A 1000e^{-.2t}\,dt$$

$$3000 = 1234.507 + \int_5^A 1000^{-.2t}\,dt$$

$$\int_5^A 1000e^{-.2t} = 1765.493$$

3. A squirrel starts at building A at time $t = 0$ and travels along a straight wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph below.



Graph of v

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a. At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

2

$t=9$ AND 15 BECAUSE THE VELOCITY SWITCHES SIGNS

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b. At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A? How far from building A is the squirrel at this time?

2

$t=9$ BECAUSE ~~MAX~~ ~~VEL~~ MAX DISTANCE IS AT $t= 9$ & 18

AND 9 WAS A GREATER INTEGRAL TOTAL

$d(9) = 140$

c. **Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.**

TOTAL DISTANCE $= 140 - (-50) + 25$

2

$= 215$

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d. **Write expressions for the squirrel's acceleration a(t), velocity v(t), and distance x(t) from building A that are valid for the time interval $7 < t < 10$.**

3

$v(t) \Rightarrow$ $M = -10$: $v - 20 = -10(t - 7)$

$v(t) = -10t + 90$

$a(t) = -10$

$x(t) = 120 + \int_7^t (-10x + 90)\, dx$

$= 120 + \left[-5x^2 + 90x\right]_7^t$

$= -5t^2 + 90t + 575$

End of

AP Calculus BC '15-16

Fall Final -- Part IIa

AP Calculus BC '15-16

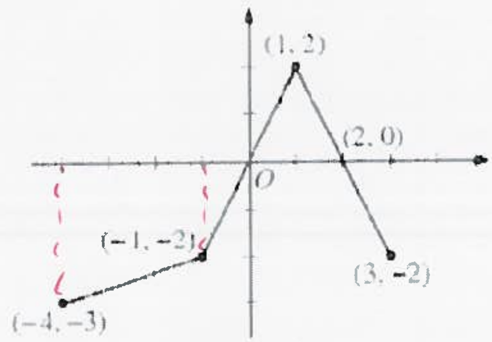
Fall Final
Part IIb

No Calculator Allowed

Name:

Solution Key

4. The graph of f below consists of three line segments:



Graph of f

Let $g(x) = \int_{-4}^{x} f(t)\,dt$.

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3 a. Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

$g(-1) = \cancel{-6.67}\ -7.5$

$g'(-1) = -2$

$g''(-1) = \text{DNE}$

½ b. Find the x-coordinate for each point of inflection of $g(x)$ on $-4 < x < 3$. Explain your reasoning.

2

$x = 1$ BECAUSE $g' = f$ AND f SWITCHES

FROM INCREASING TO DECREASING

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c. Let $h(x)=\int_x^3 f(t)\,dt$. Find all values of $x \in [-4, 3]$ for which $h(x)=0$.

$h(x)=0$ @ $x=-1, 1, 3$

$x=3 \rightarrow \int_3^3 = 0$

$x=1$ & -1 AREAS ADD TO 0

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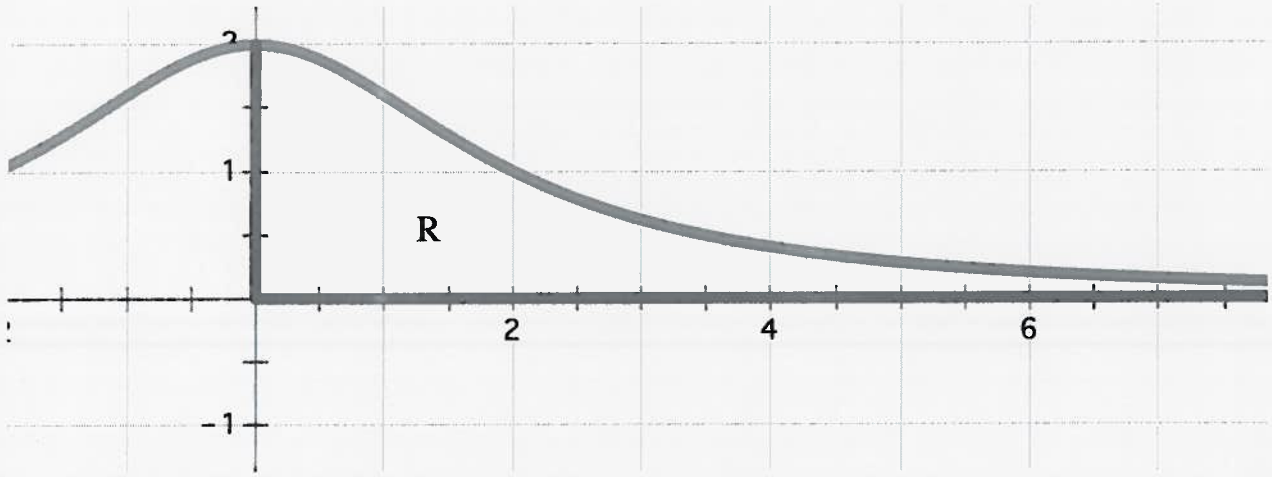
d. Find all intervals for which $h(x)$ is decreasing. Explain your reasoning.

$h' = -f$ $\therefore$ h' IS DECREASING WHEN $f > 0$

$x \in [0, 2]$



5. The picture above is the graph of $f(x)=\dfrac{8}{x^2+4}$.

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a. Find the area of the unbounded region in Quadrant I between $f(x)=\dfrac{8}{x^2+4}$ and the x-axis.

4

$$A = \int_0^{\infty} \frac{8}{x^2+4}\,dx$$

$$= \lim_{b\to\infty} 8\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^b$$

$$= \lim_{b\to\infty} 4\tan^{-1}\frac{b}{2} - \cancel{\lim}\ 4\tan^{-1} 0$$

$$= 4\left(\frac{\pi}{2}\right) - 0 = 2\pi$$

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b. The formula to find the volume of a solid formed by rotating a region about the x-axis is $V = \pi\int_a^b [f(x)]^2 dx$. Set up, but do not solve, the formula for the volume of the solid formed by revolving region R above about the x-axis.

(1) $V = \pi \int_0^{\infty} \left(\frac{8}{x^2+4}\right)^2 dx$

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c. Show that $\sum_{n=0}^{\infty} a_n$ where $a_n = f(n)$ converges. Will the volume in part b converge? Why or why not?

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{8}{n^2+4}$ BY THE INTEGRAL TEST AND a) MEANS $\sum_{n=0}^{\infty} a_n$ CONVERGES

$V \approx \sum_{n=0}^{\infty} \frac{64}{(n^2+4)^2}$ $\frac{64}{(n^2+4)^2} < \frac{64}{n^2+4} = 8\left(\frac{8}{n^2+4}\right)$

By DCT $\sum_{n=0}^{\infty} \frac{64}{(n^2+4)^2}$ CONVERGES

By THE INTEGRAL TEST, SINCE THE SERIES CONVERGES, SO DOES THE INTEGRAL THAT REPRESENTS THE VOLUME

Question 2

A 12,000-liter tank of water is filled to capacity. At time $t = 0$, water begins to drain out of the tank at a rate modeled by $r(t)$, measured in liters per hour, where r is given by the piecewise-defined function

$$r(t) = \begin{cases} \dfrac{600t}{t+3} & \text{for } 0 \le t \le 5 \\ 1000e^{-0.2t} & \text{for } t > 5 \end{cases}$$

(a) Is r continuous at $t = 5$? Show the work that leads to your answer.

(b) Find the average rate at which water is draining from the tank between time $t = 0$ and time $t = 8$ hours.

(c) Find $r'(3)$. Using correct units, explain the meaning of that value in the context of this problem.

(d) Write, but do not solve, an equation involving an integral to find the time A when the amount of water in the tank is 9000 liters.

(a) $\lim\limits_{t \to 5^-} r(t) = \lim\limits_{t \to 5^-} \left(\dfrac{600t}{t+3}\right) = 375 = r(5)$

$\lim\limits_{t \to 5^+} r(t) = \lim\limits_{t \to 5^+} \left(1000e^{-0.2t}\right) = 367.879$

Because the left-hand and right-hand limits are not equal, r is not continuous at $t = 5$.

2 : conclusion with analysis

(b) $\dfrac{1}{8}\displaystyle\int_0^8 r(t)\,dt = \dfrac{1}{8}\left(\int_0^5 \dfrac{600t}{t+3}\,dt + \int_5^8 1000e^{-0.2t}\,dt\right)$

$= 258.052$ or 258.053

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$

(c) $r'(3) = 50$

The rate at which water is draining out of the tank at time $t = 3$ hours is increasing at 50 liters/hour^2.

$2 : \begin{cases} 1 : r'(3) \\ 1 : \text{meaning of } r'(3) \end{cases}$

(d) $12{,}000 - \displaystyle\int_0^A r(t)\,dt = 9000$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{equation} \end{cases}$

6. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B)$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

2

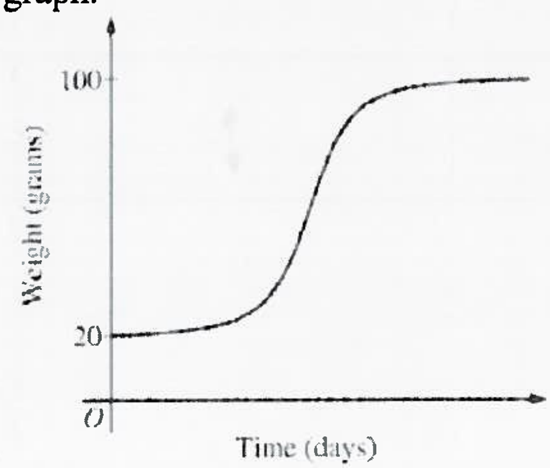
(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

$$\left.\frac{dB}{dt}\right|_{B=40} = 12 > \left.\frac{dB}{dt}\right|_{B=70} = 6$$

FASTER AT $B=40$

2

(b) Find $\frac{d^2B}{dt^2}$ in terms of B and use it to explain why the graph of B cannot resemble the following graph.



$$\frac{d}{dt}\left[\frac{dB}{dt} = 20 - \frac{1}{5}B\right]$$

$$\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt}$$

$$= -\frac{1}{5}\left(20 - \frac{1}{5}B\right)$$

$$= -4 + \frac{1}{25}B$$

$\frac{d^2B}{dt^2}$ IS ALWAYS < 0 ∴ THE CURVE SHOULD ALWAYS BE CONCAVE DOWN.

5

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

$$\frac{1}{100-B}\,dB = \frac{1}{5}\,dt$$

$$-\ln|100-B| = \frac{1}{5}t + C$$

$$\ln|100-B| = -\frac{1}{5}t + C$$

$$100 - B = e^{-\frac{1}{5}t + C} = Ke^{-1/5\,t}$$

$$(0,20) \Rightarrow \quad 80 = Ke^{0} \rightarrow K = 80$$

$$100 - B = 80e^{-\frac{4}{5}t}$$

$$B = 100 - 80e^{-\frac{1}{5}t}$$

End of

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