AP Calculus BC '16-17
Taylor Series Test

Name Solution Key

Directions: Show all work.

Score___________.

1. What is the interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}(x-3)^n$?

(A) $(-1,1)$ (B) $[-1,1)$ (C) $(-1,1]$ (D) $(2,4]$ (E) $[2,4)$

$x=2 \to \sum \frac{1}{n}$ $x=4 = \frac{(-1)^n}{n}$

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2. If the Taylor series for *g(x)* about $x=3$ is $g(x)=\sum_{n=0}^{\infty} \frac{x^n}{2n}$, then $g^{IV}(0)$ is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

$n=4 \to \frac{1}{8}x^n$

$\frac{1}{8} = \frac{g^{V}(0)}{4!}$

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3. The exact value of the series $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$ is $= \sin \pi$

(A) 0 (B) -1 (C) e^{π} (D) $\ln \pi$ (E) undefined

4. Which of the following Series diverge?

$r > 1$

I. $\sum_{n=1}^{\infty} 5^{-n}6^{n-1}$ $= \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{6}{5}\right)^n$ II. $\sum_{n=1}^{\infty} \frac{3n^5}{7n^4-1}$ DIV TEST III. $\sum_{n=1}^{\infty} \frac{1}{5^n}$ $r = 1/5 < 1$ CONV

(A) I only (B) III only (C) I and II only

(D) II and III only (E) I, II, and III

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5. What are the first three non-zero terms of the power series for xe^{-x}?

(A) $x - x^2 - \frac{x^3}{2}$ (B) $x - x^2 + \frac{x^3}{2}$ (C) $-x + x^2 - \frac{x^3}{2}$

(D) $x + x^2 + \frac{x^3}{2}$ (E) $1 - x - \frac{x^2}{2}$

$e^x = 1 + x + \frac{x^2}{2}$

$e^{-x} = 1 - x + \frac{x^2}{2}$

$xe^{-x} = x - x^2 + \frac{x^3}{2}$

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6. The Taylor series for $f(x)$ is $1 + x + x^2 + \cdots + x^n + \cdots$, and the Taylor series for $g(x)$ is $1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$. Find the 3rd degree Taylor polynomial for the product $f(x) \cdot g(x)$.

(A) $1 - x^3$ (B) $1 + x$ (C) $1 + x + x^2 + x^3$ (D) $1 + x + x^3$ (E) $1 + x - x^2$

$1 + x + x^2 + x^3$

$1 - x^2$

$-\cancel{x^2} - \cancel{x^3} - x^4$

$1 + x + \cancel{x^2} + \cancel{x^3}$

$= x(-1)^n x^{3n} = x(x^3)^n = x\left(\frac{1}{1+(x^3)}\right) = \frac{x}{1+x^3}$

7. $\sum_{n=1}^{\infty}(-1)^n x^{3n+1}$ is the Maclaurin series for which function?

(A) $\frac{1}{1+x^{3x+1}}$ (B) xe^{x^3} (C) $\frac{x}{1+x^3}$ (D) $x!e^{3x+1}$ (E) $3x+1$

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8. Which of these is the Maclaurin series for $\cos\left(\sqrt{x}\right)$? $= \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!}$

(A) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$ (B) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$ (C) $\sum_{n=0}^{\infty}\frac{x^n}{2n!}$

(D) $\sum_{n=0}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$ (E) $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!}$

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9. The radius of convergence for $\sum_{n=0}^{\infty}\frac{2^n}{3^n n^2}x^n$ is

(A) $2/3$ (B) $3/2$ (C) 3 (D) 0 (E) ∞

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}(n+1)^2}}{\frac{2^n}{3^n n^2}}|x| = \lim_{n\to\infty} \frac{2n^2}{3(n+1)^2}|x|$$

$$\frac{2}{3}|x| < 1$$

$$|x| < 3/2$$

1. Let $f(x) = \sin(x^2) + \cos x$

a. Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

4

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

b. Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$.

2

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

c. Use the series found in part (a) and part (b) to write the first four nonzero terms of the Taylor series for $f(x)$ about $x = 0$.

$$f(x) = 1 - \frac{x^2}{2} + x^2 + \frac{x^4}{4!} - \frac{x^6}{6!} - \frac{x^6}{3!}$$

2

$$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{121}{6!}x^6$$

d. Find the value of $f^{VI}(0)$.

$$\frac{f^{VI}(0)}{6!} = \frac{-121}{6!}$$

1

$$f^6(0) = -121$$

2. The Maclaurin series for the function $f(x)$ is given by $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1) x^{3n}}{n!}$ on its interval of convergence. The first four terms of the Maclaurin series for the function $f(x)$ is $M_9(f,\,0) = 1 - 2x^3 + \frac{3}{2!}x^6 - \frac{4}{3!}x^9$.

a. Find the interval of convergence for the Maclaurin series of $f(x)$. Justify your answer.

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+2)\, x^{3(n+1)}}{(n+1)!} \cdot \frac{n!}{(-1)^n (n+1) x^{3n}} \right| < 1$$

4

$$\lim_{n \to \infty} \frac{n+2}{(n+1)(n+1)}(x^3) < 1$$

$$0 < 1$$

INTERVAL IS $-\infty < x < \infty$

b. Find the approximate values of $f(.5)$. What is the maximum difference between the exact value of $f(.5)$ and the approximate value of $f(.5)$ found in part.

1 $f(.5) \approx 1 - 2(.5)^3 + \frac{3}{2}(.5)^6 - \frac{4}{6}(.5)^9 = .772$

4 ~~ERROR BOUND~~ $= \left| \frac{(-1)^4 (4+1)}{4!} (.5)^{3(4)} \right| = .00005086$

c. What is the maximum error between the exact value of $f(.5)$ and the approximate value of $f(.5)$ found in part b.

1 $\quad \left|\frac{(-1)^4(4+1)}{4!}(.5)^{3(4)}\right| = \frac{5}{24}\left(\frac{1}{2}\right)^{12} = .00005086$

d. Let $h(x) = x^2 f'(x)$. Find the first three non-zero terms and the general terms of $h(x)$.

$$f = 1 - 2x^3 + \frac{3}{2}x^6 - \frac{4}{6}x^9 + \frac{5}{24}x^{12}$$

$$f'(x) = -6x^2 + 9x^5 - 6x^8 + \frac{5}{2}x^{11}$$

2 $\quad x^2 f'(x) = -6x^4 + 9x^7 - 6x^{10} + \cancel{\frac{5}{2}x^{13}}$

$$f' \to \frac{(-1)^n (n+1)\, 3n\, x^{3n-1}}{n!} = \frac{(-1)^n (n+1)\, 3x^{3n-1}}{(n-1)!}$$

1

$$h = x^2 f' = \frac{(-1)^n (n+1)\, 3x^{3n+1}}{(n-1)!}$$