AP Calculus BC '17-18

Fall Final
Part IIa

Calculator Required

Name:
Solution Key

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

1. A metal wire of 8 cm is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius, of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

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a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate the units.

1

$$\frac{T(8)-T(6)}{8-6} = -\frac{7}{2}\ {}^\circ C/cm$$

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b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature using a right-hand Riemann sum with four subintervals indicated by the data on the table. Indicate units of measure.

3

$$\frac{1}{8}\int_0^8 T(x)\,dx$$

$$= \frac{1}{8}\left[93(1) + 70(4) + 62(1) + 55(2)\right]$$

$$= \frac{545}{8}\ {}^\circ C$$

---

c) Find $\int_0^8 T'(x)dx$ and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)dx$ in terms of the temperature of the wire.

$T(8) - T(0) = 55 - 100 = -45°C$

2 THE TEMPERATURE DROPS 45°C FROM THE HEATED END OF THE WIRE TO THE OTHER END OF THE WIRE

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d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your reasoning.

2

AVERAGE RATE OF CHANGE ON $x \in [1, 5] = -5.75$

" " " " " $x \in [5, 6) = -8.$

BY MVT $T'(c) = -5.75$ ON $[1, 5]$ & $T'(c) = -8$ ON $[5, 6)$

SO T' MUST DECREASE SOMEWHERE ON $(1, 6)$

AND $T'' \leq 0$

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1 UNITS

2. A water tank holds 150 gallons at time $t = 0$. During the time interval $0 \leq t \leq 12$, water is pumped into the tank at a rate of

$$w(t) = 4e^{\sin\left(\frac{\pi}{12}t\right)} \text{ gallons per hour.}$$

At time $t = \cancel{12}\ 8$, as second pump begins removing water at a rate of

$$R(t) = \frac{13t}{1+2t} \text{ gallons per hour.}$$

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a) How many gallons enter the tank between $t = 0$ and $t = 8$?

1

$$\int_0^8 4e^{\sin\frac{\pi}{12}t}\,dt = 68.236 \text{ GALONS}$$

---

b) At what time between $t = 0$ and $t = 8$ is the amount of water increasing most rapidly?

3

$$w'(t) = \cancel{0} = 4e^{\sin\frac{\pi}{12}t}\left(\cos\frac{\pi}{12}t\right)\frac{\pi}{12} = 0$$

$$\cos\frac{\pi}{12}t = 0$$

$$\frac{\pi}{12}t = \pm\frac{\pi}{2} \pm 2\pi n$$

$$t = \pm 6 \pm 24n$$

w'	+	0	−
t	0	6	8
		MAX	

---

c) What is the total amount of water in the tank at $t = 12$ hours?

$$= 150 + \int_0^{12} w(t)\,dt - \int_8^{12} R(t)\,dt$$

$$= 220.116 \text{ GALLONS}$$

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d) Is the amount of water increasing or decreasing at time $t = 12$? Justify your answer.

$$\frac{d}{dx}\left[150 + \int_0^{12} w(t)\,dt - \int_8^{12} R(t)\,dt\right]$$

$$= w(12) - [R(12) - R(8)]$$

$$= 2.309 \text{ GAL/HR}$$

THE AMOUNT OF WATER IS INCREASING BECAUSE THE RATE OF CHANGE IS POSITIVE

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AP Calculus BC '17-18

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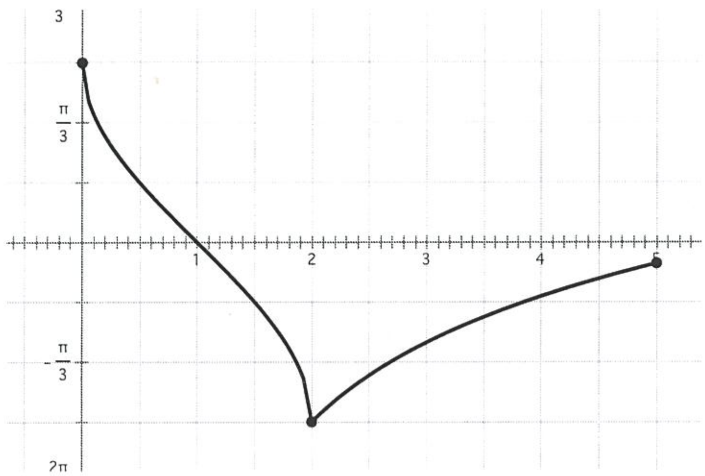
Part IIb

No Calculator Allowed

Name:

Solution Key

3. The function f is defined as $f(x)=\begin{cases}\sin^{-1}(1-x), \text{ if } 0\le x<2\\ -\dfrac{\pi}{2}, \quad \text{if } x=2\\ -\dfrac{\pi}{2}+\ln(x-1), \text{ if } 2<x\le 5\end{cases}$. The graph is below.



3

a) Prove $f(x)$ continuous at $x=2$.

i) $f(2)$ EXISTS

ii) $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2} \sin^{-1}(1-x) = \sin^{-1}(-1) = \frac{-\pi}{2}$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2} \left[-\frac{\pi}{2} + \ln(x-1)\right] = -\frac{\pi}{2} + 0 = -\frac{\pi}{2}$

$\lim\limits_{x\to 2} f(x)$ EXISTS BECAUSE $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

iii) $\lim\limits_{x\to 2} f(x) = \frac{-\pi}{2} = f(2)$

∴ CONTINUOUS

---

b) Prove $f(x)$ not differentiable at $x=2$.

$$f'(x) = \begin{cases} \dfrac{1}{\sqrt{1-(1-x)^2}} & \text{IF } 0<x<2 \\ \dfrac{1}{x-1} & \text{IF } 2<x<5 \end{cases}$$

3

i) $f(x)$ IS CONTINUOUS

ii) $\lim_{x \to 2^-} f(x) = \frac{1}{0} = \text{DNE} \neq 1 = \lim_{x \to 2^+} f(x)$

$\therefore$ $f(x)$ IS NOT DIFFERENTIABLE

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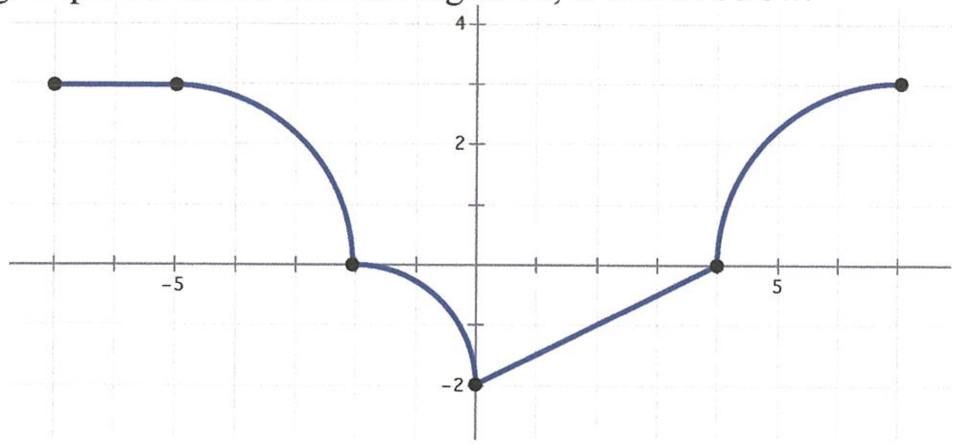
c) Set up, but **do not solve**, an expression for the arc length of $f(x)$ on $x \in [0, 5]$

3

$$L = \int_0^2 \sqrt{1 + \left(\frac{1}{\sqrt{1-(1-x)^2}}\right)^2}\, dx + \int_2^3 \sqrt{1+\left(\frac{1}{x-1}\right)^2}\, dx$$

---

4. Let $g(x)=\int_{-2}^{x} f(t)\,dt$ for $-7 \le t \le 7$, where the graph of the function f, consisting of quarter-circles and line segments, is shown below.



a) Find $g(4)$, $g'(4)$, and $g''(4)$.

$g(4) = \int_{-2}^{4} f(x)\,dx = -(4-\pi) + (-4) = \pi - 8$

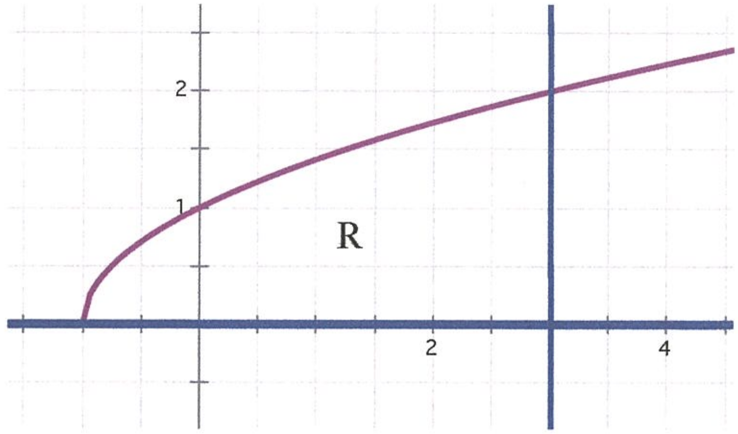
3 $g'(4) = f(4) = 0$

$g''(4) = f'(4) = \text{DNE}$

b) Find the **average rate of change** of $g(x)$ on $-2 \le t \le 4$?

1 $\dfrac{g(4) - g(-2)}{4-(-2)} = \dfrac{(\pi-8)-0}{4+2} = \dfrac{\pi-8}{6}$

5. Let R be the region bounded by the x-axis, $y=\sqrt{x+1}$, and the line $x=3$ as shown below.



a) Find the area of region R.

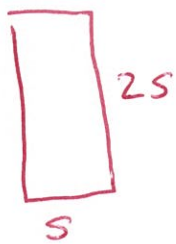
$$A = \int_{-1}^{3} \sqrt{x+1}\, dx$$

3

$$= \left. \frac{(x+1)^{3/2}}{3/2} \right]_{-1}^{3}$$

$$= \frac{2}{3}\left[4^{3/2} - 0\right] = \frac{16}{3}$$

b) Let the base of a solid be the region R. If the cross-sections of the solid perpendicular to the x-axis are rectangles that are twice as tall as they are wide, find the volume of the solid.



3

$$V = \int_{-1}^{3} 2s^2\,dx$$

$$= \int_{-1}^{3} 2\left(\sqrt{x+1}\right)^2 dx$$

$$= 2\int_{-1}^{3} (x+1)\,dx$$

$$= 2\left[\frac{x^2}{2} + x\right]_{-1}^{3} = \cancel{2}\; x^2 + 2x\Big]_{-1}^{3}$$

$$= (9+6) - (1-2)$$

$$= 16$$

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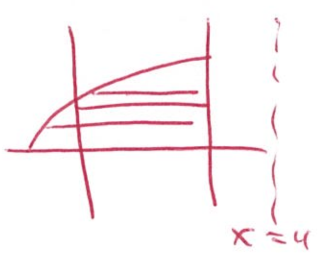
c) Set up, but **do not solve**, an expression for the volume of a solid formed if Region R is revolved about the line $x = 4$.



3

$y = \sqrt{x+1} \Rightarrow x = y^2 - 1$

$r = 1$

$$V = \pi \int_0^2 \cancel{[\ \ ]}\ [4-(y^2-1)]^2 - (1)^2\,dy$$

6. Consider the equation $\frac{dy}{dx} = \frac{xy}{x^2+4}$.

Let $y = f(x)$ be the solution to the differential equation above with initial condition $y(0) = 4$.

---

a) Find the equation of the line tangent to $y = f(x)$ at $\left(2, 4\sqrt{2}\right)$.

2

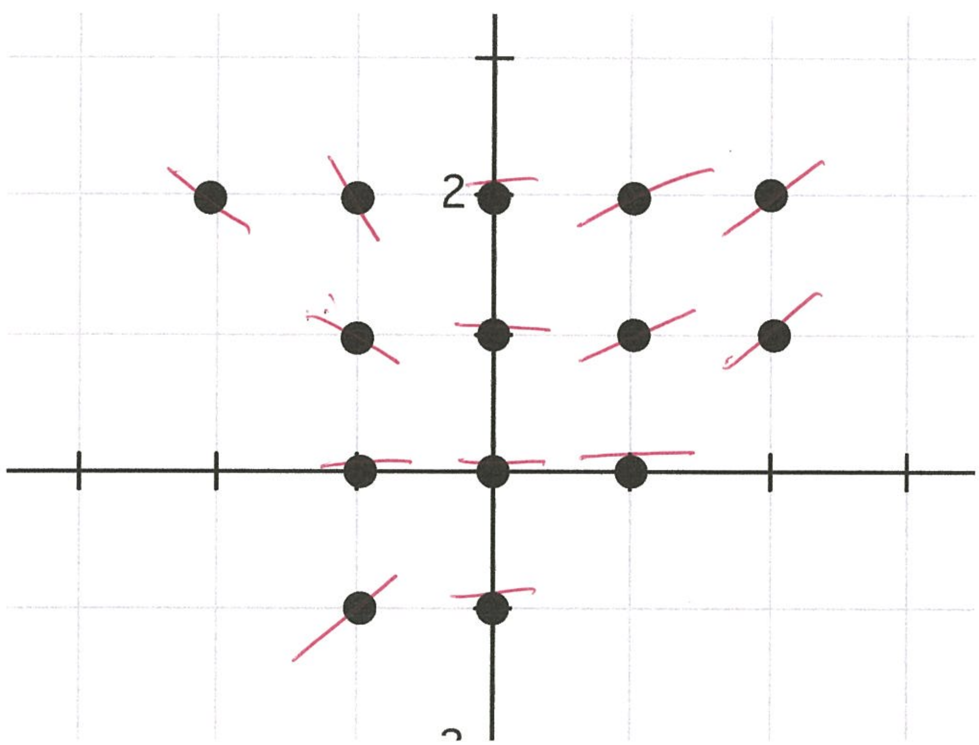
$$m = \frac{2(4\sqrt{2})}{4+4} = \sqrt{2}$$

$$y - 4\sqrt{2} = \sqrt{2}(x-2$$

---

b) On the axes provided, sketch a slope field for the given differential equation at the points indicated.

2



c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = \frac{xy}{x^2+4}$ with initial condition $y(0) = 4$.

5

$$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+4}\,dx$$

$$\Rightarrow \ln|y| = \frac{1}{2}\ln(x^2+4) + C$$

$$= \ln(x^2+4)^{1/2} + C$$

$$y = e^{\ln(x^2+4)^{1/2} + C}$$

$$y = K(x^2+4)^{1/2}$$

$$(0,4) \Rightarrow 4 = K(4^{1/2}) = 2K$$

$$2 = K$$

$$y = 2(x^2+4)^{1/2}$$

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