AP Calculus BC '17-18
Taylor Series Test

Name Solution Key

Directions: Show all work.

Score__________.

1. Which of these is the Maclaurin series expansion for the function $h(x)=\frac{\sin(x^3)}{x}$?

$\sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} \Rightarrow \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

a) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n-1}}{(2n)!}$ b) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{(2n+1)!}$ c) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+2}}{(2n+1)!}$

d) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ e) $\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}$

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2. If $\sum_{n=0}^{\infty}\frac{(-2)^n x^{2n+1}}{(2n+1)!}$ is the series expansion for $f(x)$, which of these is the series for $f'(x)$?

a) $\sum_{n=0}^{\infty}\frac{(-2)^n x^{2n}}{(2n)!}$ b) $\sum_{n=0}^{\infty}\frac{(-2)^{n-1} x^{2n}}{(2n)!}$ c) $\sum_{n=0}^{\infty}\frac{(-2)^n x^{2n}}{(2n+1)!}$

(d) $\sum_{n=0}^{\infty}\frac{(-2)^n (2n+1)x^{2n}}{(2n)!}$ e) $\sum_{n=0}^{\infty}\frac{(-2)^{n-1} x^{2n+2}}{(2n+2)!}$

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3. . Which of these series converge when $x = -1$?

I. $\sum_{n=1}^{\infty} \frac{3x^n}{\sqrt[4]{n}}$ ~~II.~~ $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ III. $\sum_{n=1}^{\infty} \frac{(1-x)^n}{3^n}$

a) I and II only b) II and III only

c) I and III only d) III only e) I, II, and III

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4. The interval of convergence for the series $\sum_{n=0}^{\infty} \frac{n}{4^n}(x-2)^n$ is

a) $x \in [-4,4)$ b) $x \in (-4,4)$ c) $x \in [-2,6)$

d) $x \in (-2,6)$ e) $x \in (-\infty,\infty)$

$x = -2 \rightarrow \sum (-1)^n n \rightarrow \text{DIV}$

$x = 6 \rightarrow \sum n \rightarrow \text{DIV}$

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5. $\sum_{n=1}^{\infty} \frac{(-2)^n}{e^{n+1}} = \; = \frac{1}{e}\left(\frac{-2}{e}\right)^n$

a) $\frac{-2}{e^2 - 2e}$ b) $\frac{-2}{e^2 + 2e}$ c) $\frac{-2}{e+2}$ d) $\frac{e}{e+2}$ e) DNE

$S = \frac{-2/e^2}{1-(-2/e)} = \frac{-2}{e^2} \cdot \frac{e}{e+2}$

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6. $1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\cdots+\frac{(-1)^n\pi^{2n}}{(2n)!}+\cdots=$ $\cos \pi$

a) 1 b) 0 c) -1.2 d) -1 e) π

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7. Let $P(x)=3-3x^2+6x^4$ be the fourth degree Taylor polynomial for the function f about $x=0$. Then $f^{IV}(0)=$

a) 0 b) $\frac{1}{4}$ c) 6 d) 24 e) 144

$6=\frac{f^{IV}(0)}{4!}$ $f^{IV}(0)=6(24)=$

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8. The radius of convergence for $\sum_{n=0}^{\infty}\frac{3^n(x-1)^n}{n!}$ is

a) 1 b) 3 c) $\frac{1}{3}$ d) 0 e) ∞

$\frac{3^{n+1}}{(n+1)!}\ \frac{n!}{3^n}\ (x-1)$

$\lim \frac{3}{n+1} = 0 < 1$

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9. If the third degree Taylor series for $f(x)$ about $x = -3$ is used to approximate $f(-2.7)$, and $f^{III}(x) \le 2$ and $f^{IV}(x) \le 5$ for all x-values, what is the Lagrange Error bound?

a) 0.001

b) 0.009

c) 0.062

d) 0.100

e) 0.208

$$\left| \frac{f^{IV}(c)}{4!} \left(-2.7 + 3\right)^4 \right] < \frac{5}{4!} (.3)^4$$

$= .002$

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10. The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty} \frac{x^n}{3n+1} = 1 + \frac{x}{4} + \frac{x^2}{7} + \cdots + \frac{x^{n\cancel{+1}}}{3n+1} + \cdots$

a) What is $g'''(0)$?

1

$$\frac{g'''(0)}{3!} = \frac{1}{10}$$

$$g'''(0) = \frac{6}{10} = \frac{3}{5}$$

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b) Write the first three nonzero terms and the general term for the Maclaurin series for $g'(3x)$

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$$\text{OR } g(3x) = 1 + \frac{3}{4}x + \frac{9}{7}x^2 + \frac{27}{10}x^3 + \cdots \frac{3^n x^n}{3n+1}$$

$$\cancel{g(3x) = 1 + \frac{3}{4}x + \frac{9}{7}x^2 + \cdots + \frac{3^n x^n}{3n+1} + \cdots}$$

$$g' = \frac{3}{4} + \frac{18}{7}x + \frac{81}{10}x^2 \quad \frac{n3^n}{3n+1}x^{n-1}$$

$$g'(x) = \frac{1}{4} + \frac{2}{7}x + \frac{3}{10}x^2 + \cdots \frac{n x^{n-1}}{3n+1} + \cdots$$

$$g'(3x) = \left[\frac{1}{4} + \frac{6}{7}x + \frac{27}{10}x^3 + \cdots \frac{n3^{n-1}x^{n-1}}{3n+1}\right]3 + \cdots$$

$$\text{OR } \frac{n3^n x^n}{3n+4}$$

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c) Write the first 3 nonzero terms of the series for $\sin x$ and use them to find the 4[th] degree Maclaurin polynomial for the product $\sin x \cdot g(x)$

1 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$

$g(x)\sin x = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(1 + \frac{3}{4}x + \frac{9}{7}x^2\right)$

2 $= \cancel{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{6}x^3 - \frac{1}{8}x^4}$

4 $\approx 1 + \frac{1}{3}x^3 - \frac{1}{8}x^4$

$= 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{3}{4}x - \frac{1}{8}x^4 + \frac{9}{7}x^2 - \frac{1}{6}x^3 \; \frac{2}{21}$

$= 1 + \frac{3}{4}x + \frac{9}{7}x^2 - \frac{1}{3}x^3 - \cancel{\frac{4}{44}x^4} - \frac{41}{168}x^4$

$x + \frac{1}{4}x^2 + \frac{1}{7}x^3 - \frac{1}{6}x^3 + \frac{1}{10}x^4 - \frac{1}{24}x^4$

$x + \frac{1}{4}x^2 - \frac{1}{42}x^3 + \frac{7}{120}x^4$

11. A function f has derivatives of all orders at $x=1$ given by

$$f^{(n)}(1)=\frac{(-1)^{n+1}n!}{2^n}. \qquad c_n = \frac{(-1)^{n+1}}{2^n}$$

a) Write the first 4 nonzero terms and the general term for the Taylor series for f centered at $x=1$ (Recall, $0!=1$)

$$\approx -\frac{1}{2} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{8}(x-1)^3 - \frac{1}{16}(x-1)^4 + \dots \frac{(-1)^{n+1}}{2^n}(x-1)^n + \dots$$

3

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b) Find the radius and interval of convergence for the series you found in part a).

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$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-1)^{n+1}}{2^{n+1}}}{\frac{(-1)^{n+1}(x-1)^n}{2^n}}\right| = \lim_{n\to\infty}\frac{1}{2}|x-1|<1$$

$$|x-1|<2 \qquad r=2$$

$$\boxed{-2<x-1<2}$$

$$\boxed{-1<x<3}$$

$x=2 \to \sum (-1)^{n+1}$ DIV

$x=-2 \to \sum (-1)^{2n+}$ DIV

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c) Use the first four nonzero terms of the Taylor series to approximate $f(1.2)$, and show that this approximation is within 0.0001 of the actual value of $f(1.2)$.

$$-1+\frac{1}{2}(.2)-\frac{1}{4}(.2)^2+\frac{1}{8}(.2)^3$$

1

$$= -.909$$

ALT SERIES ERROR $\left|\frac{(-1)^5}{2^4}(.2)^4\right| = (.1)^4 = .00001 < .0001$

1

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d) Rewrite $f(x)$ as a geometric series and find the actual value of $f(1.2)$.

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^n}(.2)^n=\sum_{n=0}^{\infty}(-1)^{n+1}(.1)^n$$

1EC

MA

$A=-1 \quad r=.1 \qquad S=\frac{-1}{1-.1}=-\frac{10}{9}$

≈ 1.111
