AP Calculus BC '18-19

Fall Final
Part IIa

Calculator Required

Name:

Solution Key

1. At time $t=0$, there are 120 gallons of oil in a tank. During the time interval $0 \le t \le 10$ hours, oil flows into the tank at a rate of $h(t)=10-\dfrac{t\cos(t)}{2}$ and out of the tank at a rate given by $g(t)=6+\dfrac{e^{0.52t}}{t+1}$. Both h and g are measured in gallons per hour.

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a) How much oil flows out of the tank during this 10-hour time period?

1

$$\int_0^{10} 6+\frac{e^{.52t}}{t+1}\,dt = 100.827 \text{ GALLONS}$$

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b) Find the value of $h(4.3)-g(4.3)$. Using correct units, explain what this value represents in the context of this problem.

2

$$h(4.3)-g(4.3)=\cancel{6.627}\ 3.096\ \frac{\text{GAL}}{\text{HR}}$$

AT $t=4.3$ HOURS, THE OIL IS FLOWING IN TO THE TANK A ~~6.627~~ 3.096 GAL/HR FASTER THAN IT IS FLOWING OUT.

THE RATE AT WHICH THE AMOUNT OF OIL IN THE TANK IS CHANGING

c) Write an expression for $A(t)$, the total amount of oil in the tank at time t.

$$A(t) = 120 + \int_0^{t} \left(\left(10 - \frac{t}{2}\cos t\right) - \left(6 + \frac{e^{.52t}}{t+1}\right) \right) dt$$

d) Find the absolute maximum and minimum amount of oil in the tank during $0 \le t \le 10$ hours.

$h(t) - g(t) = 0 \Rightarrow t \approx 5.309$

t	A
0	120 GAL
5.309	136.824 GAL
10	122.812 GAL

ABS MAX = 136.824 GAL

ABS MIN = 120 GAL

2. A tree trunk has circular horizontal cross sections. The radius $r(h)$ of the tree trunk is a differentiable function of the height h of the tree measured from the ground. Both r and h are measured in feet. Selected values of r and h are given in the table below.

h height from the ground (feet)	0	1	3	5	8
$r(h)$ radius of the tree trunk (feet)	3	2.5	2	2.5	1.5

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a) Use a Right Riemann sum with subintervals indicated by the table to approximate the value of $\frac{1}{8}\int_0^8 r(h)\,dh$. Using correct units, explain the meaning of this value in the context of this problem.

$$\frac{1}{8}\int_0^8 r(h)\,dh = \frac{1}{8}\left[2.5 + 2(2) + 2(2.5) + 3(1.5)\right] = 2$$

2 IS THE APPROXIMATE AVERAGE VALUE OF THE RADIUS OF THE TREE OVER THE FIRST 8 FEET.

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b) Must there be a height within the first eight feet of tree trunk where $\frac{dr}{dh} = 0$? Explain.

YES BECAUSE $r(1) = r(5)$ AND THE FUNCTION IS DIFFERENTIABLE, SO THE MVT HOLDS

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c) Write an expression involving one or more integrals to calculate the volume of the tree trunk from $h=0$ to $h=8$. Use a Left Riemann sum to approximate the value of this expression.

$$V = \pi \int_0^8 r^2 \, dh$$

$$= \pi \left[3^2 + 2(2.5)^2 + 2(2)^2 + 3(2.5)^2 \right]$$

$$= 42\pi$$

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d) For heights above 8 feet, the radius is given by $g(h) = \frac{1}{h^2} + \frac{95}{64}$. A squirrel climbs up the tree at a rate of $\frac{dh}{dt} = 3 \text{ ft/sec}$. How quickly is the radius of the tree changing when the squirrel is 9 feet above the ground?

$$g(h) = h^{-2} + \frac{95}{64}$$

$$\frac{dr}{dt} =$$

$$\frac{dr}{dt} = -2h^{-3} \frac{dh}{dt} = \frac{-2}{9^3}(3) = \frac{-2}{243}$$

End of

AP Calculus BC '18-19

Fall Final -- Part IIa

AP Calculus BC '18-19

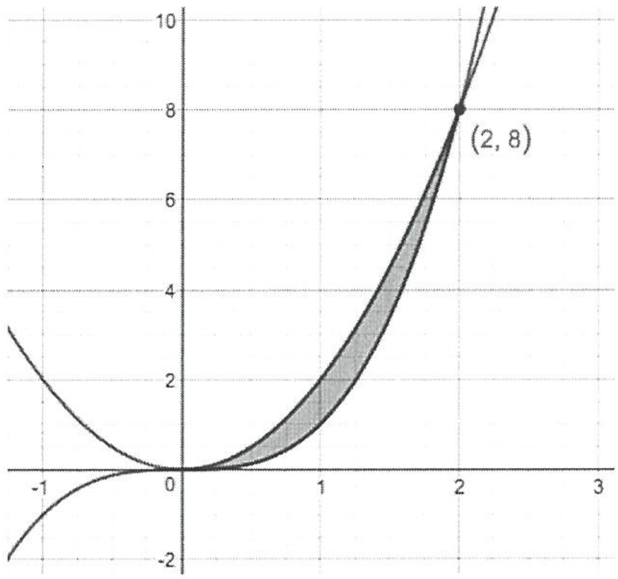
Fall Final
Part IIb

No Calculator Allowed

Name:

SOLUTION KEY

3. Let R be the region bounded by $y=2x^2$ and $y=x^3$, shaded in the picture below. The curves intersect at the origin and at the point $(2,8)$.



a) Find the area of R.

$$A = \int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{2}{3}x^3 - \frac{1}{4}x^4 \Big]_0^2$$

$$= \frac{16}{3} - 4$$

$$= \frac{4}{3}$$

b) Set up, but do not evaluate, an expression involving one or more integrals that would find the volume of the solid whose base is R and whose cross sections parallel to the y-axis are semicircles.

MEANS $\perp$ TO X-AXIS

$\cancel{x = \sqrt{\tfrac{y}{2}}}$ $\qquad$ $\cancel{x = y^{1/3}}$

$$V = \pi \int_0^2 \left(\frac{2x^2 - x^3}{2}\right)^2 dx$$

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c) Find the volume of the solid formed by revolving R around the y-axis.

$x = y^{1/3}$ $\qquad$ $x = \sqrt{\frac{y}{2}}$

$$V = \pi \int_0^8 (y^{1/3})^2 - \left(\sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi \int_0^8 \left(y^{2/3} - \frac{y}{2}\right) dy$$

$$= \pi \left[\frac{y^{5/3}}{5/3} - \frac{y^2}{4}\right]_0^8$$

$$= \pi \left[\frac{96}{5} - 16\right] = \frac{16\pi}{3}$$

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4. The temperature of water in a pond is modeled by the equation $B(t)$, which satisfies the differential equation $\frac{dB}{dt} = -\frac{\sqrt{B}}{8} + 1$. At time $t = 0$ hours, the pond's temperature is 81°F.

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a) Write the equation of the line tangent to B at $t = 0$ and use it to approximate the temperature of the pond at $t = 1$.

$m = \frac{-1}{8} + 1 = \frac{7}{8}$

$y - 81 = \frac{7}{8}x$

$B(1) \approx 81 + \frac{7}{8} = 81\frac{7}{8}$

---

b) For what value(s) of B does $B(t)$ have a critical value?

$B'(t) = -\frac{\sqrt{B}}{8} + 1 = 0$

$-\sqrt{B} = 8$

$B = 64$

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c) Find $\frac{d^2B}{dt^2}$ in terms of B.

$$\frac{dB}{dt} = -\frac{1}{8}B^{1/2} + 1$$

$$\frac{d^2B}{dt^2} = -\frac{1}{16}B^{-1/2}\frac{dB}{dt} = -\frac{B^{-1/2}}{16}\left[-\frac{1}{8}B^{1/2}+1\right]$$

$$= \frac{1}{128} - \frac{1}{16}B^{-1/2}$$

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d) A researcher suggests a different possible model for the temperature of the lake as $P(t)$, where $\frac{dP}{dt} = \frac{-10}{(2t+1)^3}$. The pond still begins with a temperature of 81°F at $t = 0$. Using this model, what temperature would the pond converge to after an infinite amount of time?

$$P = 81 + \int_0^\infty \frac{-10}{(2t+1)^3}\,dt \qquad u = 2t+1 \quad u(0)=1$$

$$du = 2dt \quad u(b) = 2b+1$$

$$= 81 + \lim_{b\to\infty}\int_1^{2b+1} -5u^{-3}\,du$$

$$= 81 + \lim_{b\to\infty}\left[\frac{-5}{-2}u^{-2}\right]_1^{2b+1}$$

$$= 81 + \frac{5}{2}\lim_{b\to\infty} u^{-2} - \frac{5}{2}\frac{1}{(1)^2} =$$

$$= 81 + 0 - \frac{5}{2} = \frac{157}{2} = 78.5^\circ$$

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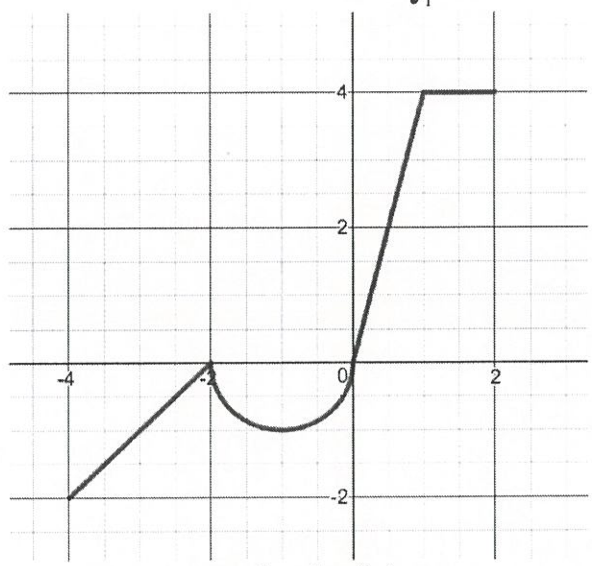
5. Below is the graph of $g(x)$, which is defined on $-4 \le x \le 2$ and consists of three line segments and a semicircle. Let $f(x) = \int_1^x g(t)\,dt$.



graph of $g(x)$

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a) Find $f(-2)$, $f'(-2)$, and $f''(-2)$

$f(-2) = -2 + \frac{\pi}{2}$

$g = f'$

$f'(-2) = 0$

$f''(-2) = $ DNE

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b) On what intervals, if any, is f(x) both decreasing and concave up? Justify your answer.

f' IS NEGATIVE AND INCREASING

$x \in (-4, -2) \cup (-1, 0)$

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c) Find the average rate of change of $f(x)$ from $x = -2$ to $x = 2$.

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{4 - (-2 + \pi/2)}{4}$$

$$= \frac{6 - \pi/2}{4}$$

$$= \frac{3}{2} - \pi/8$$

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d) Identify all x-values in the open interval $-4 < x < 2$ at which $f(x)$ has a critical point, and classify each critical point as a local minimum, local maximum, or neither. Justify your answers.

$f' = g = 0$ @ $x = -2$ & 0

$X = -2$ IS NEITHER A MAX OR MIN

$X = 0$ IS AT A MIN BECAUSE f' SWITCHES FROM $-$ TO $+$

6. A twice-differentiable function $y = f(x)$ has derivative given by the differential equation $\dfrac{dy}{dx} = \dfrac{x(y^2-1)}{y}$ and satisfies $f(1) = -\sqrt{2}$.

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a) Write the equation of the line **normal** to $y = f(x)$ at the point $(1, -\sqrt{2})$.

$m_{TAN} = \dfrac{1(2-1)}{-\sqrt{2}} = -\dfrac{1}{\sqrt{2}}$

$m_{NORMAL} = +\sqrt{2}$

$y + \sqrt{2} = \sqrt{2}(x-1)$

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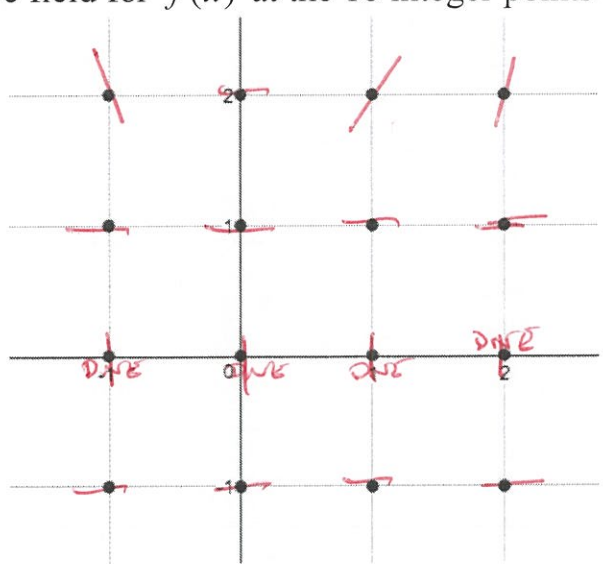
b) Sketch the slope field for $f(x)$ at the 16 integer points indicated.



c) Find the particular solution $y = f(x)$ for which $f(1) = -\sqrt{2}$.

$u = y^2 - 1$

$du = 2y\,dy$

$$\frac{1}{2}\int \frac{2y\,dy}{y^2-1} = \int x\,dx$$

$$\frac{1}{2}\ln|y^2-1| = \frac{x^2}{2} + c$$

$$\ln|y^2-1| = x^2 + c$$

$$|y^2-1| = e^{x^2+c} =$$

$$y^2 - 1 = ke^{x^2}$$

$(1, -\sqrt{2}) \Rightarrow 1 = ke^1$

$\frac{1}{e} = k$

$$y^2 - 1 = \frac{1}{e}e^{x^2} = e^{x^2-1}$$

$$y^2 = 1 + e^{x^2-1}$$

$$y = -\sqrt{1+e^{x^2-1}}$$

End of

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