Chapter 3 Overview: Derivatives and Graphs

There are two main contexts for derivatives: graphing and motion. In this chapter, we will consider the graphical applications of the derivative. Much of this is a review of material covered last year. Key topics include:

- Finding extremes
- The First and Second Derivative Tests
- Finding intervals of increasing and decreasing
- Finding intervals of concavity
- Optimization
- Graphing using the derivatives
- Making inferences regarding the original graph from the graph of its derivative graphs

Several multiple-choice questions and at least one full free response question (often parts of others), have to do with this general topic.

3.1: Extrema and the Derivative Tests

One of the most valuable aspects of Calculus is that it allows us to find extreme values of functions. The ability to find maximum or minimum values of functions has wide-ranging applications. Every industry has uses for finding extremes, from optimizing profit and loss, to maximizing output of a chemical reaction, to minimizing surface areas of packages. This one tool of Calculus eventually revolutionized the way the entire world approached every aspect of industry. It allowed people to solve formerly unsolvable problems.

OBJECTIVES

Find critical values and extreme values for functions.
Use the 1st and 2nd derivative tests to identify maxima vs. minima.

It will be helpful to keep in mind a few things from last year for this chapter (and all other chapters following).

REMINDER: *Vocabulary*:

1. *Critical Value*--The x-coordinate of the extreme
2. *Maximum Value*--The y-coordinate of the high point.
3. *Minimum Value*--The y-coordinate of the low point.
4. *Relative Extremes*--the highest or lowest points in any section of the curve.
5. *Absolute Extremes*--the highest or lowest points of the whole curve.
6. *Interval of Increasing*--the interval of x-values for which the curve is rising from left to right.
7. *Interval of Decreasing*--the interval of x-values for which the curve is dropping from left to right.

Critical Values of a function occur when

i. $\frac{dy}{dx} = 0$

ii. $\frac{dy}{dx}$ does not exist

iii. At the endpoints of its domain.

It is also helpful to remember that a **critical value** is referring specifically to the value of the x, while the **extreme value** refers to the value of the y.
The first derivative is what allows us to algebraically find extremes, and the first derivative test allows us to interpret critical values as maxima or minima. Since the first derivative of a function tells us when that function is increasing or decreasing, we can figure out if a critical value is associated with a maximum or minimum depending on the sign change of the derivative.

The 1st Derivative Test

As the sign pattern of the 1st derivative is viewed left to right, the critical value represents a
1) relative maximum if the sign changes from + to -
2) relative minimum if the sign changes from - to +
or 3) neither a max. or min. if the sign does not change.

Ex 1 Apply the first derivative test to the function $y = 5x^4 - 10x^2$

$$\frac{dy}{dx} = 20x^3 - 20x = 0$$
$$20x\left(x^2 - 1\right) = 0$$
$$x = -1, 0, 1$$

y'	$-$	0	$+$	0	$-$	0	$+$	
x		-1		0		1		

So there are critical values at x values of –1, 0, and 1. When we look at the sign pattern, we can see we have a minimum at –1, a maximum at 0, and another minimum at 1.

Ex 2 Find the maximum values for the function $g(x)=\sqrt{x^3-9x}$

Domain: $x^3-9x\geq 0$

$$\begin{array}{ccccccccc} y & - & 0 & + & 0 & - & 0 & + \\ x & & -3 & & 0 & & 3 & \end{array}$$

$$x^3-9x\geq 0 \rightarrow x\in[-3,\,0]\cup[3,\,\infty)$$

$$g'(x)=\frac{3x^2-9}{2\sqrt{x^3-9x}}$$

$g'(x)=0$ when $3x^2-9=0$, so $x=\pm\sqrt{3}$

$g'(x)$ does not exist when $x^3-9x=0$, so $x=\pm 3,0$

$$\begin{array}{ccccccccccc} y' & dne & + & 0 & - & dne & & dne & + \\ x & -3 & & -\sqrt{3} & & 0 & & \sqrt{3} & & 3 \end{array}$$

Since $x=\sqrt{3}$ is not in the domain of the function, our critical values are at $x=\pm 3,0,-\sqrt{3}$. From our sign pattern, we can conclude that at $x=-\sqrt{3}$, we have a maximum value, and by substituting this value into the function, we find that the maximum value is at $y=3.224$.

The first derivative test is not the only way to test whether critical values are associated with maximum or minimum values. If you recall, the second derivative can show us intervals of concavity. This is very useful because if we know that the curve is concave down at a critical value, it must be associated with a maximum value of the function. Similarly, if the curve is concave up, the critical value must be associated with a minimum value.

The 2nd Derivative Test

For a function f,

1) If $f'(c)=0$ and $f''(c)>0$, then f has a relative minimum at c.
2) If $f'(c)=0$ and $f''(c)<0$, then f has a relative maximum at c.

Ex 3 Use the 2nd Derivative Test to determine if the critical values of $g(t) = 27t - t^3$ are at a maximum or minimum value

$$g'(t) = 27 - 3t^2 = 0$$
$$t = \pm 3$$

So –3 and 3 are critical values.

$$g''(t) = -6t$$
$$g''(-3) = -6(-3) = 18$$
$$g''(3) = -6(3) = -18$$

Therefore, g has a minimum value at t = –3 and has a maximum value at t = 3. (Note that the numerical value "18" is irrelevant. Only the sign matters.)

EX 4 Determine if the critical values of $y = (3 - x^2)e^x$ are at a maximum or a minimum value.

First, find the critical values:

$$\frac{dy}{dx} = (3 - x^2)e^x + e^x(-2x)$$
$$= -e^x(x^2 + 2x - 3) = 0$$

The derivative is never DNE.

Critical values: $x = -3, 1$

Find the 2nd derivative and substitute the critical values:

$$\frac{dy}{dx} = -e^x(x^2 + 2x - 3)$$
$$\frac{d^2y}{dx^2} = -e^x(2x + 2) + (x^2 + 2x - 3)(-e^x)$$
$$= -e^x(x^2 + 4x - 1)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = -4e \qquad \left.\frac{d^2y}{dx^2}\right|_{x=-3} = 4e^{-3}$$

Therefore, $x = 1$ is at a maximum value and $x = -3$ is at a minimum value.

Ex 5 2004 BC #4

Ex 6 2008 BC #5

Optimization

Optimization is a practical application of finding maxima and minima for functions. As I mentioned before, this revolutionized thinking and is a critical component of all industries. You might remember this topic from chapter 2 of book 2 last year; the word problems many of you avoided on the test last year. This year, they form a much more fundamental part of what we need to be able to do, so we can no longer simply skip these problems on tests.

OBJECTIVES
Solve optimization problems.

Every optimization problem looks a bit different, but they all follow a similar progression. You must first identify your variables and any formula you need. Use algebra to eliminate variables, and **take the derivative of the function you are trying to optimize**. This is the most common mistake in optimization problems; taking the derivative of the wrong function.

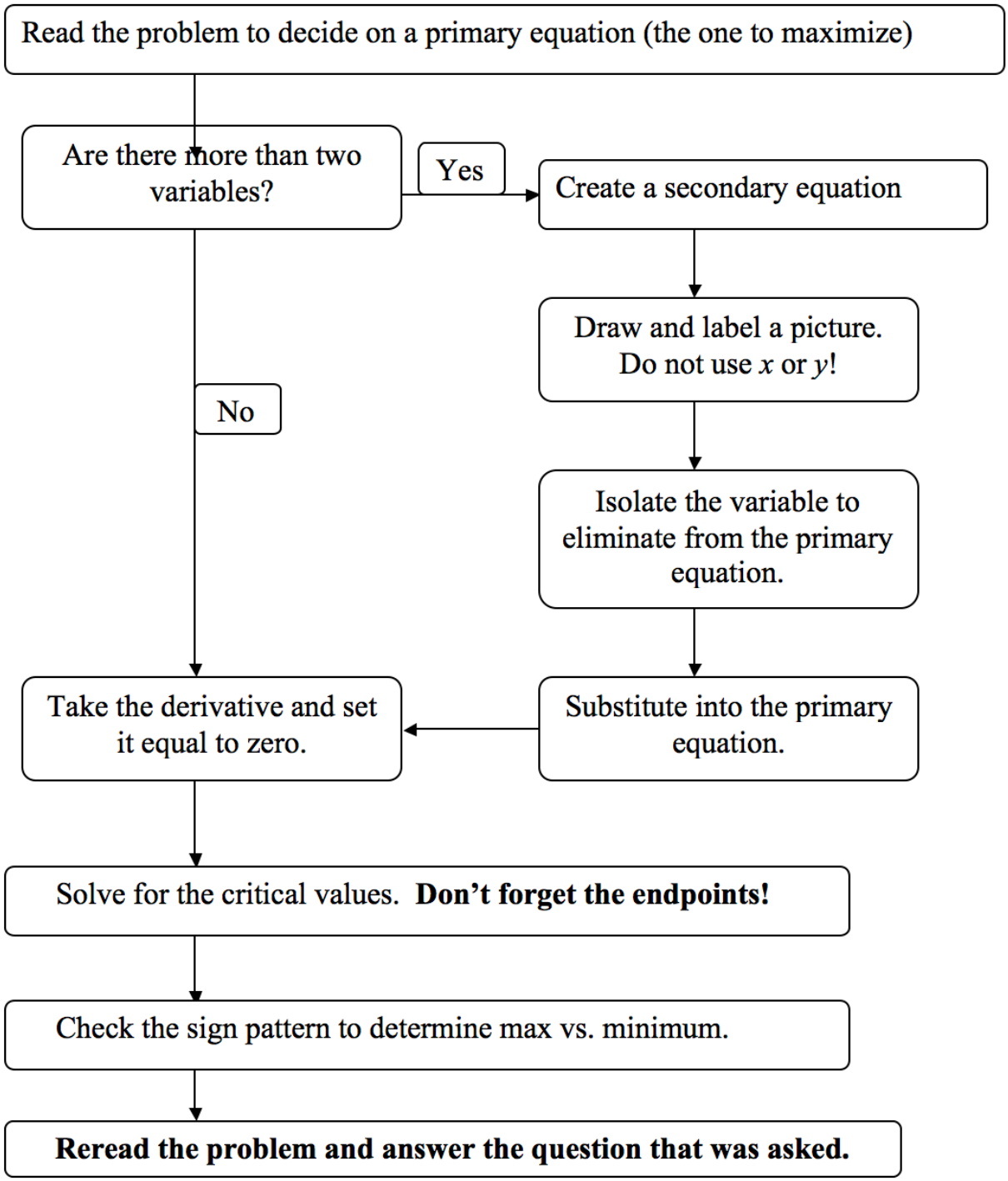
Key Ideas:

I. Find the variable to be maximized or minimized. That is the equation/variable to differentiate.

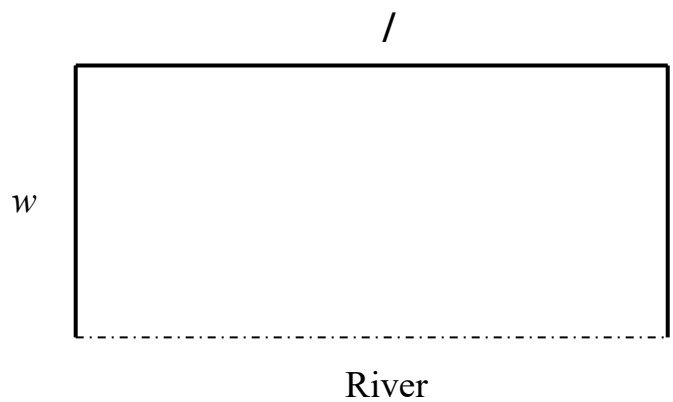
II. The equation which comes from the sentence with the superlative usually has too many variables. Other sentences help to substitute out the extra variables.

The following flowchart might help:

Strategy for Approaching Optimization Problems with Calculus



Ex 1 The owner of the Rancho Grande has 3000 yards of fencing material with which to enclose a rectangular piece of grazing land along the straight portion of a river. If fencing is not required along the river, what are the dimensions of the largest area he can enclose? What is the area?



$A = lw$ The problem with maximizing this area formula lies in the fact that we have two independent variables (l and w). We need the fact about perimeter to complete the problem.

$$A = lw$$

$$P = l + 2w$$
$$3000 = l + 2w$$
$$3000 - 2w = l$$

$$A = (3000 - 2w)w$$
$$A = 3000w - 2w^2$$

Now, since we have an equation with one independent variable, we can take the derivative easily.

$$\frac{dA}{dw} = 3000 - 4w$$
$$\frac{dA}{dw} = 3000 - 4w = 0$$
$$w = 750$$
$$l = 1500$$

So we would want a width of 750 yards and a length of 1500 yards. This would give us an area of 1,125,000 yards^2.

Ex 2 A cylindrical cola can has a volume $32\pi \text{ in}^3$. What is the minimum surface area?

$$S = 2\pi r^2 + 2\pi rh$$

$$V = \pi r^2 h$$
$$32\pi = \pi r^2 h$$
$$\frac{32}{r^2} = h$$

$$S = 2\pi r^2 + \pi r\left(\frac{64}{r^2}\right)$$
$$S = 2\pi r^2 + 64\pi r^{-1}$$
$$\frac{dS}{dr} = 4\pi r - 64\pi r^{-2} = 0$$
$$4\pi r\left(1 - \frac{16\pi}{r^3}\right) = 0$$
$$r = 0 \text{ or } 3.691$$

There is an implied domain here. You cannot have a radius of 0 inches, so 3.691 inches is the radius for the minimum area. The sign pattern verifies this:

dA/dr	0	$-$	0	$+$	
r	0		3.691		

So the minimum surface area would be

$$S = 2\pi(3.691)^2 + 2\pi(3.691)\left(\frac{32}{(3.691)^2}\right) = 140.058 \text{ in}^2$$

Ex 3 Find the point on the curve $y = \frac{e^{-x^2}}{2}$ that is closest to the origin.

We want to minimize the distance to the origin, so we will be using the Pythagorean theorem to find the distance.

$$D = \sqrt{x^2 + y^2}$$

$$y = \frac{e^{-x^2}}{2}$$

$$D = \sqrt{x^2 + \left(\frac{e^{-x^2}}{2}\right)^2}$$

$$\frac{dD}{dx} = \frac{1}{2}\left(x^2 + \frac{e^{-2x^2}}{4}\right)^{-1/2}\left(2x - xe^{-2x^2}\right) = 0$$

$$x = 0, \quad \pm .841$$

x=	D=
-.841	.876
0	.5
.841	.876

So the minimum distance from the origin is at the point (0, .5).

Given the diverse nature of optimization problems, it is helpful to remember all the formulas from geometry.

3.1 Free Response Homework

Find the critical values and extreme values for each function.

1. $y = 2x^3 + 9x^2 - 168x$

2. $y = 3x^4 + 2x^3 + 12x^2 + 12x - 42$

3. $y = \dfrac{x^2 + 1}{x^3 - 4x}$

4. $y = \dfrac{x - 5}{x^2 + 9}$

5. $y = \sqrt{9x^3 - 4x^2 - 27x + 12}$

6. $y = \sqrt{\dfrac{3x}{9 - x^2}}$

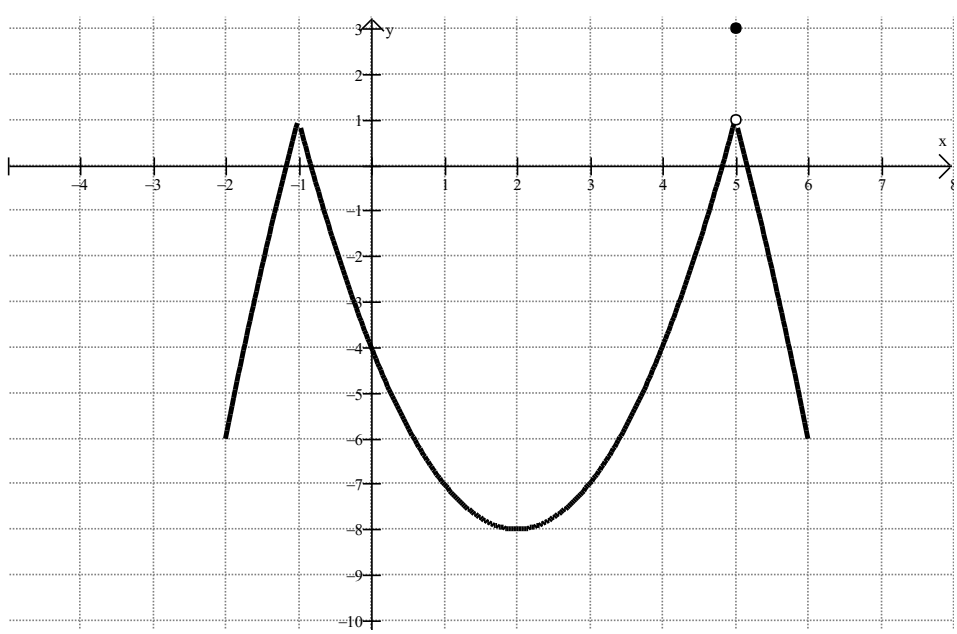
7. $y = \left(x^2\right)\sqrt[3]{9 - x^2}$

8. $y = \left(x - x^2\right)3^x$

9. $y = \dfrac{\sqrt{3}}{2}x + \cos x \ on \ x \in \left[-2\pi,\ 2\pi\right]$

10. $y = \cot^{-1}\left(\dfrac{1}{x}\right) - \tan^{-1} x$

11.



For 12 – 16, find the absolute maximum and minimum values of f on the given intervals.

12. $f(x)=\dfrac{x}{x^2+1},\ [0,2]$

13. $f(t)=\sqrt[3]{t}(8-t),\ [0,8]$

14. $f(z)=ze^{-z},\ [0,2]$

15. $f(x)=\dfrac{\ln(x)}{x},\ [1,3]$

16. $f(x)=e^{-x}-e^{-2x},\ [0,1]$

For each of the following functions, apply the 1st Derivative Test, then verify the results with the 2nd Derivative Test.

17. $f(x)=x^2\ln x$

18. $h(f)=f^3-12f+21$

19. $f(x)=xe^{-x^2}$

Solve these problems algebraically.

20. Find two positive numbers whose product is 110 and whose sum is a minimum.

21. Find a positive number such that the sum of the number and its reciprocal is a minimum.

22. A farmer with 750 feet of fencing material wants to enclose a rectangular area and divide it into four smaller rectangular pens with sides parallel to one side of the rectangle. What is the largest possible total area?

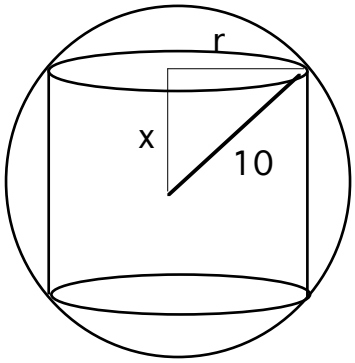
23. If 1200 cm^2 of material is available to make a box with an open top and a square base, find the maximum volume the box can contain.

24. Find the point on the line $y = 4x + 7$ that is closest to the origin.

25. Find the points on the curve $y = \frac{1}{x^2 + 1}$ that are closest to the origin.

26. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

27. Find the maximum volume of a cylinder inscribed in a sphere with radius 10.



28. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square, while the other is bent into an equilateral triangle. (a) Find where the wire should be cut to maximize the area enclosed, then (b) find where the wire should be cut to minimize the area enclosed.

3.1 Multiple Choice Homework

1. Give the value of x where the function $f(x) = x^3 - 9x^2 + 24x + 4$ has a relative maximum point.

a) 4 b) –2 c) 2 d) –4 e) 3

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2. Give the value of x where the function $f(x) = x^3 - \frac{33}{2}x^2 + 84x - 2$ has a relative minimum point.

a) –4 b) –7 c) 4 d) 5 e) 7

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3. Give the approximate location of a relative maximum point for the function $f(x) = 3x^3 + 5x^2 - 3x$.

a) $(-1.357,\ 5.779)$ b) $(0.2457,\ -0.3908)$ c) $(-1.357,\ 5.713)$

d) $(0.2457,\ -0.3216)$ e) $(-1.357,\ -0.3908)$

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4. Given this sign pattern

$\frac{dy}{dx}$	+	0		0	–	0	+	0		0 –
x		-2	$-\sqrt{2}$		0		$\sqrt{2}$		2	

, at what value of x does f have a local minimum?

a) –2 b) $-\sqrt{2}$ c) 0 d) $\sqrt{2}$ e) 2

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5. The absolute maximum of $y = -\sqrt{25 - x^2}$ on $x \in [-2, 4]$ is

a) –2 b) 0 c) –5 d) $-\sqrt{21}$ e) –3

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6. Find the absolute maximum value of $y=\sqrt{36-x^2}$ on the interval $x\in[-2,\ 2]$.

a) 5 b) 6 c) 7 d) 0 e) 1

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7. The graph of the function $f(x)=2x^{5/3}-5x^{2/3}$ is increasing on which of the following intervals

I. $1<x$ II. $0<x<1$ III. $x<0$

a) I only b) II only c) III only

d) I and II only e) II and III only

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8. If $f(x)=x^2e^{-2x}$, then the graph of f is increasing for all x such that

a) $0<x<1$ b) $0<x<\frac{1}{2}$ c) $0<x<2$

d) $x<0$ e) $x>0$

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9. If $f(x)=\frac{x}{\ln(7x)}$, then what is the interval of decreasing?

a) $\left(0,\frac{1}{7}\right)$ b) $\left(0,\frac{1}{7}\right)\cup\left(\frac{1}{7},\frac{1}{7}e\right)$ c) $(1,\ 7e)$

d) $(1,\ 7)$ e) $\left(1,\frac{1}{7}e\right)$

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10. Suppose $f'(x)=\dfrac{(x+1)^3(x-4)^2}{(x^4+1)}$. Which of the following statements must be true?

I. The slope of the line tangent to $y=f(x)$ at $x=1$ is 36.

II. $f(x)$ is increasing on $x\in(1,\ 4)$

III. $f(x)$ has a minimum at $x=-1$

a) II only b) I and III only c) II and III only

d) I and II only e) I, II, and III

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11. Given this sign pattern

$f'(x)$	$-$	0	$+$	0	$-$	0	$-$
x		-4		-1		2	

, at what value of x does f has a relative minimum point?

a) –4 b) –1 c) 2 d) 1 e) no value

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12. The sum of two positive integers x and y is 150. Find the value of x that minimizes $P=x^3-150xy$

a) $x=25$
b) $x=75$
c) $x=50$
d) $x=125$
e) $x=100$

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13. The sum of two positive integers x and y is 120. Find the value of x that minimizes $P = x^3 - 120xy$

a) $x = 20$
b) $x = 40$
c) $x = 60$
d) $x = 80$
e) $x = 100$

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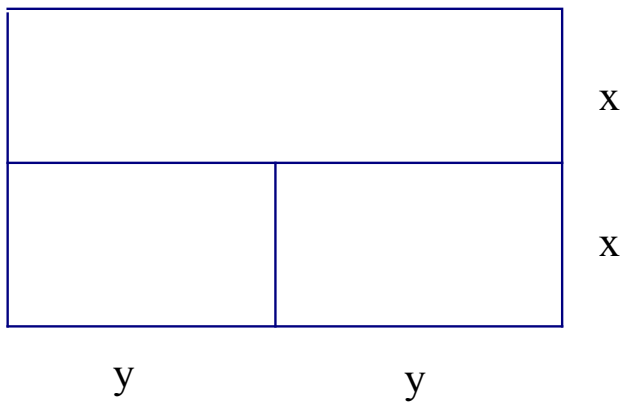
14. A farmer has 100 yards of fence to enclose a field, subdivide it into two equal pens, and further subdivide one of those pens into two equal fields as shown below.



What value of y produces the maximum total area?

a) 12.5 b) 10 c) $\frac{100}{11}$ d) $\frac{25}{3}$ e) None of these

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15. A farmer with 890 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

a) 19,825.5 ft^2 b) 19,802.5 ft^2 c) 19,801.5 ft^2

d) 19,902.5 ft^2 e) 19,791.5 ft^2

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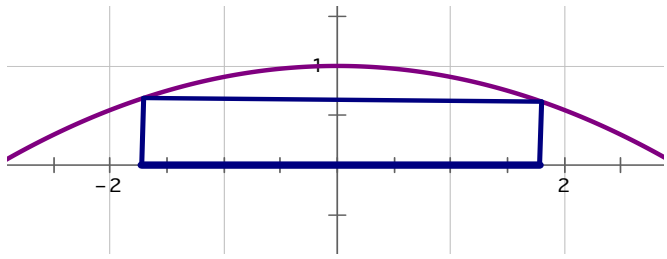
16. A rectangle with one side on the x-axis has its upper vertices on the graph of $y = 1 - \frac{x^2}{9}$, as shown in the figure below. What is the maximum area of the rectangle?



a) $\sqrt{3}$ b) $\frac{\sqrt{3}}{2}$ c) $\frac{2}{3}$ d) $\frac{4\sqrt{3}}{3}$ e) None of these

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17. In a certain community, an epidemic spreads in such a way that the percentage P of the population that is infected after t months is modeled by

$$P(t) = \frac{kt^2}{\left(C + t^2\right)^2},$$

where C and k are constants. Find t, such that P is most.

a) 0 b) $\sqrt{C}$ c) $\sqrt{k}$ d) $\sqrt{\frac{C}{3}}$ e) None of these

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x	$g(x)$	$g'(x)$	$g''(x)$	$f(x)$	$f'(x)$	$f''(x)$
0	2	3	-2	2	1	-2
2	4	2	0	4	0	3
4	1	-1	3	0	2	-1

18. At $x = 2$, $f\left(x\right)$ has a:

a) Local Minimum b) Local Maximum c) Point of Inflection

d) Zero e) None of these

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19. Let $f(x)$ and $g(x)$ be twice-differentiable functions with selected values given in the table in #18. Let $h(x) = g(f(x))$. At $x = 2$, $h(x)$ has a:

a) Local Minimum b) Local Maximum c) Point of Inflection

d) Zero e) None of these

3.1 Homework Set B

1. You need to enclose 500 cm^3 of fluid in a cylinder. If the material you are using costs 0.001 dollars per cm^3, find the size of the cylinder that minimizes the cost. If the product that you are containing is a sports drink, do you think that the size that minimizes cost is the most efficient size? Explain.

2. The height of a man jumping off of a high dive is given by the function $h(t) = -4.9t^2 + 2t + 10$ on the domain $0 \le t \le 1.64715$. Find the absolute maximum and minimum heights modeled by this function.

3. Given that the area of a triangle can be calculated with the formula $A = \frac{1}{2}ab\sin\theta$, what value of θ will maximize the area of a triangle (given that a and b are constants)?

4. You operate a tour service that offers the following rates for the tours: \$200 per person if the minimum number of people book the tour (50 people is the minimum) and for each person past 50, up to a maximum of 80 people, the cost per person is decreased by \$2. It costs you \$6000 to operate the tour plus \$32 per person.

a) Write a function that represents cost, $C(x)$.
b) Write a function that represents revenue, $R(x)$.
c) Given that profit can be represented as $P(x) = R(x) - C(x)$, write a function that represents profit and state the domain for the function.
d) Find the number of people that maximizes your profit. What is the maximum profit?

3.2 The Mean Value and Rolle's Theorems

The Mean Value Theorem is an interesting piece of the history of Calculus that was used to prove a lot of what we take for granted. The Mean Value Theorem was used to prove that a derivative being positive or negative told you that the function was increasing or decreasing, respectively. Of course, this led directly to the first derivative test and the intervals of concavity.

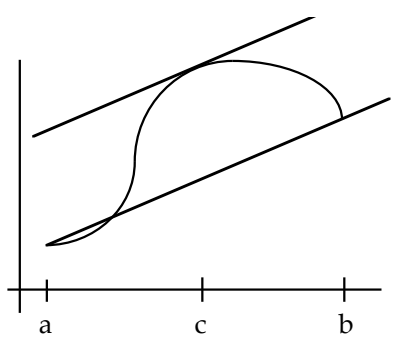
Mean Value Theorem

If f is a function that satisfies these two hypotheses

1. f is continuous on the closed interval $[a,b]$
2. f is differentiable on the closed interval (a,b)

Then there is a number c in the interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Again, translating from math to English, this just says that, if you have a smooth, continuous curve, the slope of the line connecting the endpoints has to equal the slope of a tangent somewhere in that interval. Alternatively, it says that the secant line through the endpoints has the same slope as a tangent line.



MEAN VALUE THEOREM

Ex 1 Show that the function $f(x)=x^3-4x^2+1,\ [-1,3]$ satisfies all the conditions of the Mean Value Theorem and find *c*.

Polynomials are continuous throughout their domain, so the first condition is satisfied.
Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

$$f'(c)=3c^2-8c \qquad \frac{f(3)-f(-1)}{3-(-1)}=\frac{-8-(-4)}{4}=-1$$

$$3c^2-8c=-1$$
$$3c^2-8c+1=0$$
$$c=\frac{8\pm\sqrt{8^2-4(3)(1)}}{2(3)}$$

$$c=2.535 \ or \ 0.131$$

Rolle's Theorem is a specific a case of the Mean Value theorem, though Joseph-Louis Lagrange used it to prove the Mean Value Theorem. Therefore, Rolle's Theorem was used to prove all of the rules we have used to interpret derivatives for the last couple of years. It was a very useful theorem, but it is now something of a historical curiosity.

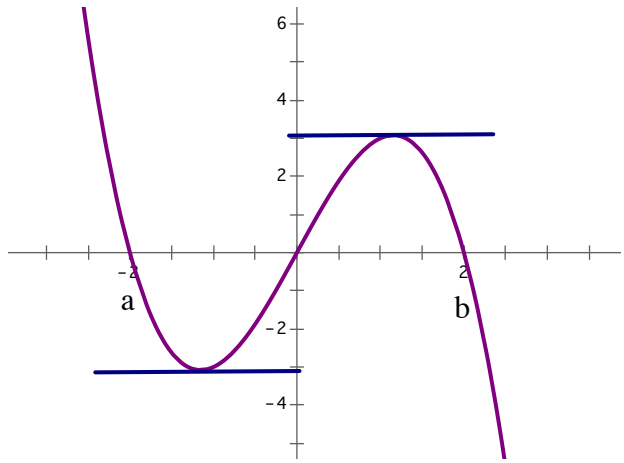
Rolle's Theorem

If f is a function that satisfies these three hypotheses

1. f is continuous on the closed interval $[a,b]$
2. f is differentiable on the closed interval (a,b)
3. $f(a)=f(b)$

Then there is a number c in the interval (a,b) such that $f'(c)=0$.

Written in this typically mathematical way, it is a bit confusing, but it basically says that if you have a continuous, smooth curve with the initial point and the ending point at the same height, there is some point in the curve that has a derivative of zero. If you look at this from a graphical perspective, it should be pretty obvious.



Ex 2 Show that the function $f(x)=x^2-4x+1,\ [0,4]$ satisfies all the conditions of the Mean Value Theorem and find c.

Polynomials are continuous throughout their domain, so the first condition is satisfied.
Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

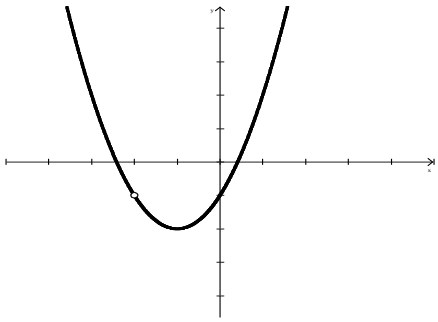
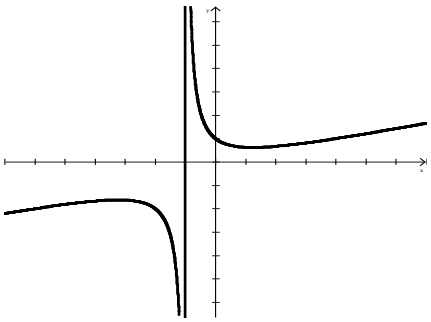
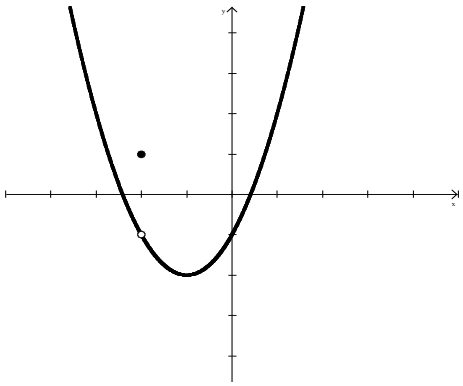
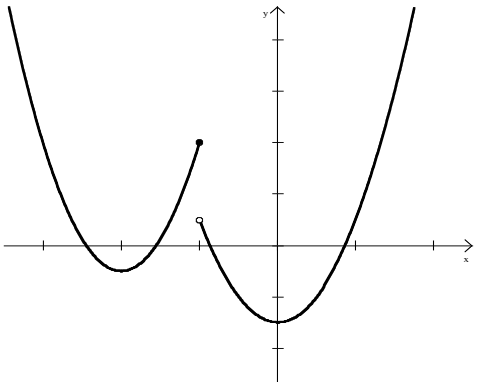
$$f'(c)=\frac{f(b)-f(a)}{b-a} \rightarrow f'(c)=\frac{f(4)-f(0)}{4-1}=0$$

$$f'(c)=2c-4=0$$

$$c=2$$

We need to consider when the MVT might not apply to a problem—that is, when is a function not continuous or not differentiable. We saw the first in PreCalculus, but did not always name it "discontinuity." Continuity basically means a function's graph has no breaks in it. The formal definition involved limits and we will explore that in the Limit Chapter.

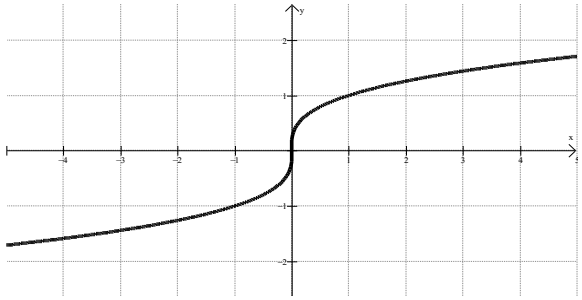
Since all the families of functions investigated in PreCalculus were continuous in their domain, it is easier to look at when a curve is discontinuous rather than continuous. There are four kinds of discontinuity:

	Removable Discontinuity ($\lim\limits_{x \to a} f(x)$ does exist)	***Essential Discontinuity*** ($\lim\limits_{x \to a} f(x)$ does not exist)
f(a) does not exist	 Point of Exclusion (POE)	 Vertical asymptotes
f(a) exists	 Point of Displacement (POD)	 Jump Discontinuity

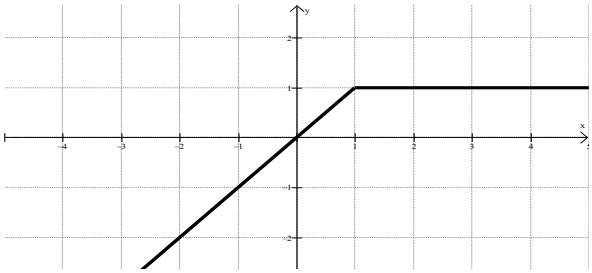
"Not differentiable" simply means the derivative does not exist. This can happen one of three ways.

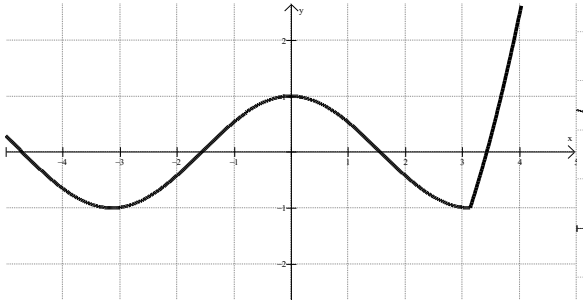
1. A function is not differentiable if it is not continuous.
   For pictures, see above.

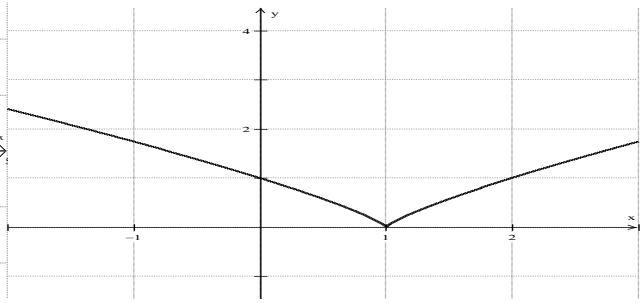
2. A function is not differentiable if the tangent line is vertical.



3. A function is not differentiable if it is not "smooth."







NB. All the families of functions studied in PreCaculus are continuous and differentiable in their domains.

Ex 3 To which of these functions does the Mean Value Theorem apply on the interval $\left[-1,3\right]$?

The MVT often arises in AP Table Problems like this one:

Ex 4 Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function.

(d) Explain why there are at least two times between $t=0$ and $t=20$ when $V'(t)=0$

(d) Explain why there are at least two times between $t=0$ and $t=20$ when $V'(t)=0$.

Because $V(t)$ is a continuous and differentiable function, the Mean Value Theorem applies. Since $V(0)=83=V(6)$, there must be a *c*-value between $t=0$ and $t=6$ where $V'(t)=0$. Similarly, since $V(13)=38=V(26)$, there must be a c-value between $t=13$ and $t=20$ where $V'(t)=0$. There might be more *c*-values where $V'(t)=0$, but the MVT guarantees at least two.

Other Theorems that can be confused with
The Mean Value Theorem

The Average Value Theorem: The average value of a function *f* on a closed interval [*a*, *b*] is defined as $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Average Rate of Change: The average rate of change of a function *f* on a closed interval [*a*, *b*] is defined as $= \frac{f(b)-f(a)}{b-a}$.

The Intermediate Value Theorem: If *f* is continuous on the closed interval [*a*, *b*] then $f(x)$ attains every height between $f(a)$ and $f(b)$.

One consequence of the Intermediate Value Theorem is that if $f(a)$ and $f(b)$ are opposite signs, there is a zero in the closed interval.

Ex 5 Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function. Explain why there are at least two times when $V(t) = 74$.

Because $V(t)$ is continuous and $68 < 74 < 83$, according to the IMT, there must be at least one time between $t = 0$ and $t = 4$ and at least one time between $t = 4$ and $t = 6$ when $V(t) = 74$.

4.2 Free Response Homework

Verify that the following functions fit all the conditions of the Mean Value Theorem, and then find all values of *c* that satisfy the conclusion of the Mean Value Theorem.

1. $f(x) = x^3 - 3x^2 + 2x + 5,\ [0, 2]$

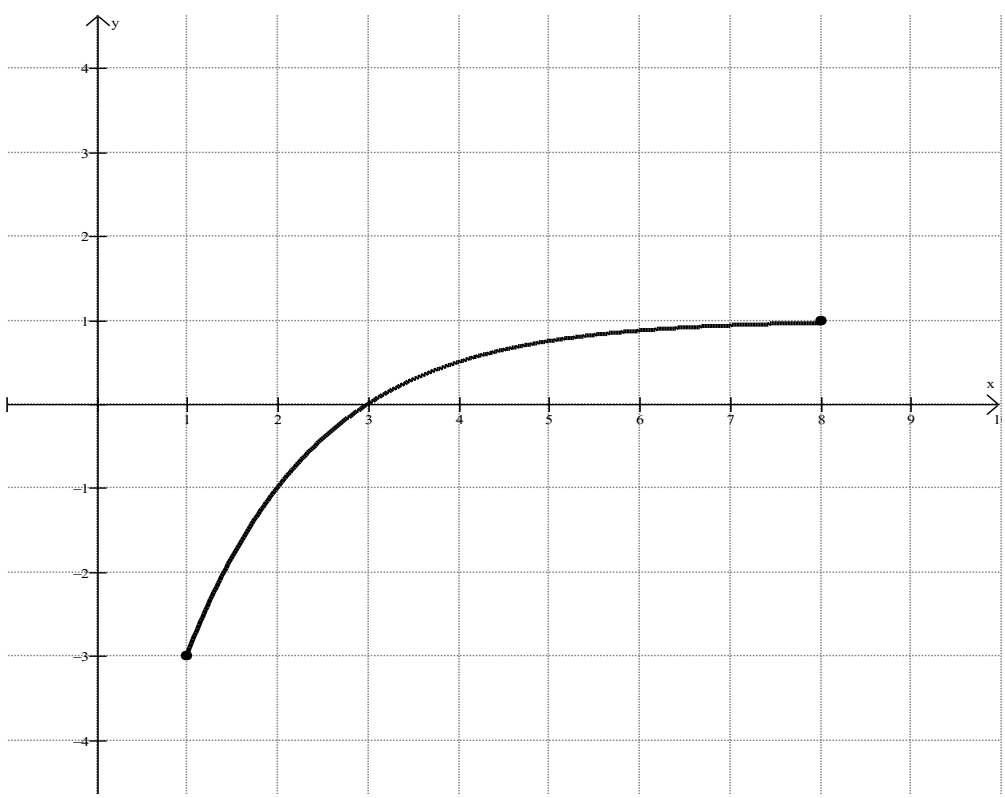
2. $f(x) = \sin(2\pi x),\ [-1, 1]$

3. $g(t) = t\sqrt{t+6},\ [-4, 0]$

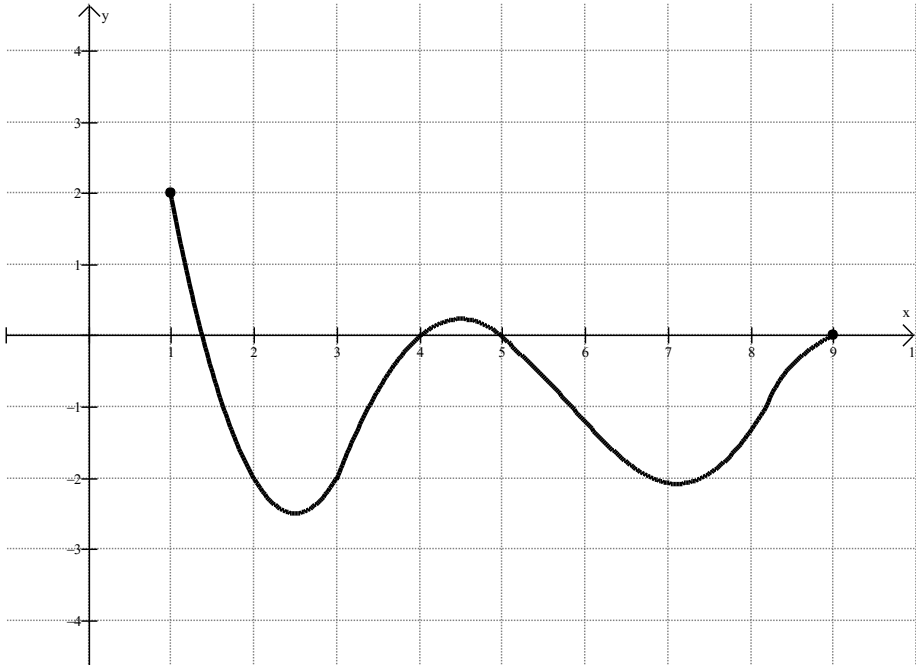
4. $H(t) = \dfrac{t}{(t-6)^2},\ [2, 8]$

Given the graph of the function below, estimate all values of c that satisfy the conclusion of the Mean Value Theorem for the interval [1,8].

5.



6. Given the graph of the function below, estimate all values of c that satisfy the conclusion of the Mean Value Theorem for the interval [1,9].



Determine if the following functions fit the conditions of the Intermediate Value Theorem, and determine the range of the results.

7. $f(x)=x^3-3x^2+2x+5,\ [0,4]$

8. $f(x)=\dfrac{\sin(2\pi x)}{x},\ [-1,1]$

9. $g(t)=t\sqrt{t+6},\ [-4,0]$

10. $H(t)=\dfrac{t}{\sqrt{t+6}},\ [2,8]$

11. On May 15, the weather in the town of Apcalc changes at a rate of $W(t)$ degrees Fahrenheit per hour. $W(t)$ is a twice-differentiable, increasing and concave up function with selected values in the table below. At midnight, t=0, the weather in Apcalc is 40 degrees Fahrenheit.

t (in hours since midnight)	0	1	3	6	8
$W(t)$ (in degrees Fahrenheit per hour)	-2.4	-2.1	-1.2	1.8	4.5

a) At approximately what rate is the rate of change of the temperature changing at 2am (t=2)? Include units.

b) Use a right Riemann sum with subdivisions indicated by the table to approximate $\int_0^8 W(t)\,dt$. Using correct units, explain the meaning of this value in the context of this problem.

c) Is there a time when the rate of change of the temperature equals 7? Justify your answer.

d) Is there a time in $0 \le t \le 8$ when $W(t) = 0$? Justify your answer.

12. Star Formation Rate (*SFR*) observations of red-shift allow scientists to track the gains and approximate future gains. Below is a table of such data:

t	0	1	2	3	4	5	6	7	8
SFR	0.0029	0.0051	0.0055	0.0049	0.0042	0.0035	0.0029	0.0025	0.0021

Assume the *SFR* data represents a continuous and differentiable function. *SFR* is measured in solar masses per giga-years (millions of years) per cubic parsecs and time *t* is measured in giga-years.

a) Use midpoint rectangles to approximate the total star formation. Using the correct units, explain the meaning of your result.

b) Is there a time when *SFR* = 0? Justify your answer.

c) Is there a time when *SFR* = 0.0056? Justify your answer.

13. AP Handout: AP Calc AB/BC 2004B #3, AP Calc AB 2002 #6

3.2 Multiple Choice Homework

1. Let f be a polynomial function with degree greater than 2. If $a \neq b$ and $f(a) = f(b) = 1$, which of the following must be true of at least one value of x between a and b?

I. $f(x) = 0$ II. $f'(x) = 0$ III. $f''(x) = 0$

a) I only b) II only c) III only

d) II and III only e) I, II, and III

---

2. Which of the following functions **fail** to meet the conditions of the Mean Value Theorem?

I. $3x^{2/3} - 1$ on $[-1, 2]$ II. $|3x - 2|$ on $[1, 2]$

III. $4x^3 - 2x + 3$ on $[-1, 2]$

a) I only b) II only c) III only d) I and II only e) II and III only

---

3. $y = -2x^2 + x - 2$ is defined on $x \in [1, 3]$. Find the c-value determined by the Mean Value Theorem.

a) $x = \frac{1}{2}$ b) $x = \frac{5}{4}$ c) $x = \frac{3}{2}$ d) $x = 2$

e) $x = \frac{9}{4}$

---

4. Let $g(x)$ be a continuous function on the interval $x \in [0, 1]$ and let $g(1) = 0$ and $g(0) = 1$. Which of the following statements is not necessarily true?

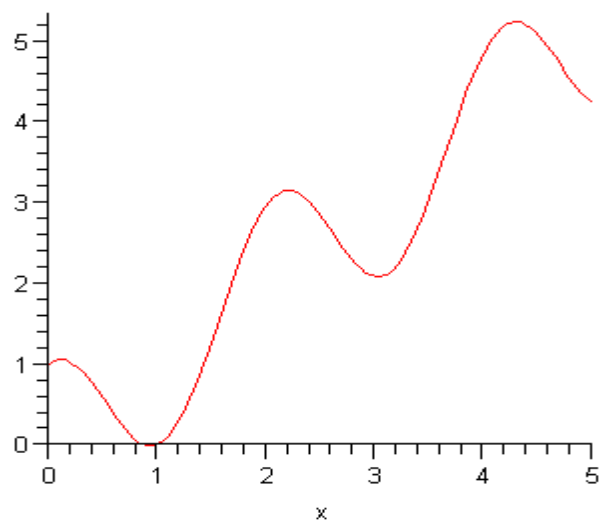
a) The exist a number c on $[0, 1]$ such that $g(c) \geq g(x)$ for all $x \in [0, 1]$

b) For all a and b in $[0, 1]$, if $a = b$, then $g(a) = g(b)$.

c) The exist a number c on $[0, 1]$ such that $g(c) = \frac{1}{2}$

d) The exist a number c on $[0, 1]$ such that $g(c) = \frac{3}{2}$

e) For all c in $(0, 1)$, $\lim_{x \to c} g(x) = g(c)$



5. The graph of a differentiable function f is shown above on the closed interval [0, 5]. How many values of x in the open interval (0, 5) satisfy the conclusion of the Mean Value Theorem for f on [0, 5] ?

a) Two b) Three c) Four d) Five

---

6. To which of the following phrases does the equation $Y = \frac{f(b) - f(a)}{b - a}$ pertain?

a) The Mean Value Theorem

b) The Average Value Theorem

c) The Average Rate of Change

d) The Intermediate Value Theorem

e) Rolle's Theorem

---

t hours	0	2	4	6	8	10	12	14
$E(t)$	10	4	10	5	7	3	7	5

7. The table above shows particular values of a differentiable function $E(t)$. At how many times on the interval $t \in [0, 12]$ does the Mean Value Theorem guarantee that $E'(t) = 0$

a) None b) One c) Two d) Three

3.3: The Second Derivative, Points of Inflection, and Concavity

Concavity--Means: The way a curve bends
Concave Up--Means: The curve bends counter-clockwise.
Concave Down--Means: The curve bends clockwise.
Inflection Value--Defn: The x-value at which the concavity of the curve switches. This can be a vertical asymptote.
Point of Inflection--Defn: The point at which the concavity of the curve switches from up to down or down to up.

The 2nd Derivative is what is used to analyze this Trait. The 2nd Derivative are to Points of Inflection what The 1st Derivative is to Extremes.

A Point of Inflection (POI) occurs if and only if

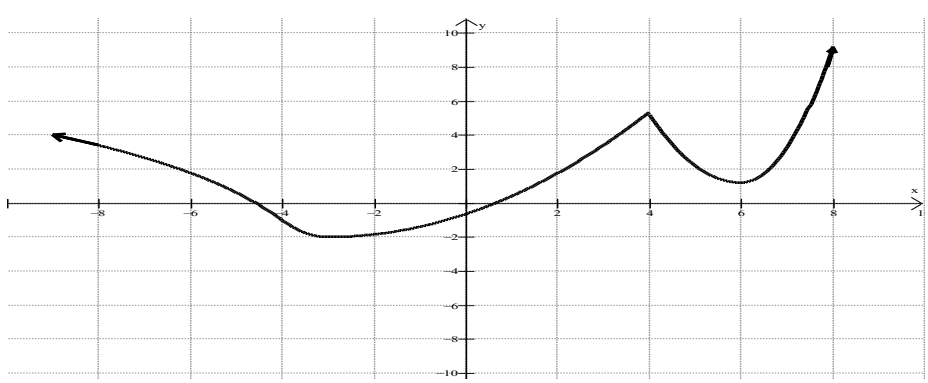
the sign of $f''(x)$ changes.

Also, Intervals of Concave Up or Down are comparable to Intervals of Increasing or Decreasing, respectively.

OBJECTIVE

Find Points of Inflection and Intervals of Concavity.

Take this graph, for instance:



x

It appears to be concave down from $(-\infty,-4)$, concave up from $(-4,4)\cup(4,\infty)$, more or less. So, $x=-4$ would be a point of inflection. $x=4$ would not be a point of inflection because the concavity does not change.

EX 1 Find the Points of Inflection of $y=x^3+4x^2-3x+1$

$$y'=3x^2+8x-3$$
$$y''=6x+8=0$$
$$x=-\frac{4}{3}$$

$$\begin{array}{c|ccc} y'' & - & 0 & + \\ \hline x & & -\frac{4}{3} & \end{array}$$

$$x=-\frac{4}{3}\rightarrow y=5.889$$

$\left(-\frac{4}{3},\ 5.889\right)$ is the POI.

EX 2 Find the Points of Inflection of $y=\dfrac{4x}{x^2+1}$

$$y'=\frac{(x^2+1)(4)-(4x)(2x)}{(x^2+1)^2}=\frac{-4x^2+4}{(x^2+1)^2}$$

$$y''=\frac{(x^2+1)^2(-8x)-(-4x^2+4)2(x^2+1)(2x)}{(x^2+1)^4}$$
$$=\frac{(8x)(x^2+1)\left[-(x^2+1)-(-2x^2+2)\right]}{(x^2+1)^4}$$
$$=\frac{(8x)(x^2-3)}{(x^2+1)^3}=0$$

$$x = 0, \pm\sqrt{3}$$

Check the sign pattern to make sure we have POIs and not something else.

$$\begin{array}{c|ccccccc} y'' & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -\sqrt{3} & & 0 & & \sqrt{3} & \end{array}$$

So all three x-values represent POIs, and those POIs are

$$\left(-\sqrt{3}, -\sqrt{3}\right), (0, 0), \textit{and } \left(\sqrt{3}, \sqrt{3}\right)$$

EX 3 Given these two sign patterns:

$$\begin{array}{c|ccccccccc} dy/dx & - & 0 & + & 0 & - & 0 & - & 0 & + \\ \hline x & & -9 & & -3 & & -1 & & 7 & \end{array}$$

$$\begin{array}{c|ccccccccc} d^2y/dx^2 & + & 0 & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -7 & & -2 & & 0 & & 5 & \end{array}$$

a) On what interval(s) is y decreasing?
b) On what interval(s) is y concave up?
c) On what interval(s) is y both increasing and concave down?

a) $\left(-\infty, -9\right) \cup \left(-3, -1\right) \cup \left(-1, 7\right)$

b) $\left(-\infty, -7\right) \cup \left(-2, 0\right) \cup \left(5, \infty\right)$?

c) $\left(-7, -3\right)$

EX 4 Find the Intervals of Concavity for $y = xe^{2x}$

$$\frac{dy}{dx} = x\left(e^{2x}(2)\right) + e^{2x}(1) = e^{2x}(2x+1)$$

$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x+1)\left(e^{2x}(2)\right)$$

$$= e^{2x}\left[(2) + (2x+1)(2)\right]$$

$$= e^{2x}(4x+4) = 0$$

The critical value for the POI is $x = -1$ and there are no Vertical Asymptotes.

So

$$\begin{array}{ccccc} y'' & - & 0 & + & \\ x & & -1 & & \end{array}$$

and

y is concave down on $x \in (-\infty,\ -1)$ *and*
y is concave up on $x \in (-1,\ \infty)$.

3.3 Homework

1. Given this sign pattern for the derivative of $F(x)$, on what interval(s) is $F(x)$ concave down?

$F''(x)$	$-$	0	$+$	0	$-$	0	$+$
x		-3		0		3	

2. The sign pattern for the derivative of $H(x)$ is given. (a) Is $x=-4$ a point of inflection? (b) Is $x=-1$ a point of inflection?

d^2H/dx^2	$+$	0	$-$	0	$-$	0	$+$
x		-4		-1		2	

Find Points Of Inflection and Intervals of Concavity.

3. $y=x^3+x^2-7x-15$

4. $y=3x^4-20x^3+42x^2-36x+16$

5. $y=\dfrac{-4x}{x^2+4}$

6. $y=\dfrac{x^2-1}{x^2-4}$

7. $y=Ln\left(4x-x^3\right)$

8. $y=x\sqrt{8-x^2}$

9. $y=\dfrac{1}{2}x+\sin x \;\; on \;\; x\in(0,\ 2\pi)$

10. $y=xe^{-x}$

11. $y=e^{-x^2}$

12. $y=\dfrac{x}{x^2-9}$

13. $y=2x-x^{2/3}$

3.3 Multiple Choice Homework

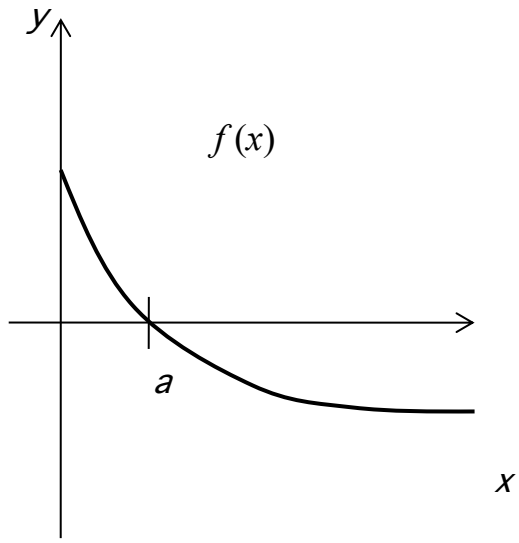
1. $f(x) = x^3 - 6x^2$ is concave up when

a) $x > 2$ b) $x < 2$ c) $0 < x < 4$

d) $x < 0$ or $x > 4$ e) $x > 6$

---

2. Given the twice differentiable function f, which of the following statements is true?



a). $f'(a) < f''(a) < f(a)$

b). $f'(a) < f(a) < f''(a)$

c). $f''(a) < f(a) < f'(a)$

d). $f''(a) < f'(a) < f(a)$

f) $f''(a) < f(a) < f'(a)$

---

3. For $x > 0$, let $f'(x) = \frac{\ln x}{x}$ and $f''(x) = \frac{1 - \ln x}{x^2}$. Which of the following is true?

a) f is decreasing for $x > 1$ and the graph of f is concave down for $x > e$

b) f is decreasing for $x > 1$ and the graph of f is concave up for $x > e$

c) f is increasing for $x > 1$ and the graph of f is concave down for $x > e$

d) f is increasing for $x > 1$ and the graph of f is concave up for $x > e$

e) f is increasing for $0 < x < e$ and is concave down for $0 < x < e^{3/2}$

---

4. Consider the function $f(x) = \left(x^2 - 5\right)^3$ for all real numbers x. The number of inflection points for the graph of f is

a) 1 b) 2 c) 3 d) 4 e) 5

---

5. If $\frac{d}{dx}\left[f(x)\right] = g(x)$ and $\frac{d}{dx}\left[g(x)\right] = f(3x)$, then $\frac{d^2}{dx^2}\left[f\left(x^2\right)\right]$ is

a) $4x^2 f\left(3x^2\right) + 2g\left(x^2\right)$ b) $f\left(3x^2\right)$ c) $f\left(x^4\right)$

d) $2xf\left(3x^2\right) + 2g\left(x^2\right)$ e) $2xf\left(3x^2\right)$

---

6. If $y = e^{kx}$, then $\frac{d^5 y}{dx^5} =$

a) $k^5 e^x$ b) $k^5 e^{kx}$ c) $5!e^{kx}$ d) $5!e^x$ e) $5e^{kx}$

---

7. The graph of $y = 3x^5 - 10x^4$ has an inflection point at

a) $(0,\ 0)$ and $(2,\ -64)$ b) $(0,\ 0)$ and $(3,\ -81)$

c) $(0,\ 0)$ only d) $(3,\ -81)$ only e) $(2,\ -64)$ only

---

8. An equation of the line tangent to the graph of $y = x^3 + 3x^2 + 2$ at its point of inflection is

a) $y = -3x + 1$ b) $y = -3x + 7$ c) $y = x + 5$

d) $y = 3x + 1$ e) $y = 3x + 7$

---

9. The number of inflection points for the graph of $y = 2x + \cos\left(x^2\right)$ in the interval $0 \le x \le 5$ is

a) 6 b) 7 c) 8 d) 9 e) 10

---

10. On which of the following intervals is the graph of the curve $y = x^5 - 5x^4 + 10x + 15$ concave up?

I. $x < 0$ II. $0 < x < 3$ III. $x > 3$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

11. Let f be a function with a second derivative given $f''(x) = x^2(x-3)(x-6)$. What are the x-coordinates of the points of inflection of the graph of f?

a) 0 only b) 3 only c) 0 and 6 only

d) 3 and 6 only e) 0, 3, and 6

---

12. The derivative of the function is given by $f'(x) = x^2 \cos(x^2)$. How many points of inflection does the graph of f have on the open interval $(-2,\ 2)$?

a) One b) Two c) Three d) Four e) Five

---

13. How many points of inflection does the graph of $y = \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x$ have on the interval $0 \le x \le \pi$?

a) 1 b) 2 c) 3 d) 4 e) 5

3.4: Graphing with Derivatives

OBJECTIVE

Sketch the graph of a function using information from its first and/or second derivatives.
Sketch the graph of a first and/or second derivative from the graph of a function.

All last year, we concerned ourselves with sketching graphs based on traits of a function. We tended to look at the one key aspect of the derivative – that is finding extremes – as it applied to a function. Toward the end of the year, we looked at the first and second derivatives as traits of the function. They gave a much wider range of information than specific details.

Remember:

Critical values representing extremes of a function occur when

i. $f'(x)=0$

ii. $f'(x)$ does not exist

or iii. at endpoints of an arbitrary domain.

If $f'(x)>0$, then $f(x)$ is increasing.

If $f'(x)<0$, then $f(x)$ is decreasing.

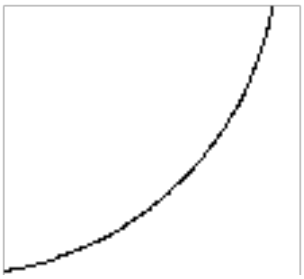
Critical values representing a Point of Inflection (POI of a function occur when

i. $f''(x)=0$

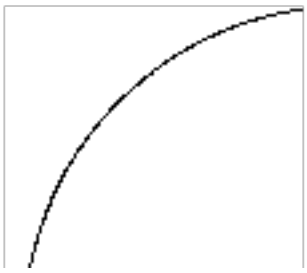
or ii. $f''(x)$ does not exist

If $f''(x)>0$, then $f(x)$ is concave up.

If $f''(x)<0$, then $f(x)$ is concave down.

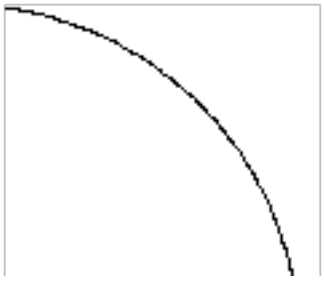


Increasing and Concave Up



Increasing and Concave Down

Decreasing and Concave Up



Decreasing and Concave Down

Ex 1 Find the sign patterns of y, y', and y'' and sketch $y = xe^{2x}$

Zeros: $xe^{2x} = 0 \rightarrow x = 0$

$x = 0 \rightarrow y = 0$

$$\begin{array}{c} y \\ x \end{array} \begin{array}{ccc} - & 0 & + \\ \longleftarrow & \underset{0}{\longrightarrow} & \end{array}$$

Extremes: $\dfrac{dy}{dx} = xe^{2x}(2) + e^{2x}(1)$

$$xe^{2x}(2) + e^{2x}(1) = 0$$

$$e^{2x}(2x+1) = 0$$

$$x = -\frac{1}{2}$$

$$y = -.184$$

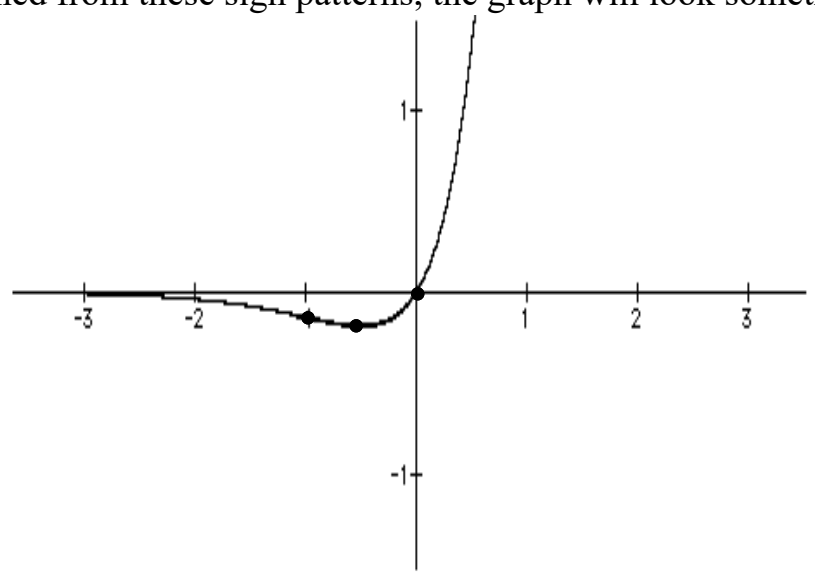
$$\begin{array}{ccc} y' & - \quad 0 & + \\ x & \longleftarrow\!\longrightarrow & \\ & -\frac{1}{2} & \end{array}$$

POI

$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x+1)\left(e^{2x}(2)\right)$$
$$= e^{2x}(4x+4) = 0$$
$$x = -1 \rightarrow y = -.135$$

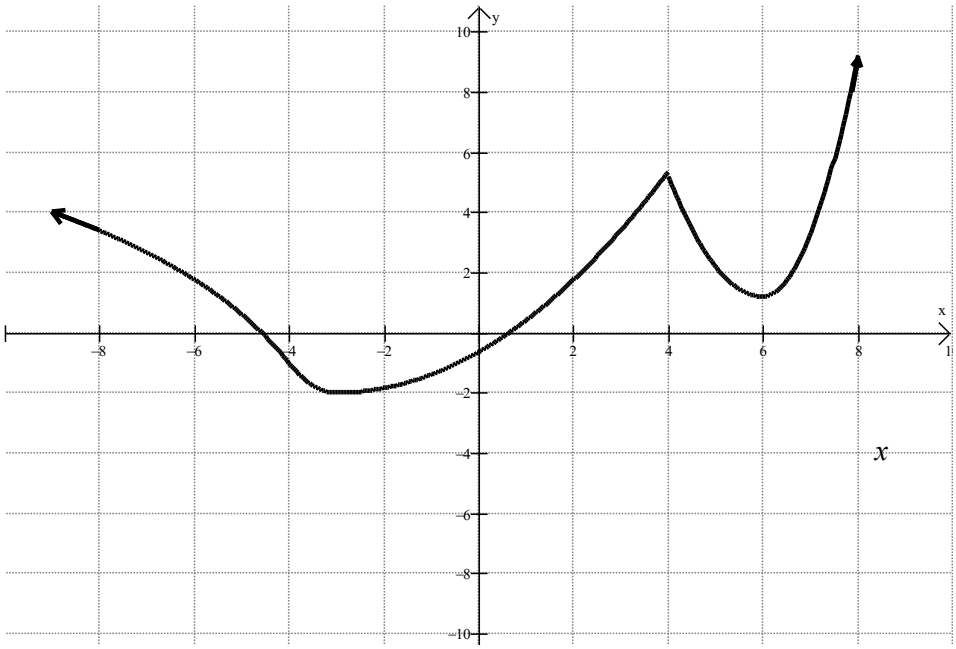
$$\begin{array}{ccc} y'' & - \quad 0 & + \\ x & \longleftarrow\!\longrightarrow & \\ & -1 & \end{array}$$

Putting together the points, increasing/decreasing and concavity that can be determined from these sign patterns, the graph will look something like this:



$y = xe^{2x}$

Ex 2 Sketch the function described as follows: decreasing from $(-\infty,-3)\cup(4,6)$, increasing from $(-3,4)\cup(6,\infty)$, concave down from $(-\infty,-4)$, concave up from $(-4,4)\cup(4,\infty)$.



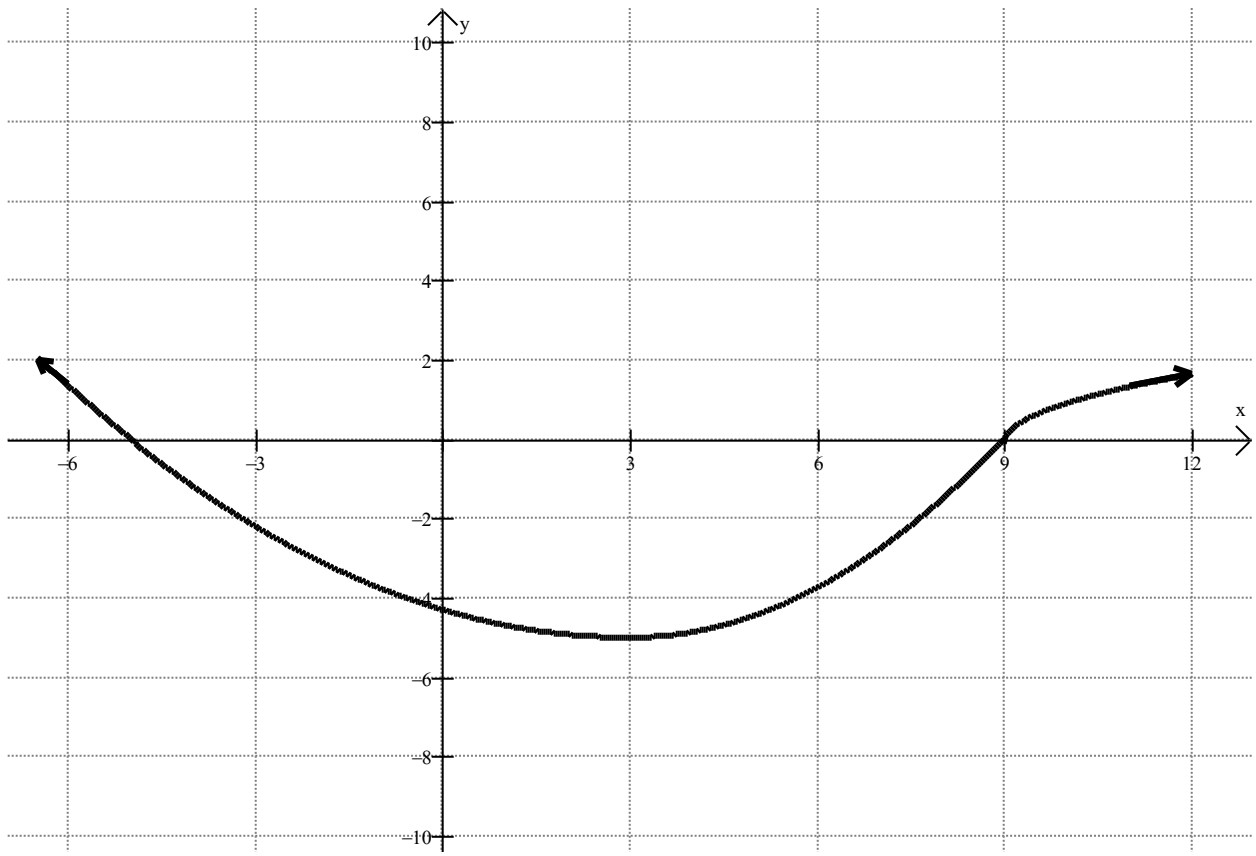
Note that this is only one possible answer. Since no y-values are given, the points could be at any height.

The trait and sign pattern information could be given in table form.

Ex 3 AP 2005 AB #4

Ex 4 Sketch the graph of the function whose traits are given below.

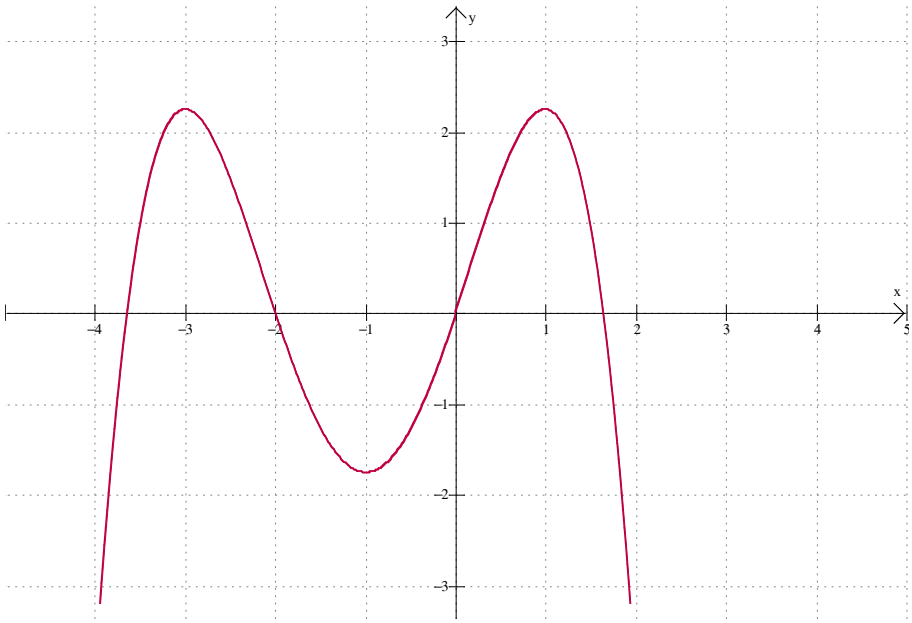
x	$f(x)$	$f'(x)$	$f''(x)$
x<–5	Positive	Negative	Positive
x= –5	0	Negative	Positive
–5<x<3	Negative	Negative	Positive
x=3	–5	0	Positive
3<x<9	Negative	Positive	Positive
x=9	0	Positive	0
9<x	Positive	Positive	Negative



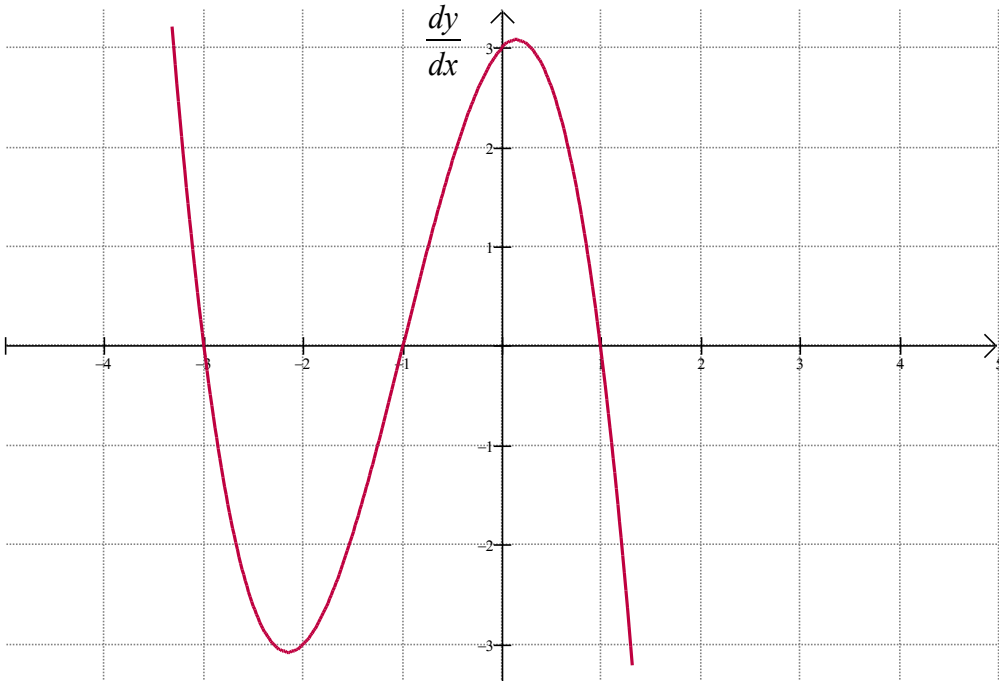
Notice that the point of inflection just happens to occur at a zero of the function. It is very possible that the traits can occur at the same point; you can conceive of functions that have maximums that are also points of inflection, minimums that are also zeroes, etc.

The derivatives also have graphs and we can discern what they look like from the information gleaned from the original graph.

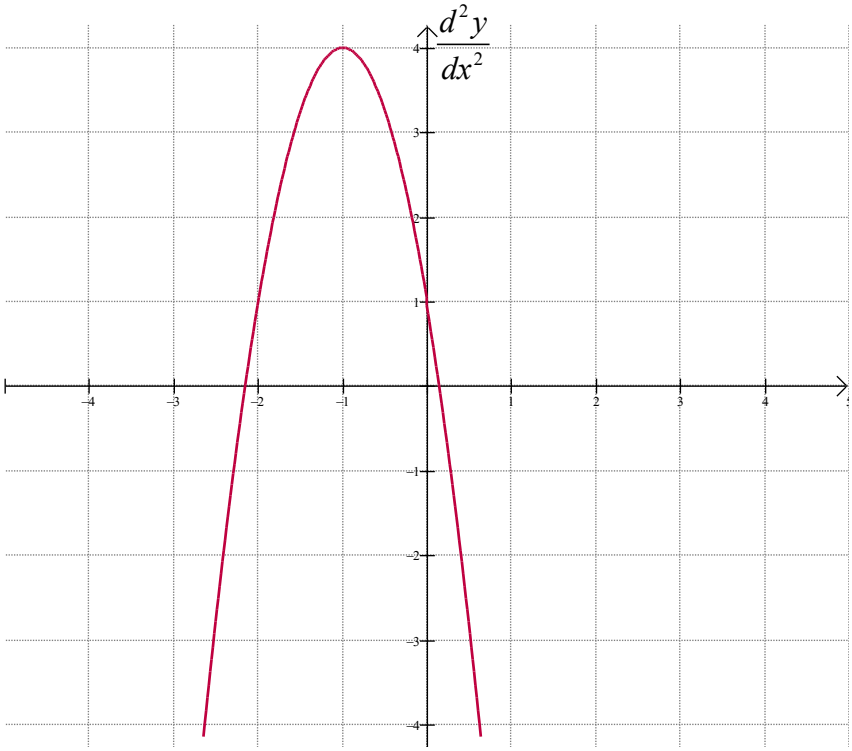
Ex 5 Sketch a possible graph of the first and second derivative of the function shown below.



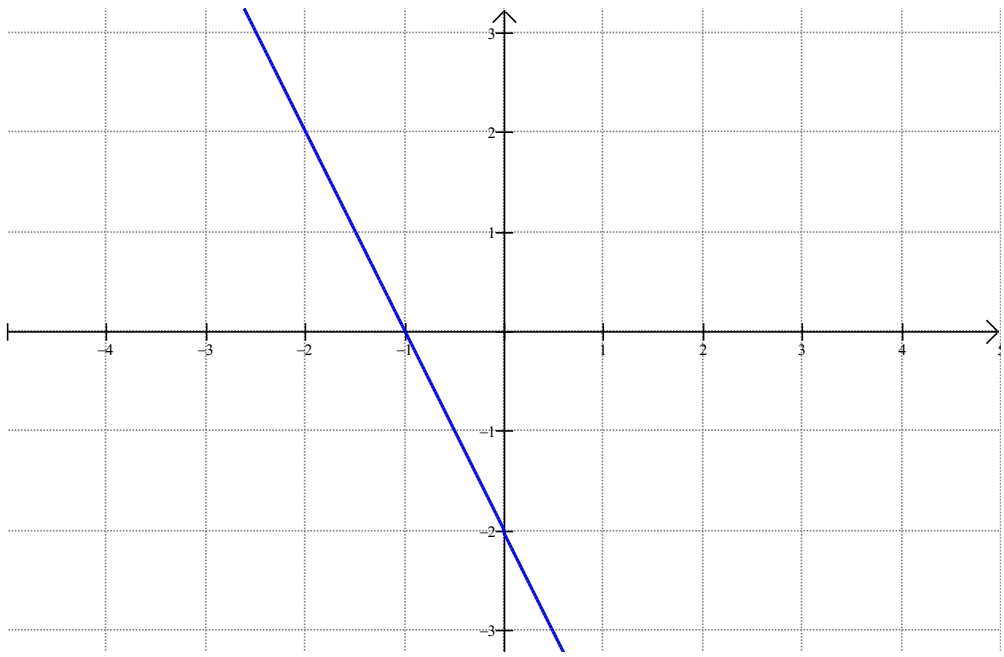
Notice we have critical values at x = –3, –1, and 1. These should be zeroes on the graph of the derivative. We can also see where the graph is increasing, and where it is decreasing; these regions correspond to where the graph of the derivative should be positive and negative, respectively.



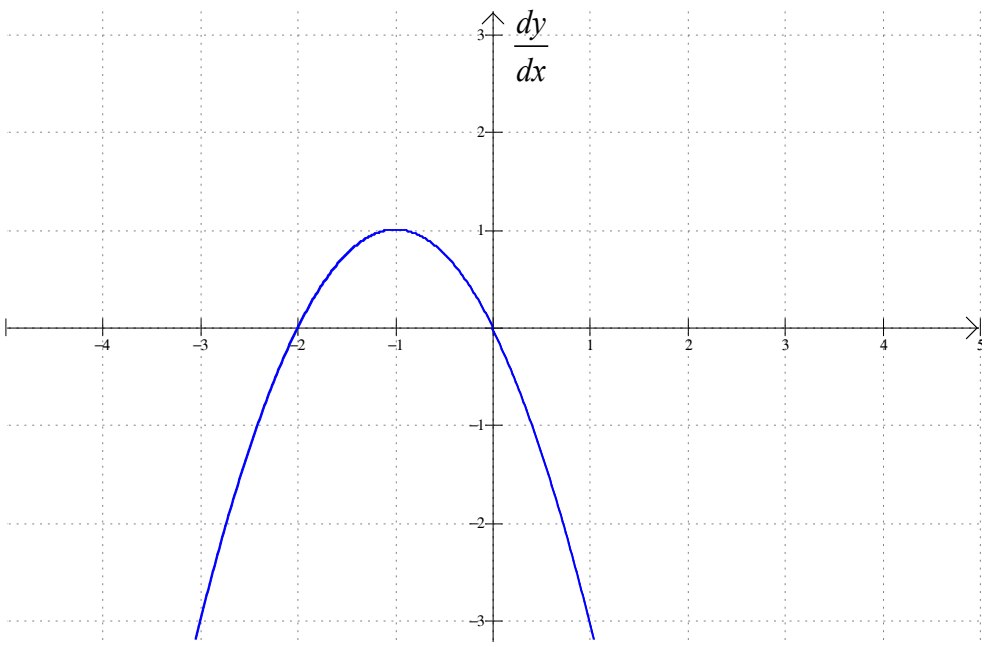
Notice that the points of inflection on the graph of y are maximums or minimums on the graph of the derivative. Those points will also be zeroes on the second derivative.



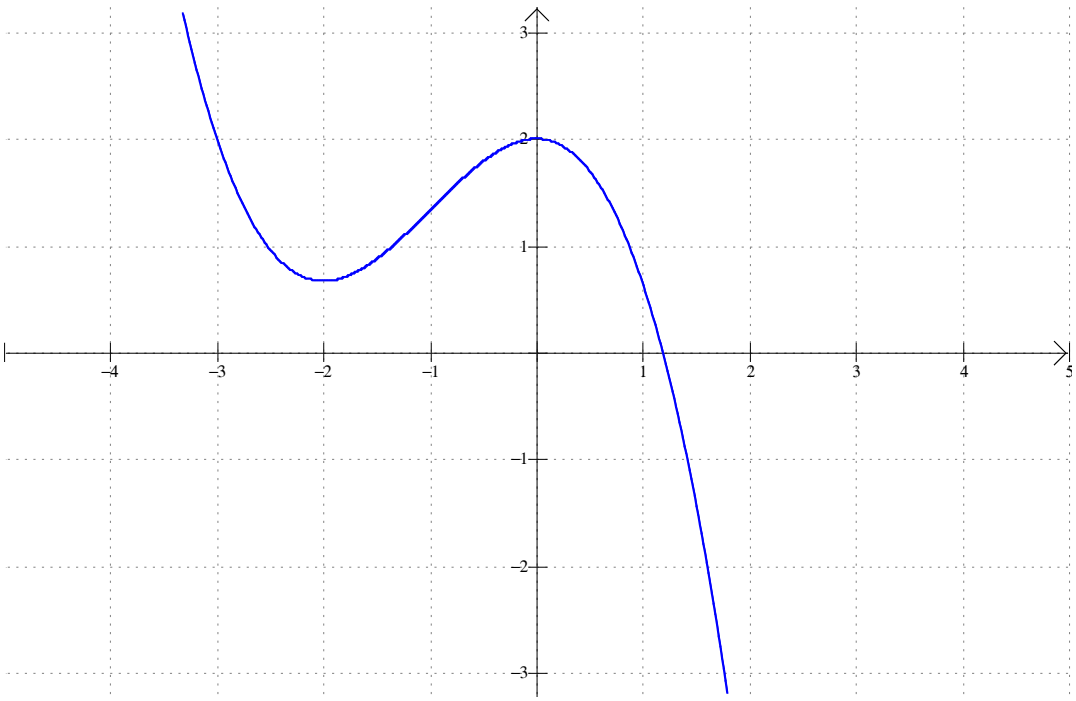
Ex 6 Sketch the possible graphs of a function and its first derivative given the graph of the second derivative below.



Looking at this graph, we can see a zero at x = –1, and the graph is positive for x < –1, and negative for x > –1. This should correspond to the graph of the first derivative increasing, then decreasing, with a maximum at x = –1.



Notice that this looks like a parabola. Since its derivative was a line this should make sense. However, we don't actually know the height of the maximum, nor do we actually know where the zeroes of the derivative are, or even if there are any. This is a sketch that is one of many possible functions that could have had the previous graph as its derivative. We will discuss this further when we discuss *integration*.



Again, the zeroes of the previous graph are extremes of this graph, and the zero on the initial graph is a point of inflection on this graph.

3.4 Free Response Homework

Sketch these graphs using the sign patterns of the derivatives.

1. $y = 3x^4 - 15x^2 + 7$

2. $y = x^3 + 5x^2 + 3x - 4$

3. $y = \dfrac{x+1}{x^2 - 2x - 3}$

4. $y = \dfrac{x^2 - 1}{x^2 + x - 6}$

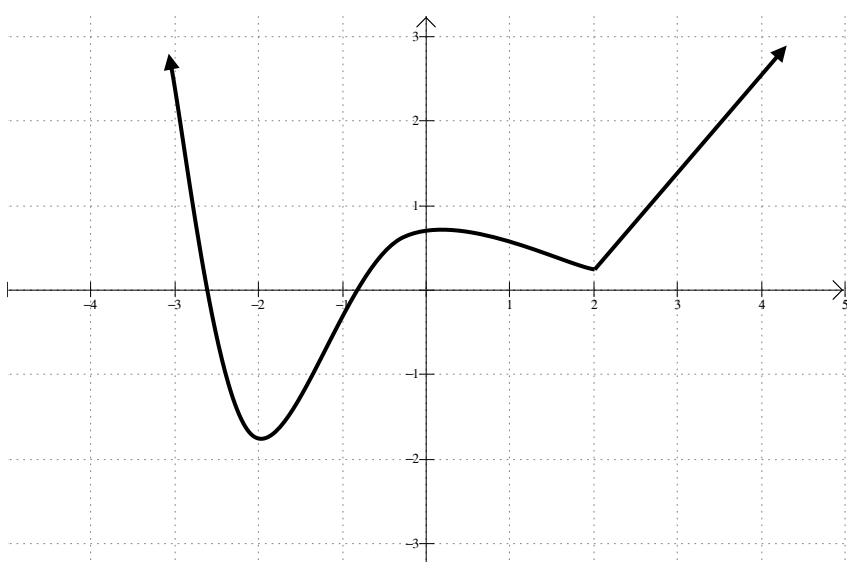
5. $y = 5x^{2/3} - x^{5/3}$

6. $y = \left(x^2\right)\sqrt{4 - x}$

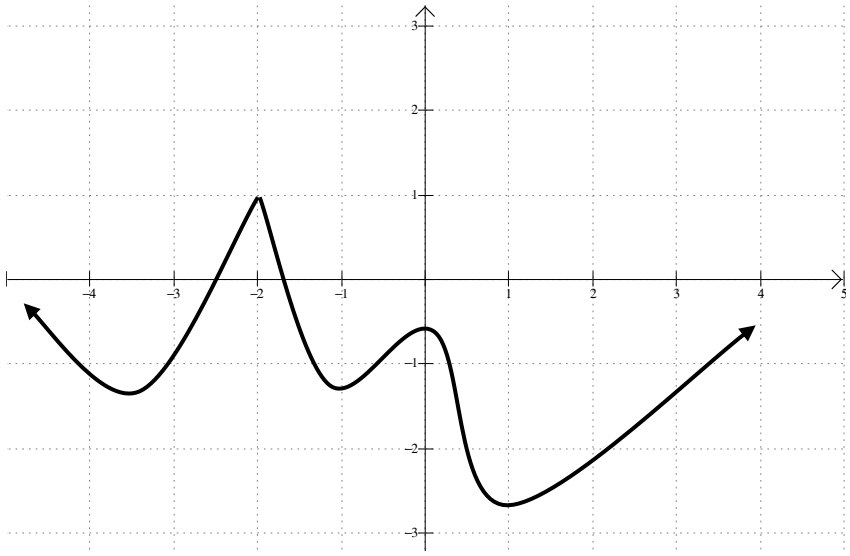
7. $y = x^2 e^{-x}$

Sketch the graph of the first and second derivatives for the function shown in the graphs below.

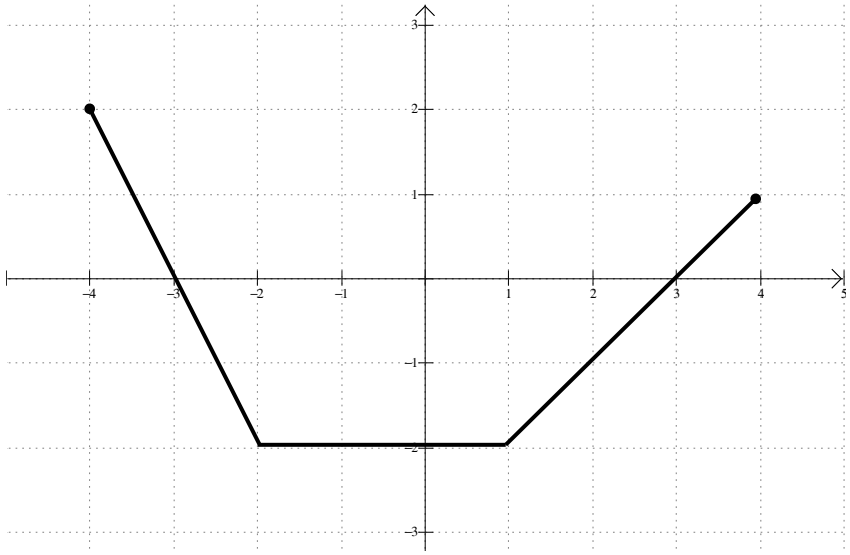
8.



9.



10.



Sketch the possible graph of a function that satisfies the conditions indicated below

11. Increasing from $(-\infty,5)\cup(7,10)$, decreasing from $(5,7)\cup(10,\infty)$

12. Increasing from $(-\infty,-3)\cup(5,\infty)$, decreasing from $(-3,5)$, concave up from $(-\infty,-2)\cup(2,\infty)$, concave down from $(-2,2)$

13. Decreasing from $(-\infty,-5)\cup(5,\infty)$, increasing from $(-5,5)$, concave down from $(-\infty,-7)\cup(-3,3)\cup(7,\infty)$, concave up from $(-7,-3)\cup(3,7)$

14. Increasing and concave up from (2,4), decreasing and concave down from (4,7), increasing and concave up from (7,10), with a domain of [2,10).

Sketch the possible graph of a function that has the traits shown below

15.

x	$f(x)$	$f'(x)$	$f''(x)$
x<2	Positive	Negative	Positive
x=2	0	Negative	Positive
2<x<3	Negative	Negative	Positive
x=3	–5	0	Positive
3<x<5	Negative	Positive	Positive
x=5	0	Positive	0
5<x<7	Positive	Positive	Negative
x=7	9	0	Negative
7<x<9	Positive	Negative	Negative
x=9	1	Negative	0
9<x<10	Positive	Negative	Positive
x=10	0	Negative	Positive
10<x	Negative	Negative	Positive

16.

x	$f(x)$	$f'(x)$	$f''(x)$
x<–1	Positive	Positive	Positive
x= –1	3	DNE	DNE
–1<x<4	Positive	Negative	Positive
x=4	1	0	Positive
4<x<9	Positive	Positive	Positive
x=9	3	DNE	DNE
9<x	Positive	Positive	Positive

17. Absolute minimum at 1, absolute maximum at 3, local minima at 4 and 7, local maximum at 6

18. Absolute minimum at 2, absolute maximum at 7, local minima at 4 and 6, local maxima at 3 and 5

19. Absolute maximum at 1, absolute minimum at 7, no local maxima

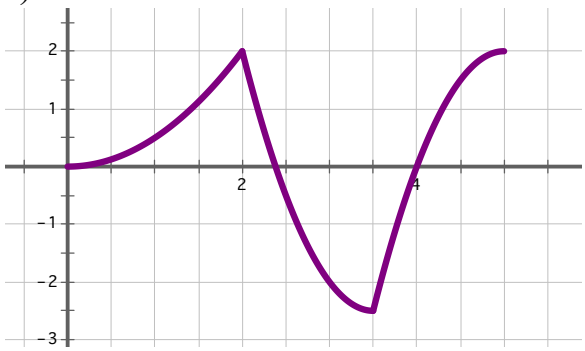
20. Absolute minima at 1 and 7, absolute maximum at 4, no local maxima

3.4 Multiple Choice Homework

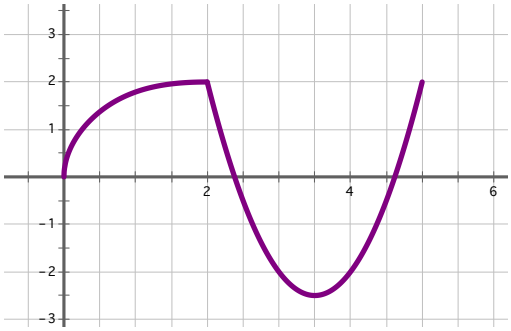
1. Which of the following graphs has these two sign patterns:

f'	+	*dne*	−	dne	+	f''	+	*dne*	+	dne	−
x		2		3.5		x		2		3.5	

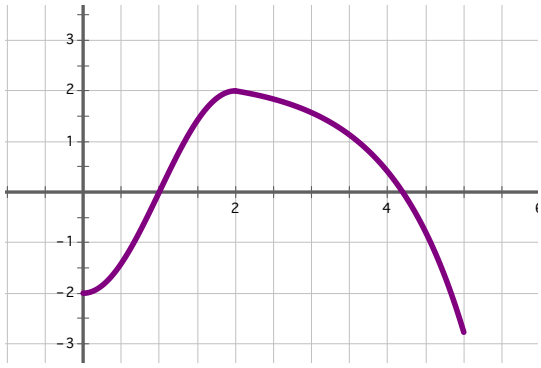
a)



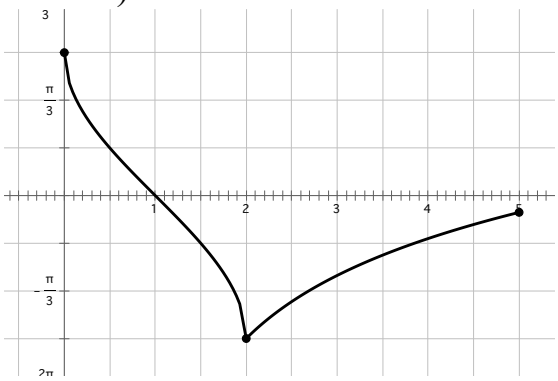
b)



c)



d)

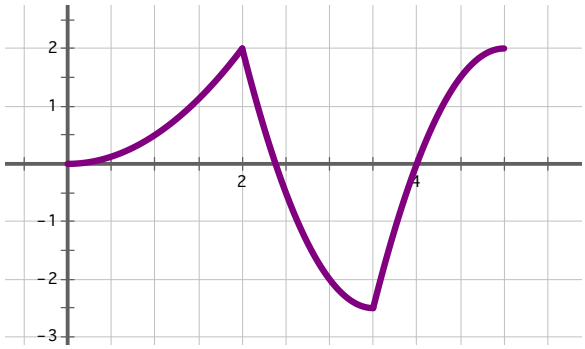


2. Which of the following graphs has these two sign patterns:

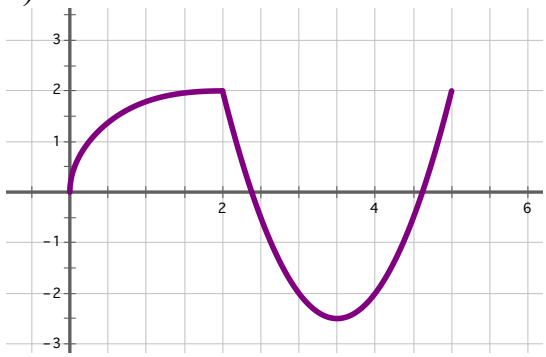
f'	$-$	dne	$+$	
x		2		

f''	$+$	0	$-$	dne	$-$
x		1		2	

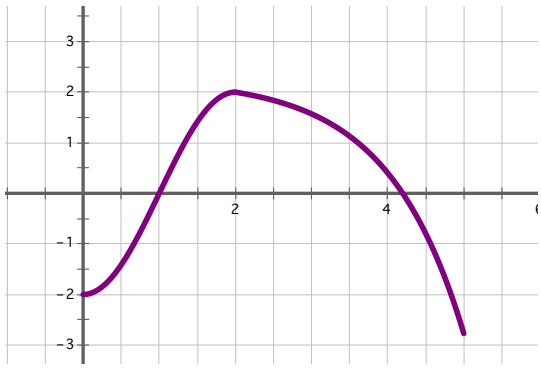
a)



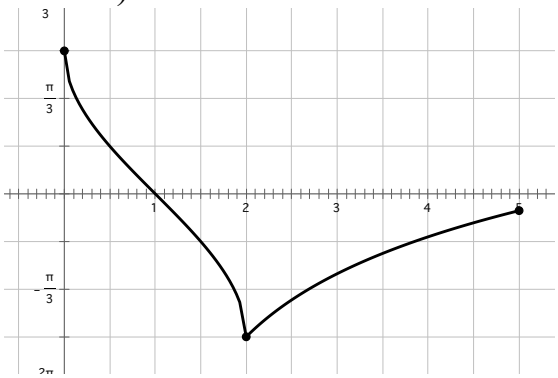
b)



c)



d)

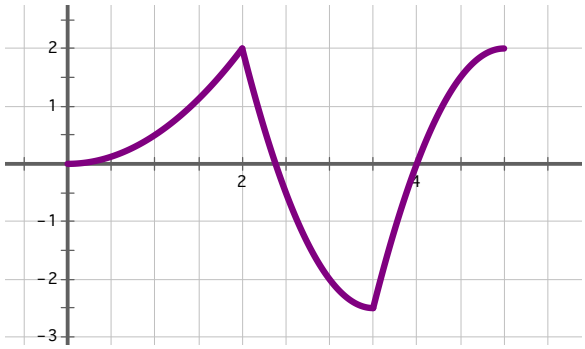


3. Which of the following graphs has these two sign patterns:

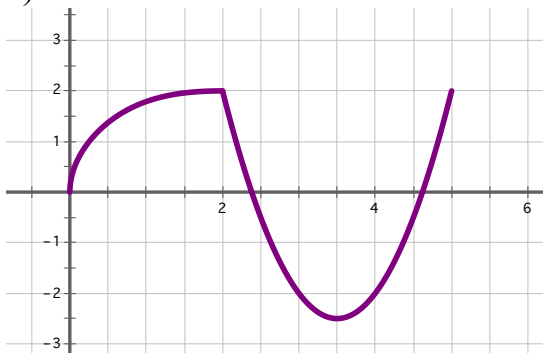
f'	$+$	dne	$-$	
x		2		

f''	$+$	0	$-$	dne	$-$
x		1		2	

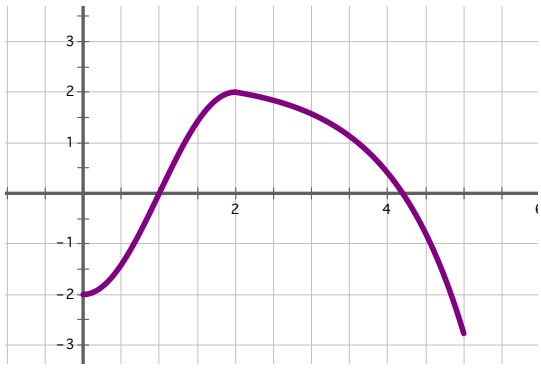
a)



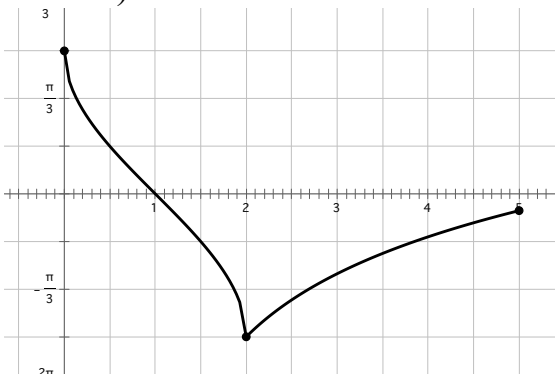
b)



c)



d)

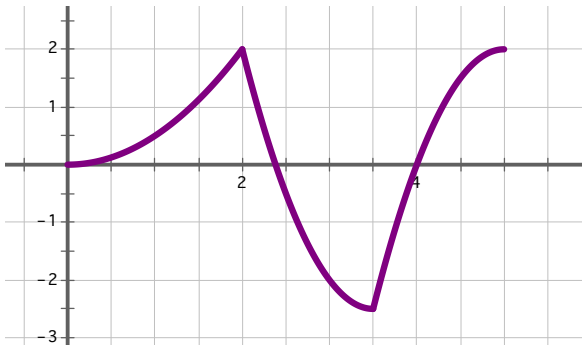


4. Which of the following graphs has these two sign patterns:

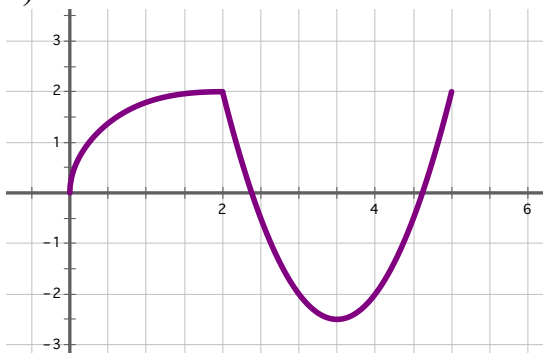
f'	$+$	*dne*	$-$	dne	$+$	
x		2		3.5		

f''	$-$	*dne*	$+$
x		2	

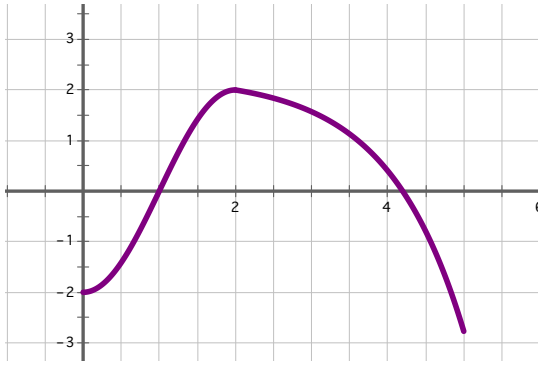
a)



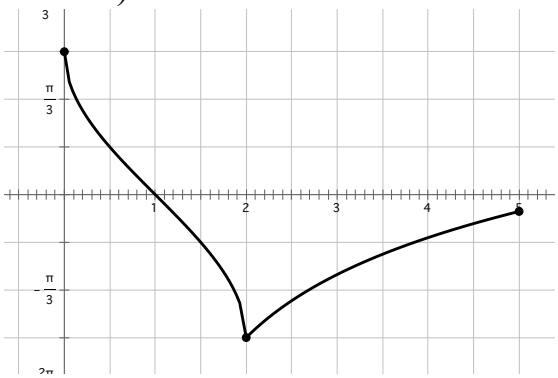
b)



c)



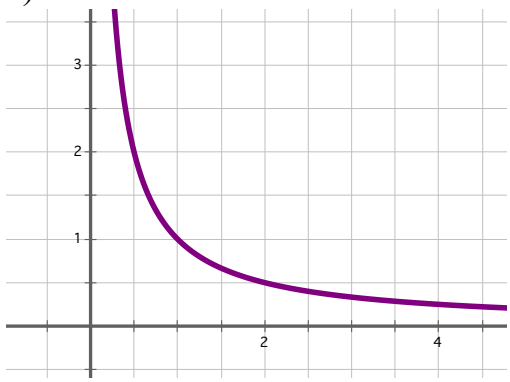
d)



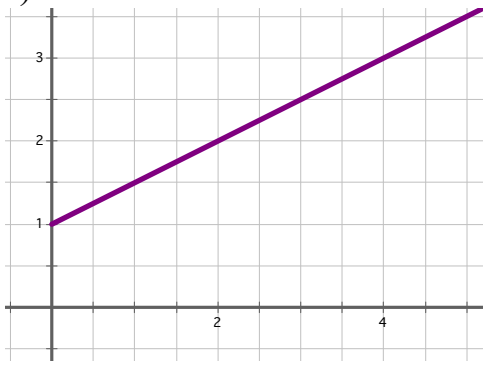
5. Which of the graphs of $y = g(x)$ below has $g'(x) < 0$ and $g''(x) > 0$?

a)

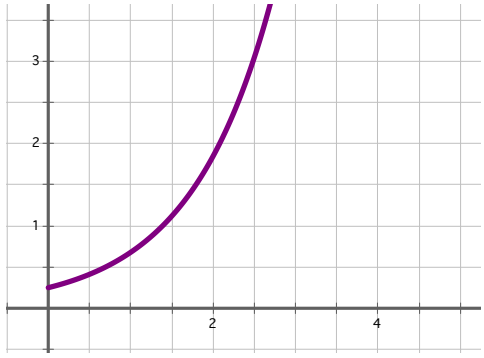
b)



c)



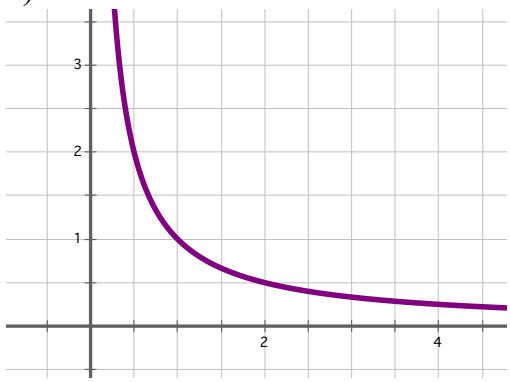
d)



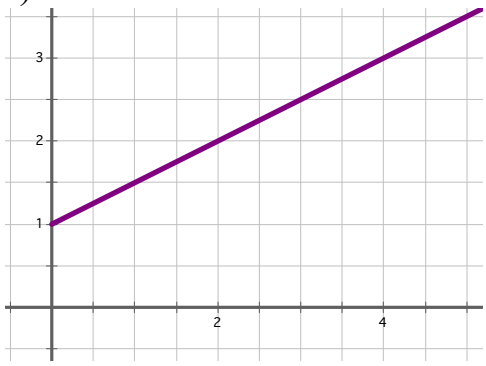
6. Which of the graphs of $y = g(x)$ below has $g'(x) > 0$ and $g''(x) > 0$?

a)

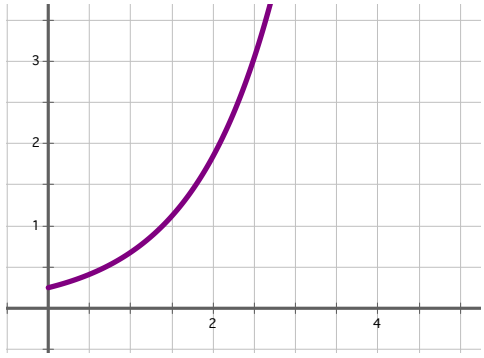
b)



c)



d)



3.5: Graphical Analysis I

In the last section, we looked at graphing functions and derivatives, but now we will reverse that process. As we noted in the last example of the last section, there is a layering and parallelism between the function, its derivative and its second derivative. The zeros and signs of one tell us about increasing, decreasing, and extremes or the concavity and POIs of another.

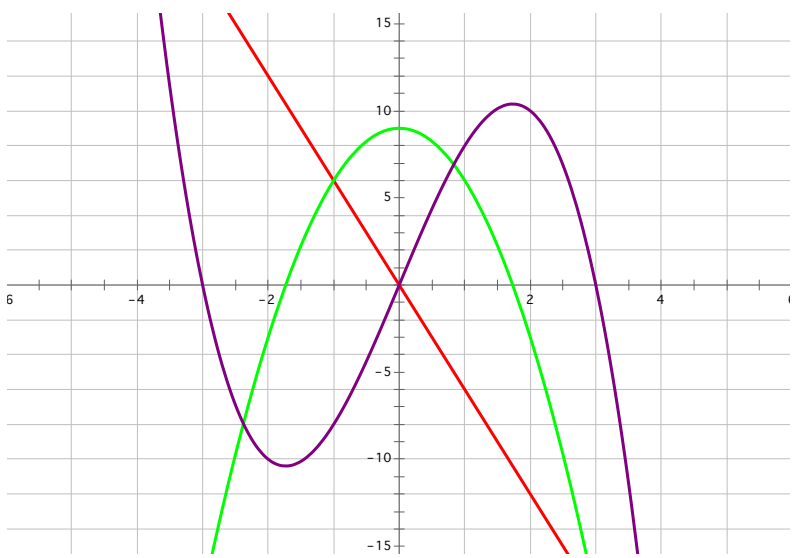
EX 1 Find the sign patterns of $f(x)=9x-x^3$, $f'(x)=9-3x^2$, and $f''(x)=-6x$ and compare them.

y	$+$	0	$-$	0	$+$	0	$-$
x		-3		0		3	

y'	$-$	0	$+$	0	$-$
x		$-\sqrt{3}$		$\sqrt{3}$	

y''	$+$	0	$-$
x		0	

Compare the graphs below of $f(x)=9x-x^3$, $f'(x)=9-3x^2$, and $f''(x)=-6x$.



It can be seen that the extreme points of $f(x)$ line up with the zeros of $f'(x)$, and the extreme point of $f'(x)$ lines up with the zero of $f''(x)$ and the POI of $f(x)$. Similarly, the positive parts of $f''(x)$ match the increasing part of $f'(x)$ and the concave up part of $f(x)$.

OBJECTIVES
Interpret information in the graph of a derivative in terms of the graph of the "original" function.

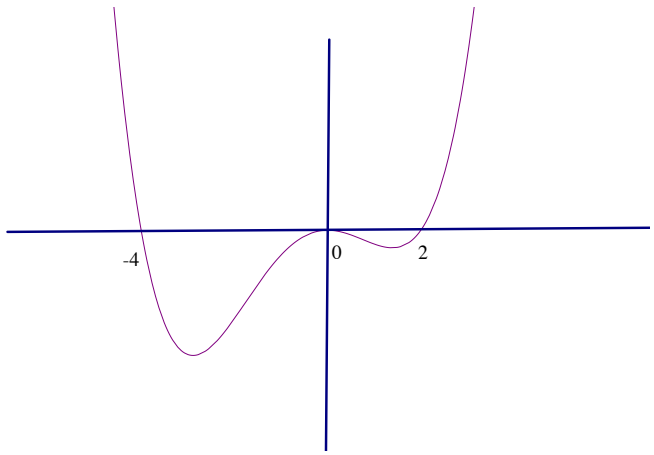
That interconnectedness between a function, its first derivative and its second derivative can be summarized thus:

	$f(x)$			$f'(x)$			$f''(x)$		
	+	**0**	-	+	**0**	-	+	**0**	-
$f(x)$	y value positive	y value zero	y value negative	Interval of increasing	max. or min.	Interval of decreasing	Concave up	POI	Concave down
$f'(x)$				y' value positive	y' value zero	y' value negative	Interval of increasing	max. or min.	Interval of decreasing
$f''(x)$							y'' value positive	y'' value zero	y'' value negative

The relationship presented here is upward. The zeros of a function refer to the extremes of the one above it and to the POIs of the one two levels up. If we had a function defined as $\int_0^x F(t)\,dt$ it would be a level above *F(x)* and it would relate to *F'(x)* in the same way that *F(x)* related to *F''(x)*. We will explore this idea further in a later chapter.

Almost any trait can be found from a known first derivative curve, but, unfortunately, the derivative is not enough to tell the exact original curve, because, though the critical values and x-values of the POI are known, their y-values cannot be found. Similarly, the zeros of the original curve cannot be found from its derivative.

Ex 2 The graph below is of $f'(x)$. Find the sign patterns of $f'(x)$ and $f''(x)$.



$\frac{dy}{dx}$	$+$	0	$-$	0	$-$	0	$+$	
x		-4		0		2		

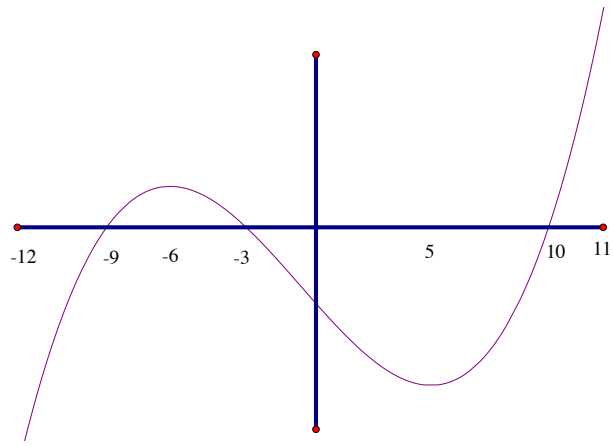
$\frac{d^2y}{dx^2}$	$-$	0	$+$	0	$-$	0	$+$	
x		-3		0		1.7		

These would then tell us the intervals of increasing and decreasing and of concave up and concave down for $f(x)$. Those relationships can be summarized in the table below.

LEARNING OUTCOME

Determine information about a function from the graph of its derivative.

EX 3 If the curve below is $y = f'(x)$, a) where are the relative maximums and relative minimums of $y = f(x)$ and b) where are the points of inflection of $y = f(x)$? Justify your answer.



What is pictured here is a two dimensional representation of the sign patterns of the first AND second derivatives.

y'		$-$	0	$+$	0	$-$	0	$+$	
x			-9		-3		10		

y''		$+$	0	$-$	0	$+$		
x			-6		5			

a) Where are the relative maximums and relative minimums of $f(x)$? Justify your answer.

Relative maximums are at $x = -12, -3,$ and 11.

- At $x = -12$, $f(x)$ starts and is decreasing because $f'(x)$ is negative.
- At $x = -3$, the derivative switches from positive to negative.

- At $x = 11$, $f(x)$ has been increasing before hitting the boundary because the derivative has been positive.

Relative minimums are at $x = -9$, and 10.

- At $x = -9$ and 10, the derivative switches from negative to positive.

b) Where are the points of inflection of $f(x)$? Justify your answer.

POIs at $x = -6$ and 5

- $f(x)$ has at points of inflection where $f'(x)$ switches from increasing to decreasing or vice versa.

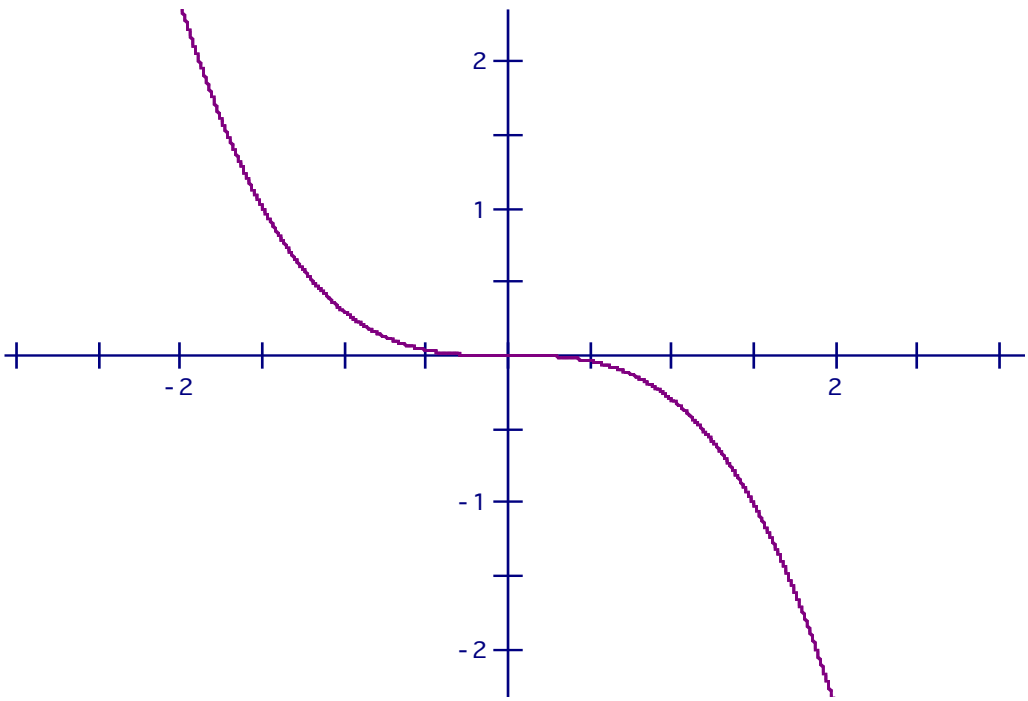
NB. Endpoints cannot be points of inflection.

Key Idea #1: The positives and negatives of *f'* inform about the increasing and decreasing (therefore the extremes) of *f*.

Key Idea #2: The increasing and decreasing of *f'* inform about the concavity of *f*.

EX 4 Given the same graph of $y = f'(x)$ in EX 1 and $f(0) = 0$, sketch a likely curve for $f(x)$ on $x \in [-2, 2]$.

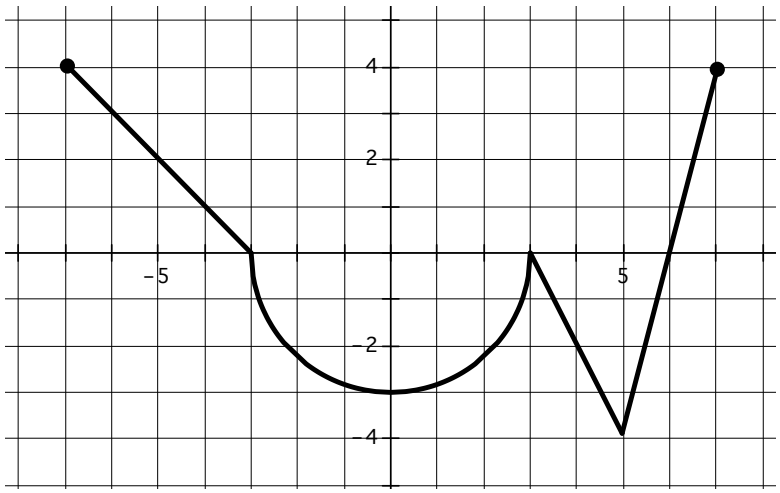
On $x \in [-2, 2]$, $f'(x)$ is negative so $f(x)$ in decreasing on that interval. Since $f(x)$ is decreasing and $f(0)$ is the zero, the curve must be above the x-axis on $x \in [-2, 0)$ and below the x-axis on $x \in (0, 2]$. On $x \in (-2, 0)$, the slope of $f'(x)$ $\left(\text{which is } f''(x)\right)$ is positive, so $f(x)$ is concave up. Similarly, on $x \in (0, 2)$, $f''(x)$ is negative since $f'(x)$ is decreasing, so $f(x)$ is concave down. So $(0, 0)$ is not only a zero, it is also a point of inflection. Putting it all together, this is a likely sketch:



Note that there are not markings for scale on the y-axis. This is because the y-values of the endpoints of the given domain cannot be known, from the given information.

The graph of $y = f'(x)$ need not be a member of any family of functions either. The sign patterns of the first and second derivatives can be deduced from any graph.

EX 5 The graph below is of $f'(x)$ on $x \in [-7, 7]$.



a) Where are the relative maximums and relative minimums of $y = f(x)$?

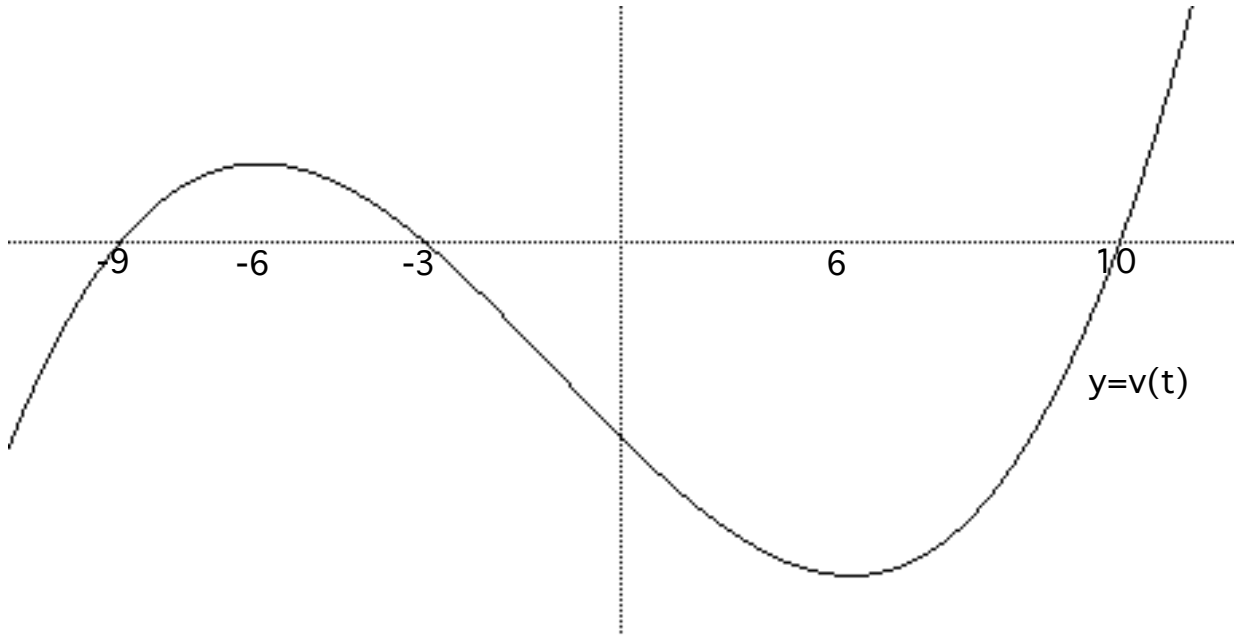
b) Where are the points of inflection of $y = f(x)$? Justify your answer.

a) $y = f(x)$ has a relative maximum at $x = -3$ because $f'(x)$ switches from positive to negative, and $y = f(x)$ has a relative maximum at $x = 7$ because $f(x)$ is increasing— $f'(x)$ is positive— and then ends.

$y = f(x)$ has a relative minimum at $x = 6$ because $f'(x)$ switches from negative to positive, and $y = f(x)$ has a relative minimum at $x = -7$ because $f(x)$ starts and is decreasing— $f'(x)$ is negative.

b) $y = f(x)$ has POIs at $x = 0, 3,$ and 5 because $f'(x)$ switches from increasing to decreasing or vice versa.

EX 6 If the following graph is the velocity of a particle in rectilinear motion, what can be deduced about the acceleration and the distance?

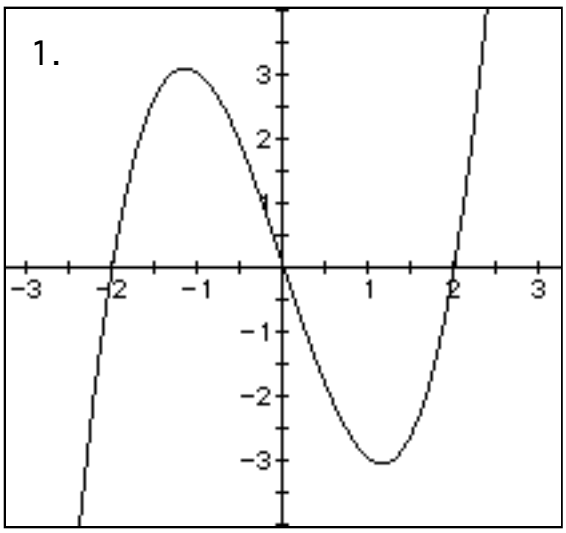


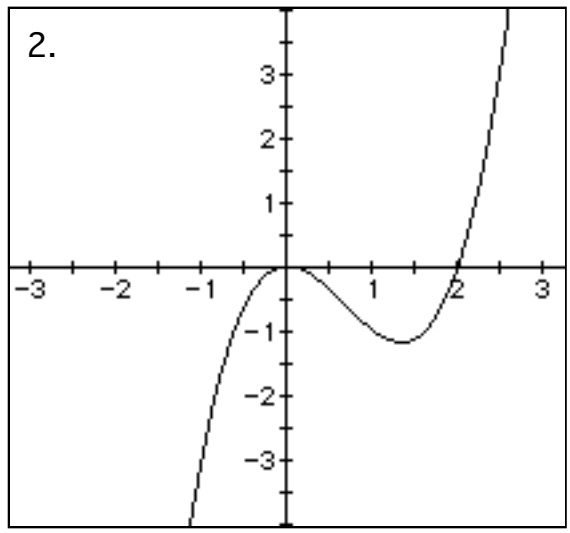
The particle is accelerating until $t=-6$, decelerating from $t=-6$ to $t=6$, and then accelerating again. The distance is at a relative maximum value at $t=-3$ and at a relative minimum value at $t=-9$ and $t=10$, but what the distances from the origin are cannot be known.

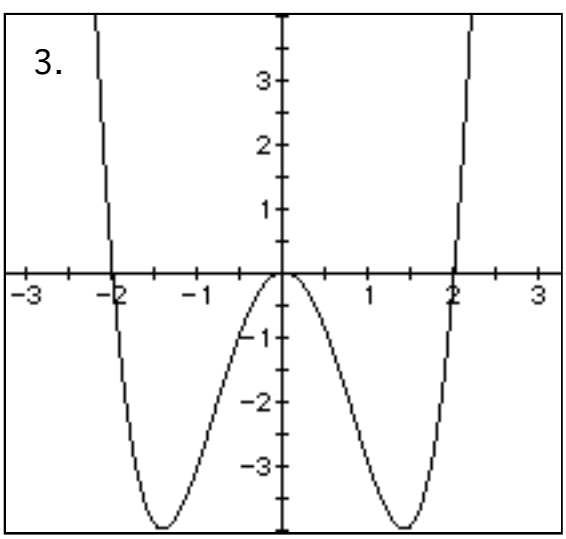
3-5 Free Response Homework

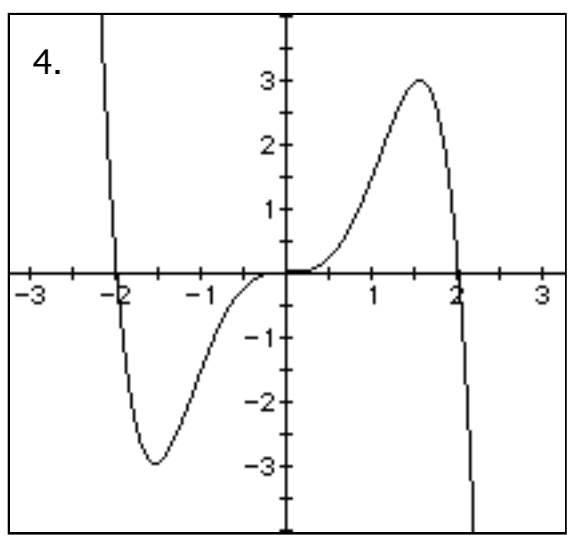
If the curve below is $y = f'(x)$, show the sign patterns of the first and second derivatives. Then find

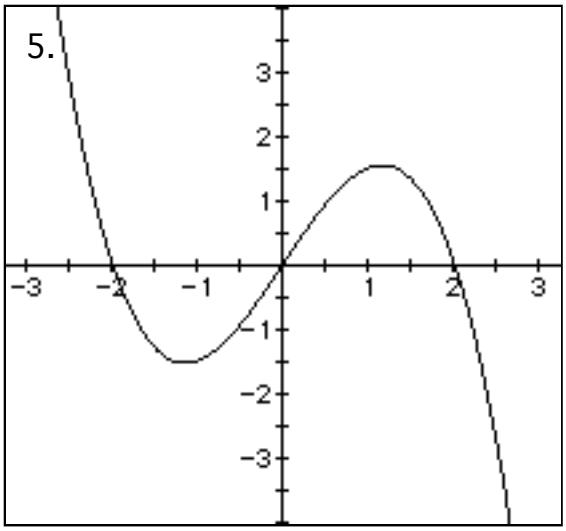
a) the critical values for the maximum and minimum points,
b) x-coordinates of the POIs,
c) intervals of increasing and decreasing,
d) the intervals of concavity of $y = f(x)$, and
e) sketch a possible curve for $f(x)$ with y-intercept $(0,\ 0)$.

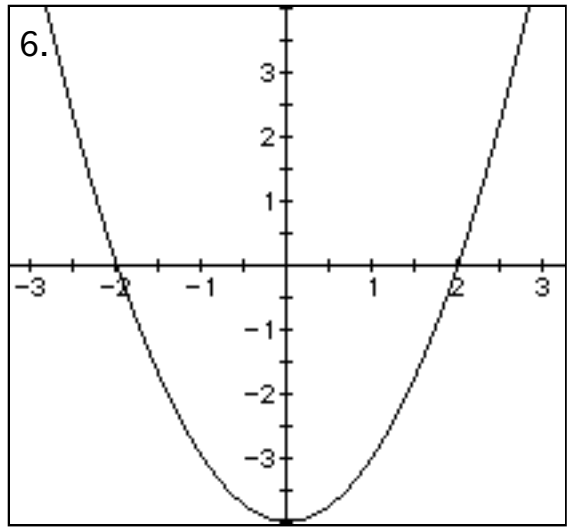




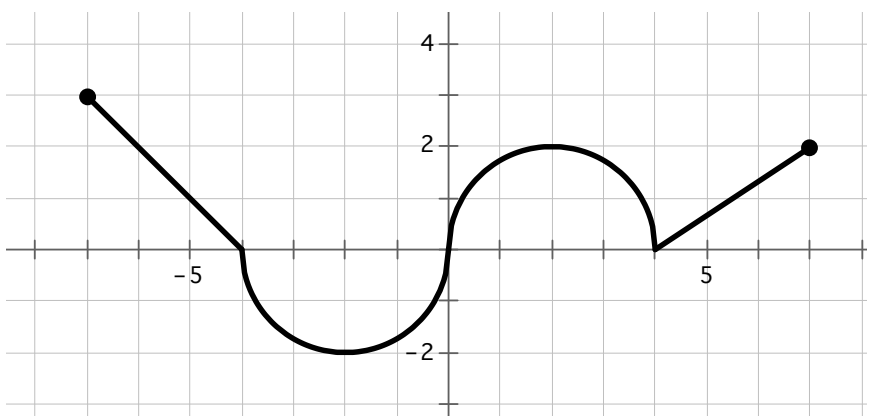






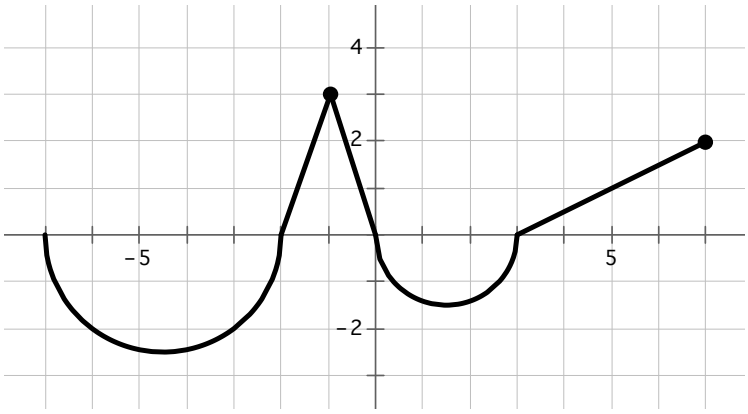


The graphs below are of $f'(x)$ on $x \in [-7, 7]$.



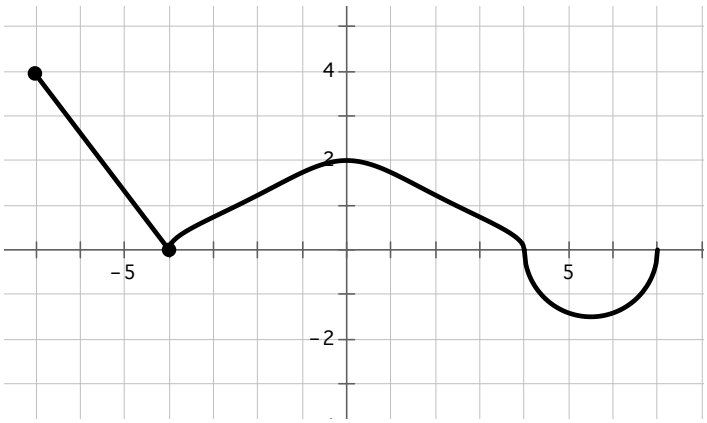
7a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.

7b) Where are the points of inflection of $y = f(x)$? Justify your answer.



8a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.

8b) Where are the points of inflection of $y = f(x)$? Justify your answer.



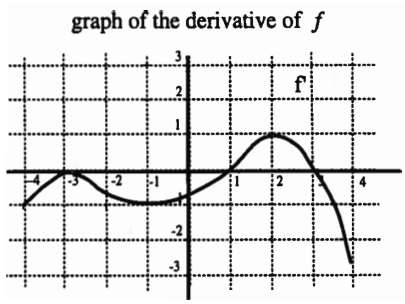
9a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.

9b) Where are the points of inflection of $y = f(x)$? Justify your answer.

AP Handout: BC2003#4, AB2000#3, AB1996 # 1

3.5 Multiple Choice Homework

1. The figure shows the graph of f', the *derivative* of a function *f*. The domain of is the interval $-4 \le x \le 4$. Which of the following are true about the graph of *f*?

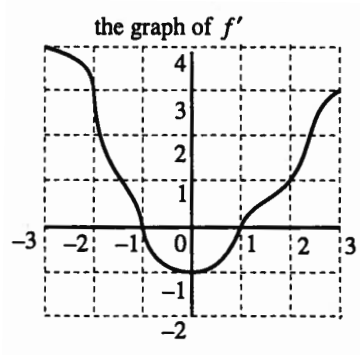


I. At the points where $x = -3$ and $x = 2$ there are horizontal tangents.
II. At the point where $x = 1$ there is a relative minimum point.
III. At the point where $x = -3$ there is an inflection point.

a) None b) II only c) III only d) II and III only e) I, II, and III

---

2. The graph of f', the derivative of a function *f*, is shown below. Which of the following statements are true about the function *f*?

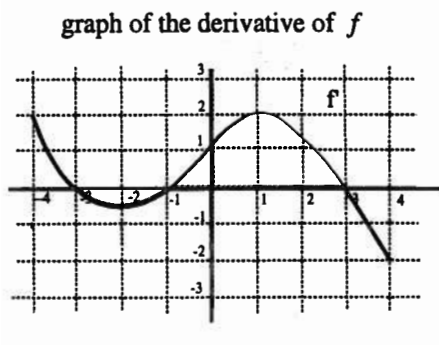


I. *f* is increasing on the interval $(-2, -1)$.
II. *f* has an inflection point at $x = 0$.
III. *f* is concave up on the interval $(-1, 0)$.

a) I only
b) II only
c) III only
d) I and II only
e) II and III only

---

3. The graph of the **derivative** of a function is *f* shown below. Which of the following is true about the function *f*?

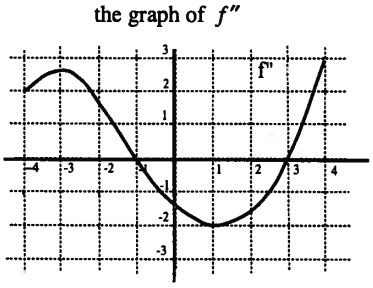


I. *f* is increasing on the interval $(-2,1)$.
II. *f* is continuous at $x = 0$.
III. *f* has an inflection point at $x = -2$.

a) I only
b) II only
c) III only
d) II and III only
e) I, II, and III

---

4. The graph of the **second derivative** of a function *f* is shown below.



Which of the following is true?

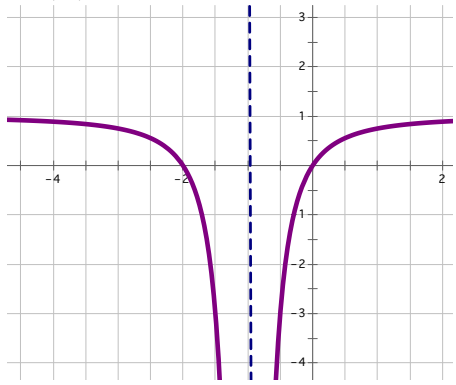
I. The graph of *f* has an inflection point at $x = -1$.
II. The graph of *f* is concave down on the interval $(-1,3)$.
III. The graph of the derivative function f' is increasing at $x = 1$.

a) I only b) II only c) III only

d) I and II only e) I, II, and III

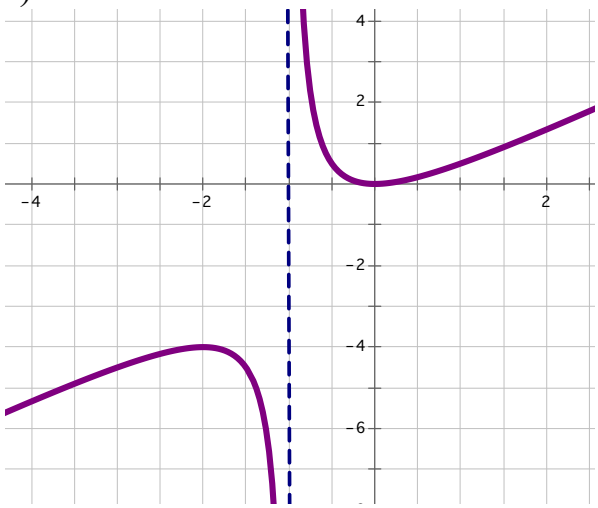
---

5. This is the graph of $f'(x)$.

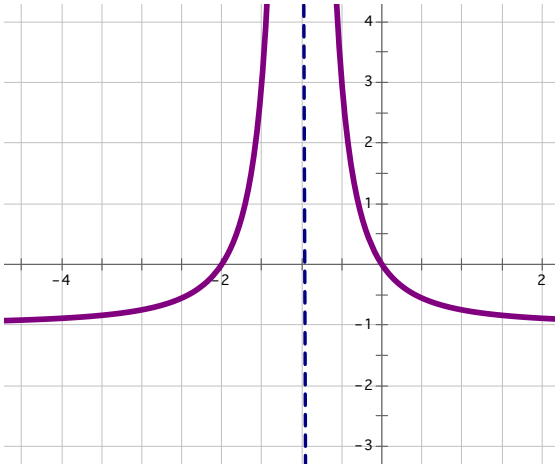


Which of the following is the graph of $f(x)$?

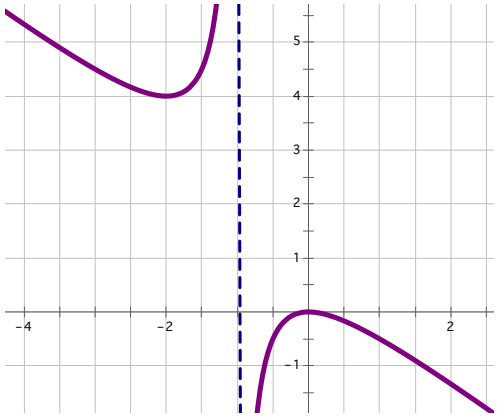
a)



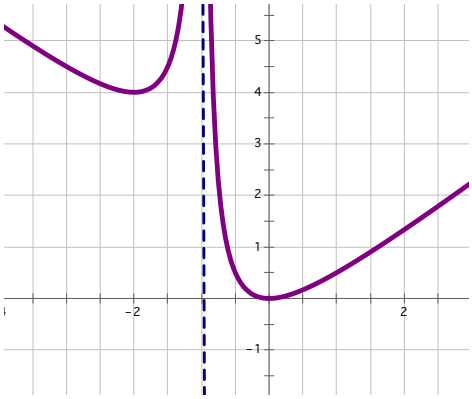
b)



c)



d)



6. The derivative of the function g is $g'(x) = \cos(\sin x)$. At the point where $x = 0$ the graph of g

I. is increasing
II. is concave down
III. attains a relative maximum point

(a) I only (b) II only (c) III only

(d) I and III only (e) I, II, and III

3.6: Rectilinear Motion

A key application of the derivative as a rate of change is the application to motion. Typically, we refer to horizontal position in terms of x(t) and vertical position in terms of y(t). Since the derivative is a rate of change of a function and the rate of change of position is velocity, it should be pretty obvious that the derivative of position of velocity. Likewise, the derivative of velocity is acceleration since the rate of change of velocity is acceleration.

Vocabulary:

1. ***Rectilinear Motion*** **–** movement that occurs in a straight line
2. ***Parametric Motion*** **–** movement that occurs in two dimensions.
3. ***Velocity*** – Defn: directed speed

   Means: how fast something it is going and whether it is moving right or left, up or down
4. ***Average Velocity*** – Defn: distance traveled divided by time or $\frac{x_2 - x_1}{t_2 - t_1}$

   Means: the average rate, as used in algebra
5. ***Instantaneous Velocity*** – Defn: velocity at a particular time t

   Means: $\frac{ds}{dt}$, $\frac{dx}{dt}$, or $\frac{dy}{dt}$, or the rate at any given instant
6. ***Acceleration*** – the rate of change of the velocity or $\frac{dv}{dt}$

Three things are implied in the definitions:

1. The derivative of a position or distance equation is the velocity equation.

2. The derivative of velocity is acceleration. Therefore, acceleration is known as the second derivative of the position or distance equation.

3. The sign of the velocity determines the direction of the movement:

   Velocity > 0 means the movement is to the right (or up)
   Velocity < 0 means the movement is to the left (or down)
   Velocity = 0 means the movement is stopped.

There is a fourth fact to know about motion::

4. Speeding up and slowing down is not determined by the sign of the acceleration.

An object is speeding up when $v(t)$ and $a(t)$ have the same sign.

An object is slowing when $v(t)$ and $a(t)$ have opposite signs.

This was actually much of the basis for Isaac Newton's exploration of the Calculus. Much of his thinking is still used in both physics and Calculus today, although his notation was not convenient. Function notation is particularly useful when dealing with motion.

OBJECTIVES

Use the derivative to make conclusions about motion.
Relate the position, velocity, and acceleration functions.
Sketch the graphs of parametric equations.
Eliminate the parameter of parametric equations.

In PreCalculus, Motion was approached

EX 1 The position of a particle is described by $x(t) = t^3 - 3t^2 - 24t + 3$.

a) Where is the particle at $t = 3$?
b) When the particle is stopped?
c) Which direction it is moving at $t = 3$ seconds?
d) Find $a(3)$.
e) Is the particle speeding up or slowing down at $t = 3$ seconds?

a) "Where it is at $t = 3$" means find $x(3)$

$$x(3) = 3^3 - 3(3)^2 - 24(3) + 3 = -69$$

b) "When the particle is stopped" means "at what time is the velocity zero?"

$$v(t) = x'(t) = 3t^2 - 6t - 24 = 0$$

$$\frac{3t^2 - 6t - 24 = 0}{3}$$

$$t^2 - 2t - 8 = 0$$

$$(t+2)(t-4) = 0$$

$$T = -2 \text{ or } 4$$

c) "Which direction it is moving at $t = 3$ seconds" means what is the sign of the velocity?"

$$v(3) = 3(3)^2 - 6(3) - 24 = -15$$

The particle is moving left because the velocity is negative.

d) $a(3)$ means plug 3 into the acceleration equation.

$$a(t) = v'(t) = 6t - 6$$

$$a(3) = v'(t) = 6(3) - 6 = 12$$

e) From parts c) and d) above, $v(3) = -15$ and $a(3) = 12$. Since these have opposite signs, the particle is slowing down.

The sign pattern of the velocity function is used to answer questions about an object moving left/right/up/down or at rest.

EX 2 A particle's distance $x(t)$ from the origin at time $t \geq 0$ is described by $x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$. Where is it when it stops to switch directions?

$$x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$$
$$v(t) = 4t^3 - 6t^2 - 22t + 12 = 0$$
$$= 2t^3 - 3t^2 - 11t + 6 = 0$$
$$= (2t-1)(t+2)(t-3) = 0$$
$$t = \frac{1}{2}, -2, \text{ and } 3$$

But the problem specifies that $t \geq 0$, so $t = \frac{1}{2}$ and 3.

$$x\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 11\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) + 1 = 4.063$$
$$x(3) = (3)^4 - 2(3)^3 - 11(3)^2 + 12(3) + 1 = -35$$

Therefore, this particle stops 4.063 units to the right of the origin $\left(\text{at } \frac{1}{2} \text{ seconds}\right)$ and 35 units left of the origin (at 3 seconds).

Ex 3 Find the acceleration of a particle at $t = \frac{\pi}{2}$ whose position is given by the function $y(t) = t^3 \sin t$ for $t \in [0, 2\pi)$, if y is in meters and t is in seconds

$$y'(t) = t^3 \cos t + 3t^2 \sin t$$
$$y''(t) = -t^3 \sin t + 3t^2 \cos t + 3t^2 \cos t + 6t \sin t$$
$$y''\left(\frac{\pi}{2}\right) = -\left(\frac{\pi}{2}\right)^3 \sin\frac{\pi}{2} + 3\left(\frac{\pi}{2}\right)^2 \cos\frac{\pi}{2} + 3\left(\frac{\pi}{2}\right)^2 \cos\frac{\pi}{2} + 6\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$
$$y''\left(\frac{\pi}{2}\right) = -\frac{\pi^3}{8} + 3\pi = 5.549$$

The acceleration at $t = \frac{\pi}{2}$ is 5.549 meters/second^2.

Ex 4 Find the position of a particle whose position is given by $x(t)=t^3-t^2+2t$ for $t>0$ when its velocity is 38.

$$x'(t)=v(t)=3t^2-2t-2$$

$$3t^2-2t-2=38$$

$$3t^2-2t-40=0$$

$$t=\frac{-10}{3}\ or\ 4$$

$$x(4)=4^3-4^2+2(4)=56$$

At $t=4$, the velocity of the particle is 38, and the position is 56 units to the right of the origin.

Ex 5 Suppose that a particle is moving along the x-axis such that the velocity is described by $v(t)=t\cos t^2$. At $t=0$, the position is $x(0)=4$.

a) For what values of $t\in[0,3]$ is the particle moving left.

b) What is the acceleration at $t=3$?

c) Find the particular position equation. What is the position of the particle at the first <u>positive</u> time when it stops to switch directions?

a) For what values of $t\in[0,4]$ is the particle moving left.

$$v(t)=t\cos t^2=0\rightarrow t=0,\sqrt{\frac{\pi}{2}},\sqrt{\frac{3\pi}{2}},\sqrt{\frac{5\pi}{2}}$$

v	0	+	0	−	0	+	0	−
t	0		$\sqrt{\frac{\pi}{2}}$		$\sqrt{\frac{3\pi}{2}}$		$\sqrt{\frac{5\pi}{2}}$	

$$t\in\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{3\pi}{2}}\right),\left(\sqrt{\frac{5\pi}{2}},3\right)$$

b) What is the acceleration at $t=3$?

$$a(t)=v'(t)=t(-\sin t^2)(2t)+\cos t^2(1)$$

$$a(3) = -18\sin 9 + \cos 9 = -8.329$$

c) Find the particular position equation. What is the position of the particle at the first <u>positive</u> time when it stops to switch directions?

$$\begin{aligned} x(t) &= \int t\cos t^2\, dt \\ &= \frac{1}{2}\int \cos t^2 (2t\,dt) \\ &= \frac{1}{2}\sin t^2 + c \end{aligned}$$

$$x(0) = 4 \rightarrow 4 = \frac{1}{2}\sin 0 + c \rightarrow c = 4$$

$$x(t) = \frac{1}{2}\sin t^2 + 4$$

$$x\left(\sqrt{\frac{\pi}{2}}\right) = \frac{1}{2}\sin\frac{\pi}{2} + 4 = 4.5$$

Ex 6: Pat takes her bike on a 4-hour ride. She records her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 4$ hours, as shown in the table below. For $0 \le t \le 4$, $v(t) > 0$.

t	0	.3	.7	1.3	1.7	2.2	2.8	3.3	4
$v(t)$	0	14.1	9.5	17.1	13.3	15.6	12.7	13.7	12.0

(a) Use the data in the table to approximate Pat's acceleration at time t = 1.5 hours. Show the computations that lead to your answer. Indicate units of measure.

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\,dt$ in the context of the problem. Approximate $\int_0^4 v(t)\,dt$ using a left-hand Riemann sum using the values from the table.

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given by $f(t) = 9\sqrt{\dfrac{3\sin(2\pi t)+8t}{t^2+2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time $t = 1.7$? Give a reason for your answer.

(a) Use the data in the table to approximate pat's acceleration at time $t = 1.5$ hours. Show the computations that lead to your answer. Indicate units of measure.

$$a(1.5) \approx \frac{v(1.7)-v(1.3)}{1.7-1.3} = \frac{13.3-17.1}{1.7-1.3} = -\frac{3.8}{.4} = -9.5\ {}^{mi}\!/_{hr^2}$$

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\,dt$ in the context of the problem. Approximate $\int_0^4 v(t)\,dt$ using a left-hand Riemann sum using the values from the table.

$\int_0^4 v(t)\,dt$ would be the approximate number of miles Pat traveled during her four-hour ride.

$$\int_0^4 v(t)\,dt \approx .3(0)+.4(14.1)+.5(9.5)+.4(17.1)+.5(13.3)+.6(15.6)+.5(12.7)+.7(13.7)$$
$$= 49.18\ miles$$

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given

by $f(t)=9\sqrt{\frac{3\sin(2\pi t)+8t}{t^2+2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

$$AveVelocity=\frac{1}{4-0}\int_0^4 9\sqrt{\frac{3\sin(2\pi t)+8t}{t^2+2}}dt=13.350\ mph$$

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time $t=1.7$? Give a reason for your answer.

$a(1.7)=f'(1.7)=-3.288\ {}^{mi}\!/_{hr^2}$

Since it was stated that "For $0 \le t \le 4$, $v(t)>0$," Pat's speed is decreasing because the velocity and acceleration have opposite signs at $t=1.7$.

<u>Summary of Key Phases</u>

When = solve for *t*
Where = solve for position
Which direction = is the velocity positive or negative
Speeding up or slowing down = are the velocity and acceleration in the same direction or opposite (do they have the same sign or not)

3.6 Homework

The position of a particle is described by the following distance equations. For each, find:

a) when the particle is stopped,
b) which direction it is moving at $t = 3$ seconds,
c) <u>where</u> it is at $t = 3$
d) $a(3)$
e) whether the particle is slowing down or speeding up at $t = 3$.

1. $x(t) = 2t^3 - 21t^2 + 60t + 4$

2. $x(t) = t^3 - 6t^2 + 12t + 5$

3. $y(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48$

4. $y(t) = 12t^5 - 15t^4 - 220t^3 + 270t^2 + 1080t$

5. $s(t) = 5t^3 - t^5$

6. $s(t) = t^4 - 5t^2 - 36$

Find maximum height of a projectile launched vertically from the given height and with the given initial velocity.

7. $h_0 = 25$ feet, $v_0 = 64$ ft/sec

8. $h_0 = 10$ meters, $v_0 = 294$ m/sec

9. The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars, $s = 11.44t^2$ on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec on each planet?

Find $x(t)$ when $v = 0$.

10. $x(t) = t^2 - 5t + 4$

11. $x(t)=t^3-6t^2-63t+4$

12. $x(t)=6t^5-15t^4-8t^3+24t^2+12$

Find $x(t)$ and $v(t)$ when $a(t)=0$.

13. $x(t)=2t^3-21t^2+60t+4$

14. $x(t)=t^3-6t^2+12t+5$

15. $x(t)=9t^4-4t^3-240t^2+576t-48$

For the given velocity equations and initial values, answer the following questions:

a) For what values is the particle moving right.

b) What is the acceleration at $t=3$?

c) Find the particular position equation.

16. $v(t)=t^2-4t-12;\ x(1)=0$

17. $v(t)=t^5-16t^3;\ x(0)=2$

18. $x(t)=t\sqrt{9-t^2};\ x\left(\sqrt{5}\right)=2$

19. $v(t)=\csc^2 t;\ x\left(\frac{\pi}{3}\right)=1$

20. $v(t)=\frac{3}{t^2+4};\ x(2)=4$

21. $v(t)=\frac{t}{t^2+4};\ x(0)=0$

t	0	1	4	6	9	10	13	15	18
$v(t)$	55	70	68	55	40	38	46	50	70

22. A car is traveling on a straight road. Values of the continuous and differentiable function $v(t)$ are given on the table above. $v(t)$ is measured in feet per minute and time t is measured in minutes.

a) Approximate the acceleration at $t=7$. Indicate the units.

b) Using left-hand rectangles, approximate $\int_0^{18} v(t)\,dt$. Using the correct units, explain the meaning of the approximation.

c) How many times on the interval $0 \le t \le 18$ is $a(t)=0$? Explain your reasoning.

d) Assume the data are modeled by $P(t)=.05t^3-1.07t^2+3.89t+62$. Use the model to find the average velocity of the car on the interval $0 \le t \le 18$.

---

t in minutes	0	8	15	23	33	45	53
$v(t)$ in mph	0	8.5	7.5	10.1	9.3	7.1	4.1
$v(t)$ in mi/min	0	0.142	0.125	0.168	0.155	0.118	0.068

23. Mr. Evans has run the 7.4-mile Bay-to-Breakers many times (always with cloths on). His best time was 53 minutes. The table above shows estimates of his velocity at different times along the course from The Embarcadero to Ocean Beach. Assume the data represents a continuous and differentiable function.

a) Approximate Mr. Evans' acceleration at $t=30$.

b) Given you result in a), was his speed increasing or decreasing at $t=10$? Explain, using the correct units.

c) Find a trapezoidal approximation for $\int_0^{53} v(t)\,dt$. Why isn't the answer 7.4 miles?

d) Mr. Lannan did the same run. (He likes to take his time and gawk at the costumes.) His velocity is modeled by $L(t)=0.023\left(5+4\sin\frac{\pi}{16}t\right)$. According to this model, does Mr. Lannan finish in under 65 minutes? Explain your reasoning.

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24. Mr. Alverado takes the Cross Country team out for a morning run and tracks his pace. The data table below shows his pace $p(t)$ in minutes per mile and his velocity $v(t)$ miles per minute at 15-minute intervals.

t (in minutes)	0	15	30	45	60
$p(t)$ (in min/mile)	8:07	7:34	8:16	8:07	7:14
$v(t)$ (in mi/min)	0.123	0.132	0.121	0.123	0.138

Both $p(t)$ and $v(t)$ are continuous and differentiable functions.

a) Find an approximation for $\int_0^{60} v(t)\,dt$ using midpoint rectangles. Explain the meaning of the result, using the correct units.

b) Using correct units, explain the meaning of $\frac{1}{60}\int_0^{60} p(t)\,dt$.

c) Approximate the acceleration at $t=37$ minutes.

d) Is there a time during which the pace reaches a maximum? Explain your reasoning.

25. Below is a chart of your speed driving to school in meters/second. Use the information below to find the values in a) and b) below.

t (in seconds)	0	30	90	120	220	300	360
$v(t)$ (in m/sec)	0	21	43	38	30	24	0

a) Approximate your acceleration at $t = 60$.
b) Given you result in a), are you speeding up or slowing down at $t = 100$? Explain, using the correct units.
c) Find an approximation for $\int_0^{360} v(t)\,dt$ using right Riemann rectangles. Using the correct units, explain the meaning of your result.

3.6 Multiple Choice Homework

1. A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Find the value of t at which the acceleration is zero.

a) $-\frac{2}{3}$ b) $-\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{1}{3}$ e) $-\frac{1}{2}$

---

2. A particle moves on the x-axis with velocity given by $v(t) = 3t^4 - 11t^2 + 9t - 2$ for $-3 \le t \le 3$. How many times does the particle change direction as t increases from –3 to 3?

a) Zero b) One c) Two

d) Three e) Four

3. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

4. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the acceleration of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

5. Find the acceleration at time $t = 9$ seconds if the position (in cm.) of a particle moving along a line is $s(t) = 6t^3 - 7t^2 - 9t + 2$.

a) 310 cm/sec^2 b) 310 cm/sec c) 1323 cm/sec^2

d) 1323 cm/sec e) -1323 cm/sec

6. The acceleration of a particle is given by $a(t) = 4e^{2t}$. When $t = 0$, the position of the particle is $x = 2$ and $v = -2$. Determine the position of the particle at $t = \frac{1}{2}$.

a) $e - 3$ b) $e - 2$ c) $e - 1$ d) e e) $e + 1$

7. A particle moves along a straight line with its position at any time $t \geq 0$ given by $s(t) = \int_0^t (x^3 - 2x^2 + x)\,dx$, where s is measured in meters and t is in seconds. The maximum velocity attained by the particle on $0 \leq t \leq 3$ is

a) $1/3$ m/s b) $4/27$ m/s c) $27/4$ m/s d) 12 m/s

---

8. A particle moves along the *x*-axis with acceleration at any time *t* given as $a(t) = 3t^2 + 4t + 6$. If the particle's initial velocity is 10 and its initial position is 2, what is the position function?

a) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 12$ b) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 10t + 2$

c) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 2$ d) $x(t) = 3t^4 + t^3 + t^2 + 10t + 2$

---

9. A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Which of the following statements is/are true?

I. The particle is moving right at $t = \frac{2}{3}$.

II. The particle is paused at $t = \frac{1}{3}$

III. The particle is speeding up at $t = 1$.

a) I only b) II only c) III only

d) I and II only e) I and III only

3.6 Homework B

1. Find out when the particle whose position is described by the equation $y(t) = t^4 - 5t^2 + 12$ is moving up.

2. Find the velocity and acceleration at $t = \frac{\pi}{3}$ for the particle whose position is described by the equation $x(t) = \tan\ (3t)$.

3. Find out when the velocity of the particle whose position is described by $y(t) = t^4 - 5t^2 + 12$ is decreasing.

4. Find the equations for the velocity and acceleration of the particle whose position is described by the equation $x(t) = \ln(t^2 + 1)$.

5. For the particle in problem 4, find when the particle stops, when it is moving left, and when it is moving right.

6. If the velocity of a horizontally moving particle is given by the equation $v(t) = \sec^2 t$, find when the particle stops and find the acceleration at $t = \frac{5\pi}{3}$. What could be a possible equation for the position of the particle?

7. Find the position of a particle whose position is given by $x(t) = -t^3 - t^2 + 21t - 12$ for $t > 0$ when its velocity is -18.

8. Find the acceleration of a particle at $t = \frac{\pi}{6}$ whose position is given by the function $y(t) = t^3 \cos\ (2t)$ for $t \in [0, 2\pi)$. Also, find when the particle changes direction.

3.7: Related Rates

Derivatives have primarily been interpreted as the slope of the tangent line. But, as with rectilinear motion, there are other contexts for the derivative. One overarching concept is that a derivative is a **Rate of Change**. The tendency is to think of rates as distance per time unit, like miles/hour or feet/second, but even slope is a rate of change—it is just that rise and run are both measured as distances.

LEARNING OUTCOME
Solve related rates problems.

The idea behind related rates is two-fold. First, change is occurring in two or more measurements that are related to each other by the geometry (or algebra) of the situation. Second, an implicit chain rule situation exists in that the x and y-values are functions of time, which may or may not be a variable in the problem. Therefore, when taking the derivative of an x or y, an **Implicit Rate Term** $\left(\frac{dx}{dt} \text{ or } \frac{dy}{dt}\right)$ occurs. A classic problem is the falling ladder.

Common Formulas for Related Rates Problems

Pythagorean Theorem:

$$x^2 + y^2 = r^2$$

Area Formulas:

Circle $A = \pi r^2$ Rectangle $A = lw$

Trapezoid $A = \frac{1}{2}h(b_1 + b_2)$

Volume Formulas:

Sphere $V = \frac{4}{3}\pi r^3$ Right Prism $V = Bh$

Cylinder $V = \pi r^2 h$ Cone $V = \frac{1}{3}\pi r^2 h$

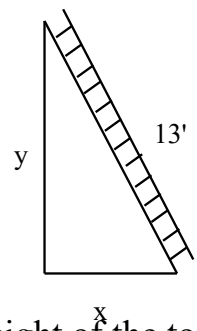
Right Pyramid $V = \frac{1}{3}Bh$

Surface Area Formulas:

Sphere $S = 4\pi r^2$ Cylinder $S = 2\pi r^2 + 2\pi rl$

Cone $S = \pi r^2 + \pi rl$ Right Prism $S = 2B + Ph$

EX 1 A 13-foot-tall ladder is leaning against a wall. The bottom of the ladder slides away from the wall at 4 ft/sec. How fast is the top of the ladder moving down the wall when the ladder is 5 feet from the wall?



As can be seen in the picture, the height of the top of the ladder and the distance the bottom of the ladder is from the wall are related by the Pythagorean theorem. Both are variables, because the ladder is moving. Therefore,

$$x^2 + y^2 = 13^2$$

4 ft/sec is the rate at which the x-value is changing—i.e. $\frac{dx}{dt}$. To find $\frac{dy}{dt}$, differentiate $x^2 + y^2 = 13^2$ to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

This is essentially an equation in four variables. But x and $\frac{dx}{dt}$ are known, and $y = 12$ (by the Pythagorean theorem).
So,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$2(5)(4) + 2(12)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{5}{3} \text{ ft/sec}$$

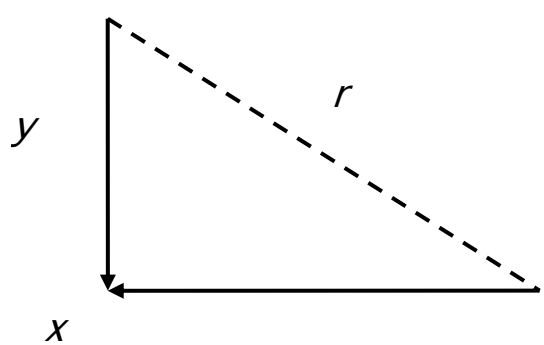
It should make sense that $\frac{dy}{dt}$ is negative since the top of the ladder is falling.

Another common related rate problem is where a tank of a particular shape is filling or draining.

Process for Related Rates Problems:

1. Determine what is being asked.
   a. Look at the units to determine what is given and what is asked.
2. Determine the equation that relates the variable to each other variable. (NB. This will be the one to be differentiated)
3. Determine what is given.
   a. Look at the units to determine what is given and what is asked.
4. If there is a product of two variables, eliminate the product by either multiplying or substituting a secondary equation.
   a. Note: This is only because we have not learned the Product Rule yet.
5. Differentiate in terms of time.
   a. Do not forget the implicit fractions.
6. Substitute and solve for the missing variable.

Ex 2 Two cars approach an intersection, one traveling south at 20 mph and the other traveling west at 30 mph. How fast is the direct distance between them decreasing when the westbound car is .6 miles and the southbound car is .8 miles from the intersection?



As we can see in the picture, the distance between the two cars are related by the Pythagorean Theorem.

$$x^2 + y^2 = r^2$$

We know several pieces of information. The southbound car is moving at 20 mph; i.e. $\frac{dy}{dt} = -20$. By similar logic we can deduce each of the following:

$$\frac{dy}{dt} = -20 \qquad \frac{dx}{dt} = -30$$
$$y = 0.8 \qquad x = 0.6$$

And, by the Pythagorean Theorem, $r = 1.0$

Now we take the derivative of the Pythagorean Theorem and get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

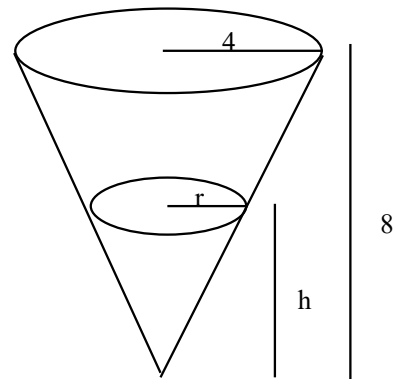
This is essentially an equation in six variables. But we know five of those six variables, so just substitute and solve.

$$2(0.8)(-30) + 2(0.6)(-20) = 2(1.0)\frac{dr}{dt}$$

$$\frac{dr}{dt} = -36 \ \ miles/hour$$

It should make sense that $\frac{dr}{dt}$ is negative since the two cars are approaching one another. We know the units based on the fraction, . Since *r* was in miles and *t* was in hours, our final units must be miles/hour.

EX 3 A tank shaped like an inverted cone 8 feet in height and with a base diameter of 8 feet is filling at a rate of 10 ft^3/minute. How fast is the height changing when the water is 6 feet deep?



The units on the 10 tell us that it is the change in volume, or $\frac{dV}{dt}$. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. But this equation has too many variables for us to differentiate it as it stands. Since the rate of change of the height—i.e., $\frac{dh}{dt}$—was what the question is, eliminate the *r* from the equation. By similar triangles, $\frac{r}{h} = \frac{4}{8}$ and $r = \frac{1}{2}h$. Substitution gives a volume equation in terms of height only:

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h \\ &= \frac{\pi}{12}h^3 \end{aligned}$$

Differentiate and plug into to solve for $\frac{dh}{dt}$.

$$V = \frac{\pi}{12} h^3$$
$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$
$$10 = \frac{\pi}{4}(6)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{10}{9\pi} \text{ ft/min}$$

Ex 4 If two resistors are connected in parallel, then the total resistance is given by the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$, where all values for R are in Ohms (Ω). If R_1 and R_2 are increasing at rates of $0.3\ \Omega/\text{sec}$ and $0.2\ \Omega/\text{sec}$, respectively, find how fast R is changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$.

$$\frac{d}{dt}\left[\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\right]$$
$$-R^{-2}\left(\frac{dR}{dt}\right) = -R_1^{-2}\left(\frac{dR_1}{dt}\right) - R_2^{-2}\left(\frac{dR_2}{dt}\right)$$
$$-\left(\frac{400}{9}\right)^{-2}\left(\frac{dR}{dt}\right) = -(80)^{-2}(0.3) - (100)^{-2}(0.2)$$
$$\frac{dR}{dt} = \frac{107}{810} \text{ or } 0.132\ \Omega/\text{sec}$$

3.7 Homework:

1. A spherical balloon is being inflated so that its volume is increasing at a rate of 6 ft^3/min. How fast is the radius changing when $r = 10$ ft?

2. A 25-foot tall ladder is leaning against a wall. The bottom of the ladder is pushed toward the wall at 5 ft/sec. How fast is the top of the ladder moving up the wall when it is 7 feet up?

3. A circular ink stain is spreading (i.e. the radius is changing) at half an inch per minute. How fast is the area changing when the stain has a 1-inch diameter?

4. A cylindrical tank of height 30' and radius 10' is leaking at a rate of 300 ft^3/min. How fast is the oil level dropping?

5. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the intersection and watches an eastbound train traveling 60m/sec. At how many m/sec is the train moving away from the observer 4 second after it passes the intersection?

6. A street light is mounted at the top of a 15-foot-tall pole. A 6-foot-tall man walks away from the pole at a speed of 5 ft/sec in a straight line. How fast is the tip of his shadow moving away from the pole when the man is 40 feet from the pole?

7. Two cars start moving away from the same point. One travels south at 60 mph, and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

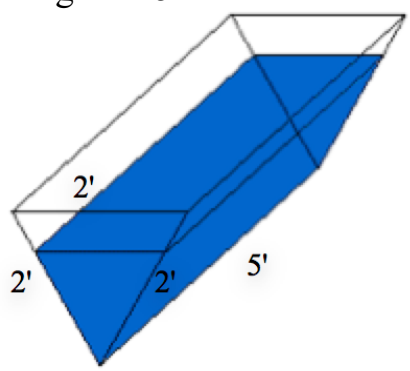
8. Sand is dumped onto a pile at 30 ft^3/min. The pile forms a cone with the height always equal to the base diameter. How fast is the base area changing when the pile is 10 feet high?

9. According to Boyle's Law, gas pressure varies directly with temperature and inversely with volume $\left(\text{or } P = \frac{kT}{V}\right)$. Suppose that the temperature is held constant while the pressure increases at 20 kPa/min. What is the rate of change of the volume when the volume is 600 in^3 and the pressure is 150 kPa?

10. Adiabatic Law: According to the adiabatic law for expansion of air, $P \cdot V^{1.4} = \frac{4}{81}$, where P is pressure and V is volume. If, at a specific instant, P = 108 lb/in^2 and is increasing at 27 lb/in^2 per second, what is the rate of change of the volume?

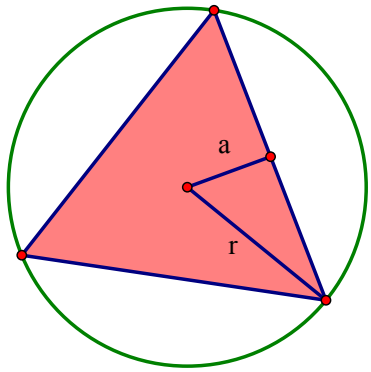
11. Water is leaking out of an inverted conical tank at a rate of 5000 cm^3/min. If the tank is 8 m tall and has a diameter of 4 m. Find the rate at which the height is decreasing when the water level is at 3 m. Then find the rate of change of the radius at that same instant.

12. Water is siphoned out of an equilateral triangular prism of length 5' and triangular base edges 2' at a rate of 2 ft^3/min. What is the rate of change of the height of the water when the trough is 25% full?



13. A man starts walking north at 5 feet/sec from a point P. 3 minutes later a woman starts walking east at a rate of 4 feet/sec from a point 500 feet east of point P. At what rate are the two moving apart at 12 minutes after the woman starts walking.

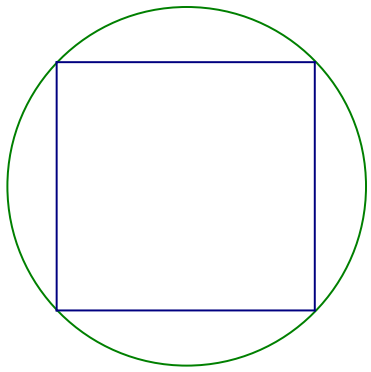
14. An equilateral triangle is inscribed in a circle. The circle's circumference is expanding at 6π in/sec and the triangle maintains the contact of its corners with the circle.



Given that the area of an equilateral triangle is equal to half the apothem a times the perimeter p, find out how fast the area inside the circle but outside the triangle is expanding when the area of the circle is $64\pi\ in$. [Hint: Find p and a in terms of r.]

15. The altitude of a triangle is increasing at a rate of 2 cm/sec at the same time that the area of the triangle is increasing at a rate of 5 cm^2/sec. At what rate is the base increasing when the altitude is 12 cm and the area is 144 cm^2?

16. A square is inscribed in a circle as shown. As the square expands, the circle expands to maintain the four contact points. The perimeter of the square is increasing at a rate of 4 inches per second. (Note: A square with side = x has a diagonal = $x\sqrt{2}$.)



a. Find the rate of change of the circumference of the circle.

b. At the instant when the area of the square is 16 square inches, find the rate at which the area between the square and the circle is increasing.

3.7 Multiple Choice Homework:

1. The width of a square is increasing at a constant rate of 0.5 cm per second. In terms of the perimeter P, what is the rate of change of the area of the square in cm^2 per second?

a) $-\frac{1}{2}P$ b) $-P$ c) $\frac{1}{4}P$ d) $\frac{1}{2}P$ e) P

---

2. When $t = 18$, the rate at which $y = \sqrt{0.5t}$ is increasing is k times the rate at which t is increasing. What is the value of $\frac{1}{k}$?

a) $\frac{1}{12}$ b) $\frac{1}{6}$ c) 1 d) 6 e) 12

---

3. The side of a cube is expanding at a constant rate of 6 inches per second. What is the rate of change of the volume, in in^3 per second, when the total surface area of the cube is 54 in^2?

a) 324 b) 108 c) 18 d) 162 e) 54

---

4. If the volume of a cube is increasing at 20 cubic inches per second when each edge is 10 inches long, how fast is the surface area increasing?

a) $\frac{4}{3}$ b) 2 c) 4 d) 6 e) 8

---

5. When the height of a cylinder is 12cm and the radius is 4 cm, the circumference is increasing at a rate of $\frac{\pi}{4}$ cm/min and the height increasing four times faster than the radius. How fast is the volume changing?

a) $.5\pi$ b) 4π c) 12π

d) 20π e) 80π

---

6. At what approximate rate (in cubic meters per minute) is the volume of a sphere changing at the instant when the surface area is 3 square meters and the radius is increasing at the rate of 1/5 meters per minute?

a) 1.228 b) 1.905 c) 0.649 d) 0.6 e) 0.62

---

7. The radius of a sphere is decreasing at a rate of 2 centimeters per second. At the instant when the radius of the sphere is 3 cm, what is the rate of change, in square centimeters per second, of the surface area of the sphere? (The surface area S of a sphere with radius r is $S = 4\pi r^2$.)

a) -108π b) -72π c) -48π d) -24π e) -16π

---

8. Water is flowing into a spherical tank with 6-foot radius at the constant rate of 30π cu ft per hour. When the water is h feet deep, the volume of the water in the tank is given by $V = \frac{\pi h^2}{3}(18 - h)$. What is the rate at which the depth of the water in the tank is increasing the moment when the water is 2 feet deep?

a) 0.5 ft/hour b) 1.0 ft/hour c) 1.5 ft/hour

d) 2.0 ft/hour e) 2.5 ft/hour

---

9. The radius of a circle is in increasing at a constant rate of 0.2 meters per second. What is the rate of increase in the area of the circle at the instant when the circumference of the circle is 20π meters?

a) 0.04π m^2/sec b) 0.4π m^2/sec c) 4π m^2/sec

d) 20π m^2/sec e) 100π m^2/sec

---

10. If the rate of change of a number x with respect to time t, is x, what is the rate of change of the reciprocal of the number when $x = -\frac{1}{4}$?

a) -16 b) -4 c) $-\frac{1}{48}$ d) $\frac{1}{48}$ e) 4

---

11. Gravel is being dumped from a conveyor belt at a rate of 35 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 15ft high?

a) 0.27 ft/min b) 1.24 ft/min c) 0.14 ft/min

d) 0.2 ft/min e) 0.6 ft/min

Derivative Applications Test

1. The position of a particle moving along a horizontal line is given by

$$x(t) = 3(t-4)^3$$

What is the maximum speed of the particle for $0 \le t \le 10$?

a) 108 b) 324 c) 144 d) 576 e) 48

---

2. If the radius of a sphere is increasing at 2 in/second, how fast, in cubic inches per second, is the volume increasing when the radius is 10 inches?

a) 40π b) 80π c) 800

d) 800π e) 3200π

---

3. A model train's velocity is modeled by $v(t) = \sin(2x) + 2$ feet per second. What is the train's average velocity, in feet/sec, from $t = 0$ to $t = \pi/2$ seconds?

a) $\frac{2}{\pi} + 2$ b) $\frac{1}{\pi} + 2$ c) 2 d) $\frac{2}{\pi}$ e) $\frac{1}{\pi}$

---

4. A particle is moving along the x-axis and its position is given by $x(t) = te^{-2t}$. For what values of t is the particle at rest?

a) No values b) 0 only c) $\frac{1}{2}$ only d) 1 only e) 0 and $\frac{1}{2}$

---

5. Two cars start moving from the same point. One travels south at 28 mi/h and the other travels west at 70 mi/h. At what rate is the distance between the cards increasing 5 hours later?

a) 75.42 mi/h b) 75.49 mi/h c) 76.4 mi/h

d) 75.39 mi/h e) 75.38 mi/h

---

6. A particle moves along the y-axis so that at any time $t \geq 0$, it velocity is given $v(t) = \sin(2t)$. If the position of the particle at time $t = \frac{\pi}{2}$ is $y = 3$, the particle's position at time $t = 0$ is

a) –4 b) 2 c) 3 d) 4 e) 6

---

7. If $g'(x) = 3xe^{2x} + 5$, then $g(x)$ has a point of inflection at:

a) $x = -1$ b) $x = 1/2$ c) $x = -1/2$

d) $x = 1$ e) Nowhere

---

8. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that they have values given on the table below.

x	$f'(x)$	$f''(x)$	$g'(x)$	$g''(x)$
2	0	-2	8	0
4	8	0	0	3
8	0	-12	0	4

Then at $x = 8$, $g(x)$ has a:

a) Relative Maximum

b) Relative Minimum

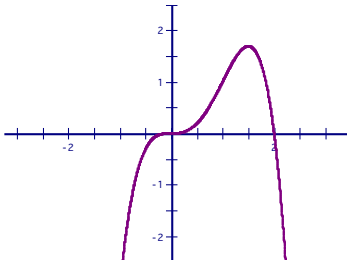
c) Point of Inflection

d) Zero

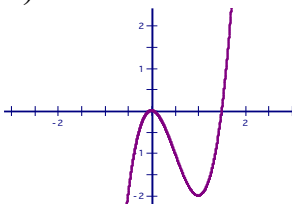
e) None of these

---

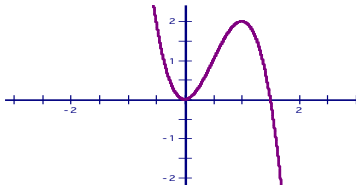
9. Suppose the function $f(x)$ has the graph shown below. Which of the following could be the graph of $f'(x)$?



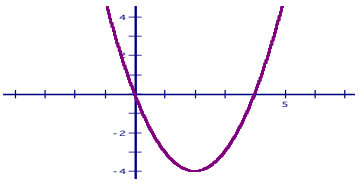
A)



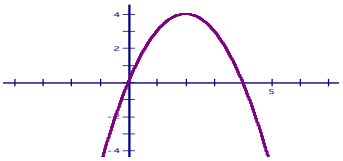
B)



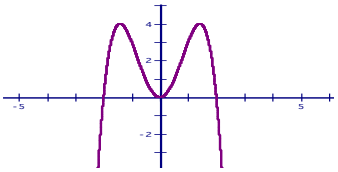
C)

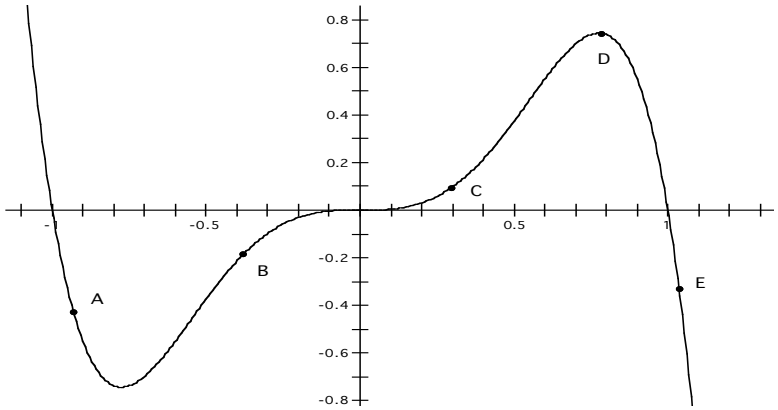


D)



E)



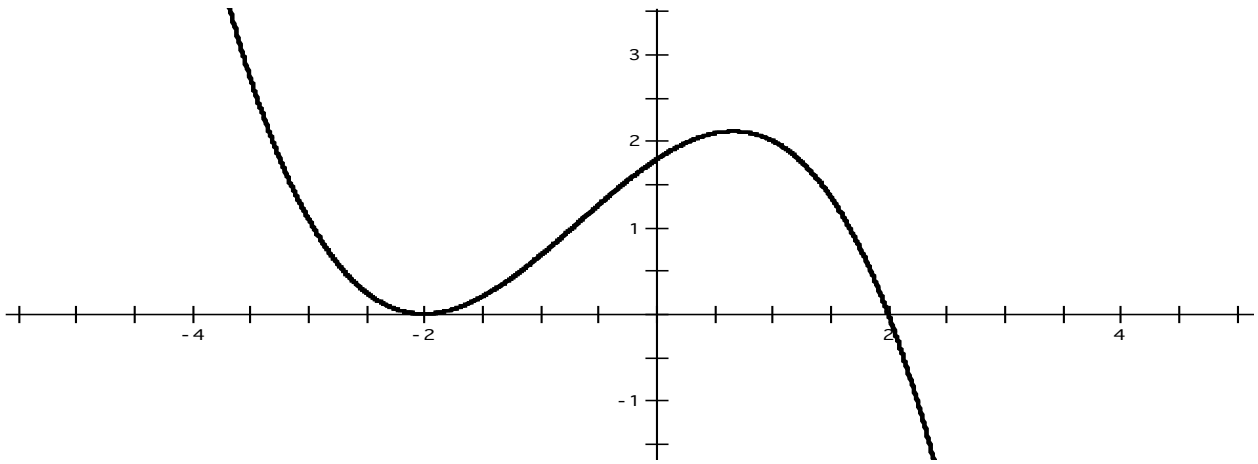


10. The graph of the function $f(x)$ is shown above. At which point on the graph of $f(x)$ is $f'(x) < 0$ and $f''(x) > 0$?

a) A b) B c) C d) D e) E

---

11. The graph of the **second** derivative of f is shown below.



Which of the following statements are true about f?

I. The graph of f has a point of inflection at $x=-2$.
II. The graph of f is concave down on $x \in (0, 4)$
III. If $f'(0)=0$, then f is increasing at $x=2$.

a) I only b) II only c) III only

d) I and II only e) I, II and III

---

12. Which of the following statements is true about the function $f(x)$ if its derivative $f'(x)$ is defined by $f'(x)=x(x-a)^3$ for $a>0$.

I. The graph of $f(x)$ is increasing at $x=2a$.
II. The graph of $f(x)$ has a local maximum at $x=0$.
III. The graph of $f(x)$ has a point of inflection at $x=a$.

a) I, II and III b) II and III only c) I and III only

d) I and II only e) I only

---

13. The table below gives values of the derivative of a function f.

x	0.998	0.999	1.000	1.001	1.002
$f'(x)$	0.980	0.995	1.000	0.995	0.980

Based on this information, it appears that in the interval covered in the table,

a) f is increasing and concave up everywhere.
b) f is increasing and concave down everywhere.
c) f has a point of inflection.
d) f is decreasing and concave up everywhere.
e) f is decreasing and concave down everywhere.

Free Response:

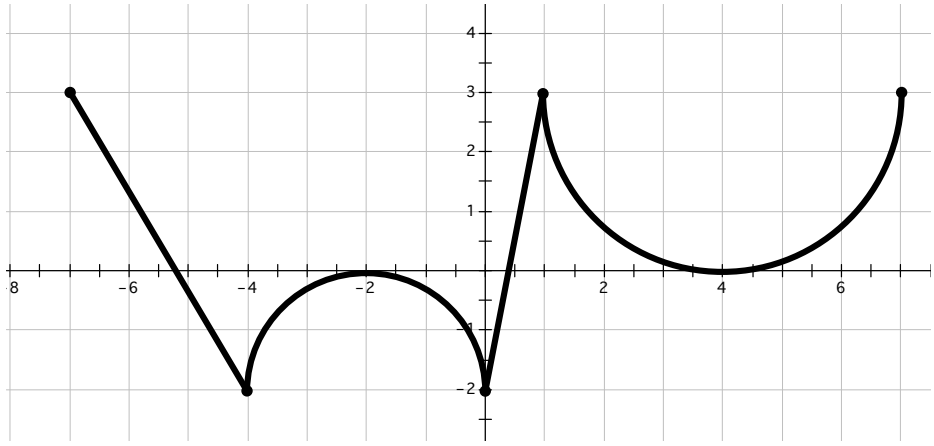
1. $v(t) = t^2 \sin t^3$ describes the velocity of a particle moving along the *y*-axis at any time $t \geq 0$. At time *t*=0, the *y*-position is 3.

---

a. For what values of t, $0 \leq t \leq 2$, is the particle moving upward?

b. Write the expression for the acceleration of the particle in terms of *t*.

c. Write the expression for the position *y(t)* of the particle.



2. The graph above is $f'(x)$ on $x \in [-7, 7]$.

a) Identify the x-value(s) of the relative maximums of $y = f(x)$? Justify your answer.

b) Identify the x-value(s) of the points of inflection of $y = f(x)$? Justify your answer.

c) On what interval(s) is f deceasing and concave up? Justify your answer.

3. Consider the curve given by $x^2 + 4xy + y^2 = -12$.

---

a) Show that $\frac{dy}{dx} = -\frac{x+2y}{2x+y}$.

b) Find the equations of all the tangent lines which are horizontal.

c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

3.1 Free Response Answers

1. -400,931 2. -45.063 3. .606, -.606 4. .046, -.602

5. 0, 5.151 6. 0 7. 0, 8.845 8. -.722, .466

9. $y = -4.441, 6.441, -4.034, -4.128, 1.407, 1.314$ 10. 0

11. $y = -6, 1, -8, 3$

12. $f(0) = 0 \therefore abs.\min., \; f(1) = \frac{1}{2} \therefore abs.\max., \; f(2) = \frac{2}{5} \therefore rel.\min.$

13. $f(0) = 0 \therefore abs.\min., \; f(2) = 7.560 \leftarrow \therefore abs.\max., \; f(8) = 0 \therefore abs.\min.$

14. $f(0) = 0 \therefore abs.\min., \; f(1) = \frac{1}{e} \therefore abs.\max., \; f(2) = \frac{1}{e^2} \therefore rel.\min.$

15. $f(1) = 0 \therefore abs.\min., \; f(e) = \frac{1}{e} \therefore abs.\max., \; f(3) = .366 \therefore rel.\min.$

16. $f(0) = 0 \therefore abs.\min., \; f(\ln 2) = .25 \therefore abs.\max., \; f(1) = .233 \therefore rel.\min.$

17. Min: $x = e^{-1/2}$

18. Max: $x = -2$; Min: $x = -2$

19. Max: $x = \frac{1}{\sqrt{2}}$; Min: $x = -\frac{1}{\sqrt{2}}$

20. $x = y = \sqrt{110}$ 21. x=1 22. $14{,}062.5\,ft^2$

23. $4000\,cm^3$ 24. $x = \frac{-28}{17}, \; y = \frac{7}{17}$ 25. $(\pm .510, .794)$

26. 12 27. $\frac{4000\pi}{3\sqrt{3}} \approx 2418.418$

28. max is all the square, min is x = 2.795cm

3.1 Multiple Choice Answers

1.	E	2.	E	3.	A	4.	C	5.	E	6.	B
7.	7	8.	A	9.	B	10.	E	11.	A	12.	C
13.	B	14.	D	15.	B	16.	D	17.	D	18.	A
19.	A										

3.2 Free Response Answer Key

1. $x = .423, 1.577$ 2. $x = 1/4, 3/4$ 3. $t = -2.150$

4. The MVT does not apply.

5. $\dfrac{f(8)-f(1)}{8-1} \approx f'(3.2)$ 6. $\dfrac{f(9)-f(1)}{9-1} \approx f'(2.4) = f'(4.6) = f'(6.9)$

7. $y \in [5, 29]$ 8. The IVT does not apply. 9. $y \in \left[-4\sqrt{2}, 0\right]$

10. $y \in \left[\dfrac{1}{\sqrt{2}}, \dfrac{8}{\sqrt{14}}\right]$

11a) 0.45 °F/hr^2

b) The total temperature change between midnight and 8am is 9.9°F.

c) Yes.

d) Yes

12a) 0.032

b) Yes.

c) While there might be a time when *SFR* = 0.0056, it is not guaranteed.

13. See APCentral.

3.2 Multiple Choice Answer Key

1. B 2. A 3. D 4. D 5. D

6. C 7. C

3.3 Free Response Answers

1. CD: $x \in (-\infty,-3) \cup (0,3)$

2a. yes 2b. No

3. CU $x \in \left(-\frac{1}{3}, \infty\right)$, CD: $x \in \left(-\infty, -\frac{1}{3}\right)$, POIs: $\left(-\frac{1}{3}, -12.593\right)$

4. CU $x \in (-\infty, 1) \cup \left(\frac{7}{3}, \infty\right)$, CD: $x \in \left(1, \frac{7}{3}\right)$, POIs: $(1, 5), \left(\frac{7}{3}, -4.481\right)$

5. CU $x \in (-\infty, -1.464) \cup (0, 5.464)$, CD: $x \in (-1.464, 0) \cup (5.464, \infty)$,

POIs: $(-1.464, .953), (0, 0), (5.464, -.646)$

6. CU $x \in (-\infty, -2) \cup (2, \infty)$, CD: $x \in (-2, 2)$, POIs: None

7. CD: $x \in (-\infty, -2) \cup (0, 2) \cup (2, \infty)$, POIs: None

8. CU $x \in \left(-2\sqrt{2}, 0\right)$, CD: $x \in \left(0, 2\sqrt{2}\right)$, POIs: $(0, 0)$

9. CU $x \in (\pi, 2\pi)$, CD: $x \in (0, \pi)$, POIs: $\left(\pi, \frac{\pi}{2}\right)$

10. CU $x \in (2, \infty)$, CD: $x \in (-\infty, 2)$, POIs: $(2, .271)$

11. CD $x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, CU: $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$,

POIs: $\left(-\frac{1}{\sqrt{2}}, .607\right), \left(\frac{1}{\sqrt{2}}, .607\right)$

12. CU $x \in (-3,0) \cup (3,\infty)$, CD: $x \in (-\infty,0) \cup (0,\infty)$, POIs: $(0, 0)$

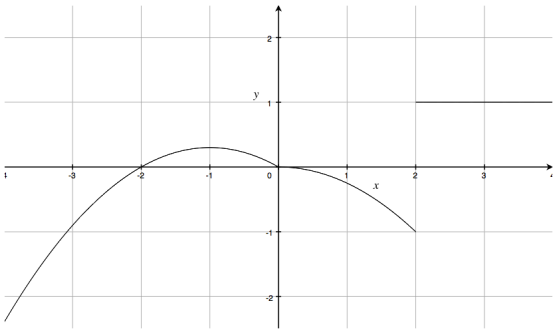
13. CU $x \in (-\infty,0) \cup (0,\infty)$, No POIs

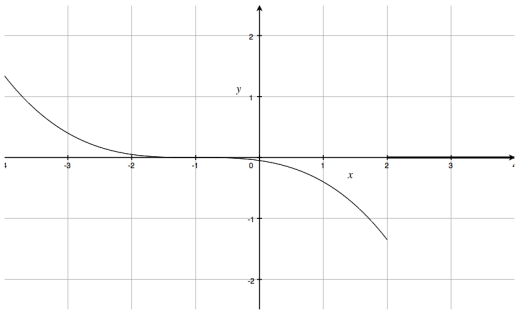
3.3 Multiple Choice Answer Key

1.	A	2.	B	3.	C	4.	D	5.	A	6.	B
7.	E	8.	D	9.	C	10.	D	11.	D	12.	C
13.	C										

3.4 Free Response Answers

1 – 7 See Calculator

8. 



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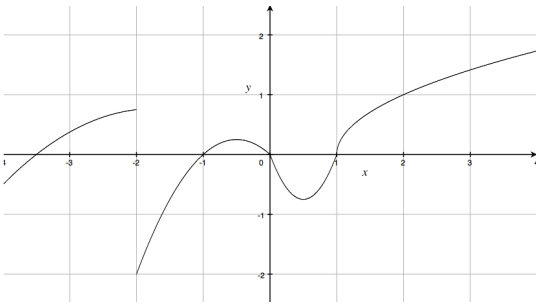

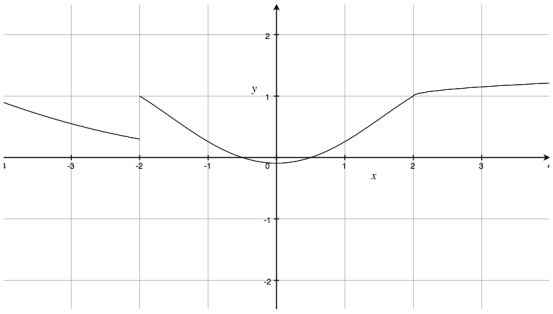

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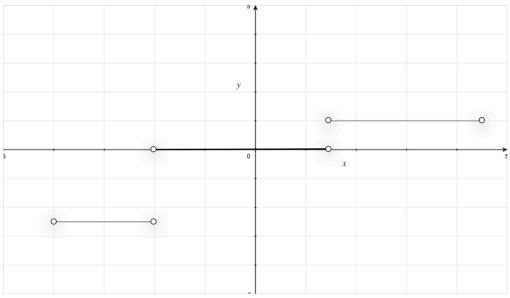

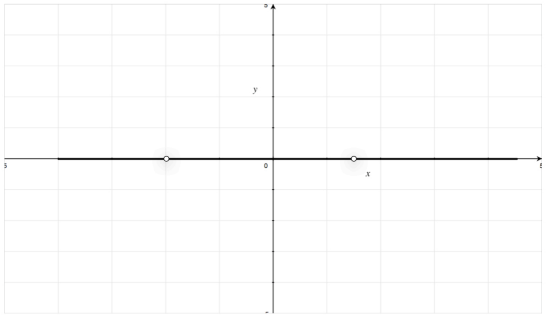

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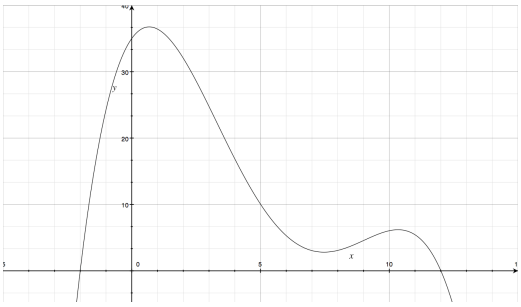

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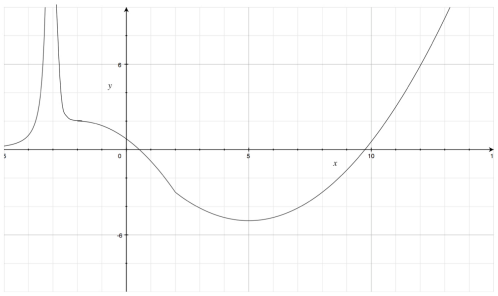

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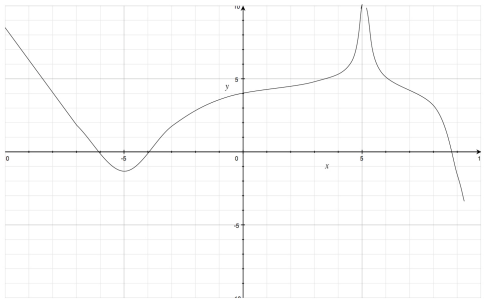

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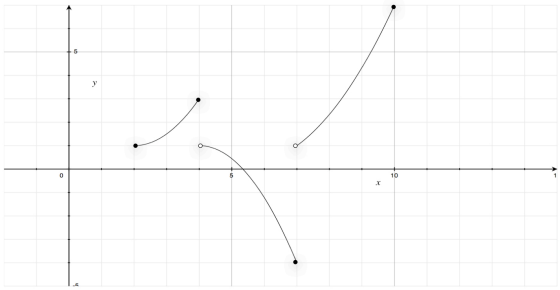

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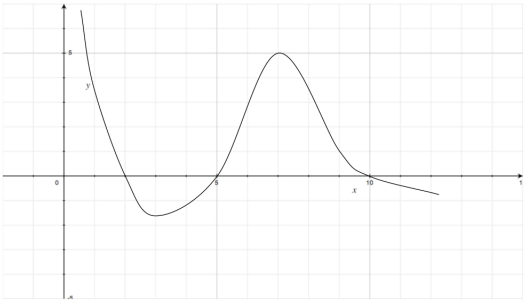
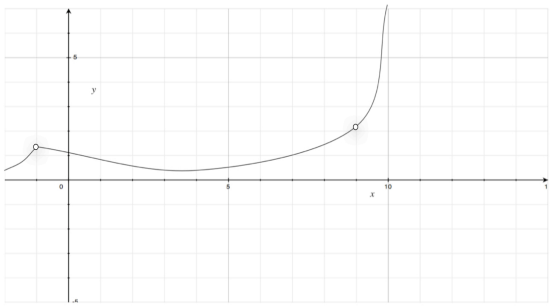
16.





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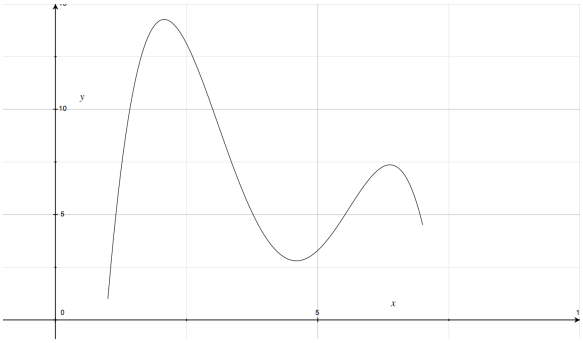

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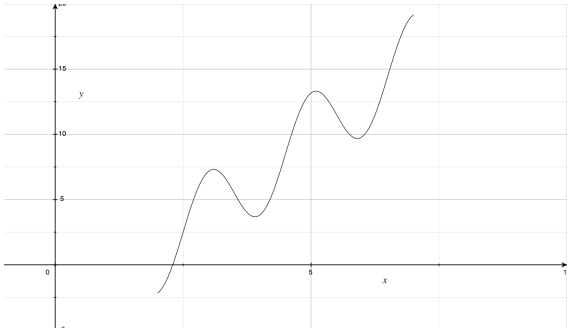

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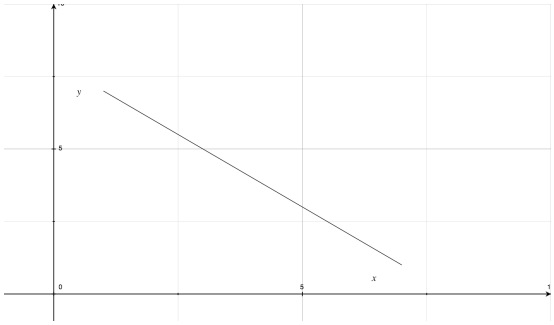

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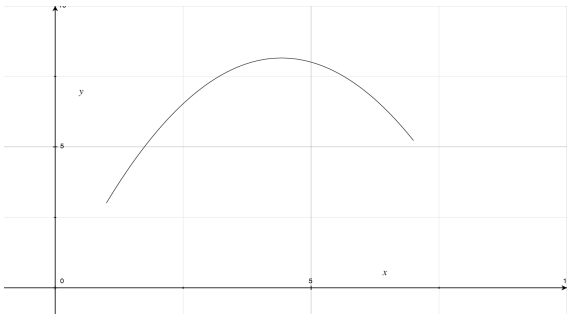


3.4 Multiple Choice KEY

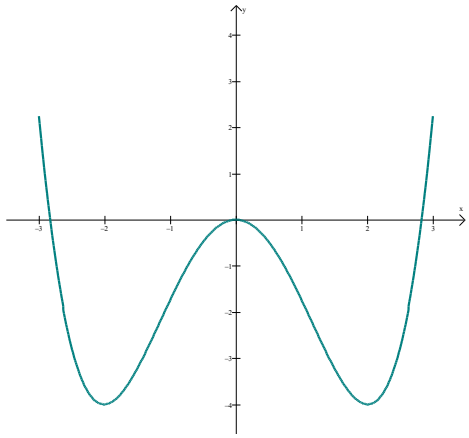
1. A 2. D 3. C 4. B 5. A 6. D

3.5 Free Response Answers

1a. min pts. @ $x = \pm 2$, max pt. @ $x = 0$ 1b. $x = \pm 1$

1c. inc: $x \in (-2,\ 0) \cup (2, 3)$; dec: $x \in (-3,\ -2) \cup (0,\ 2)$

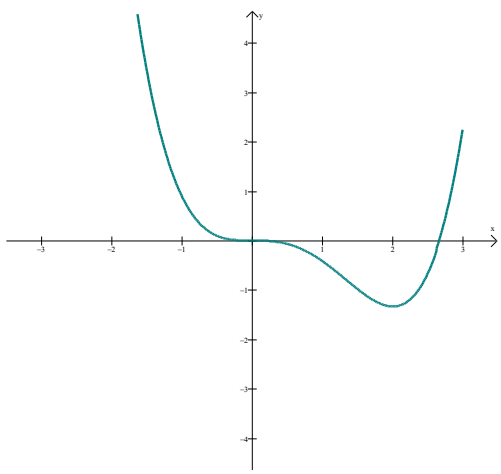
1d. CU: $x \in (-3,\ -1) \cup (1,\ 3)$; CD: $x \in (-1,\ 1)$



2a. min pt. @ $x = 2$ 2b. $x = 0,\ 1.3$

2c. inc: $x \in (2,\ 3)$; : $x \in (-3,\ 0) \cup (0,\ 2)$

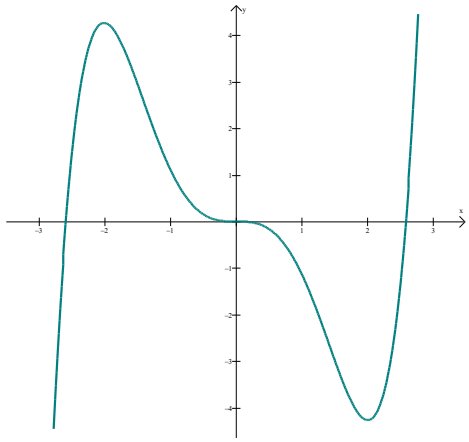
2d. CU: $x \in (-3,\ 0) \cup (1.3,\ 3)$; CD: $x \in (0,\ 1.3)$



3a. min pt. @ $x = 2$; pt. @ $x = -2$ 3b. $x = \pm 1.4,\ 0$

3c. inc: $x \in (-3,\ -2) \cup (2,\ 3)$; : $x \in (-2,\ 0) \cup (0,\ 2)$

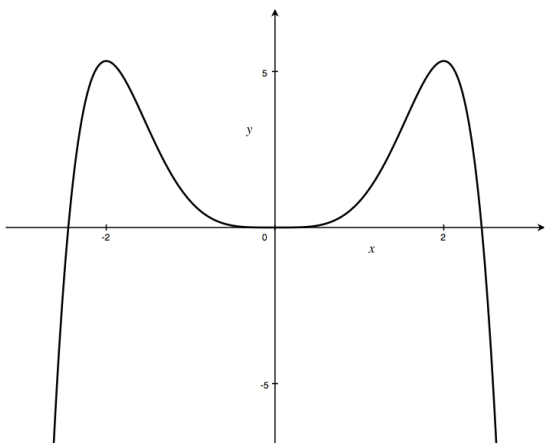
3d. CU: $x \in (-1.4,\ 0) \cup (1.4,\ 3)$; CD: $x \in (-3,\ -1.4) \cup (0,\ 1.4)$



4a. max pts. @ $x = \pm 2$; pt. @ $x = 0$ 4b. $x = \pm 1.5$

4c. inc: $x \in (-3,\ -2) \cup (0,\ 2)$; : $x \in (-2,\ 0) \cup (2,\ 3)$

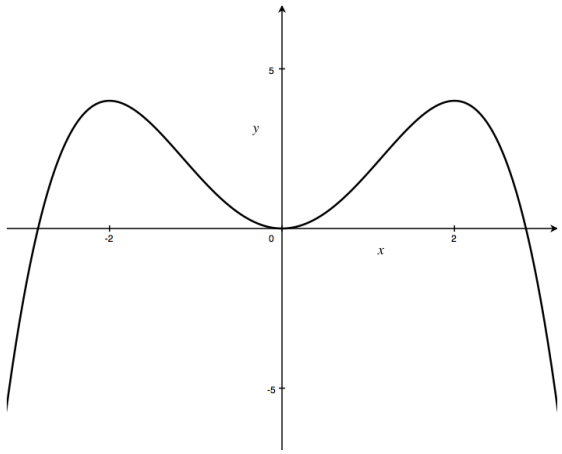
4d. CU: $x \in (-1.5,\ 1.5)$; $x \in (-3,\ -1.5) \cup (1.5,\ 3)$



5a. max pt. @ $x = \pm 2$; min pt. @ $x = 0$ 5b. $x = \pm 1.3$

5c. inc: $x \in (-3,\ -2) \cup (0,\ 2)$; dec: $x \in (-2,\ 0) \cup (2,\ 3)$

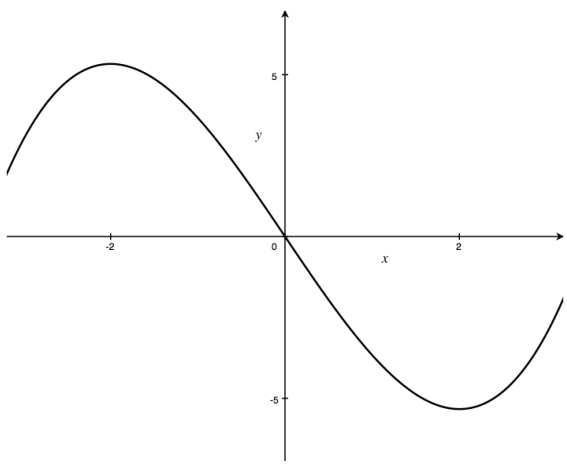
5d. CU: $x \in (-1.3,\ 1.3)$; : $x \in (-3,\ -1.3) \cup (1.3,\ 3)$



6a. min pt. @ $x = 2$; pt. @ $x = -2$ 6b. $x = 0$

6c. inc: $x \in (-3,\ -2) \cup (2,\ 3)$; dec: $x \in (-2,\ 2)$

6d. CU: $x \in (0,\ 3)$; CD: $x \in (-3,\ 0)$



7a. max pt. at $x = -4$ because $f'(x)$ switches from + to – and at $x = 7$ because $f'(x)$ is positive and the function stops; min pt. $x = 0$ because $f'(x)$ switches from – to + and at $x = -7$ because the function starts and $f'(x)$ is positive.

7b. $f(x)$ has at points of inflection at $x = \pm 2$ and 4 where $f'(x)$ switches from increasing to decreasing or vice versa.

8a. max pt. at $x = 0$ because $f'(x)$ switches from + to – , at $x = -7$ because the function starts and $f'(x)$ is positive , and at $x = 7$ because $f'(x)$ is positive and the function stops; min pt. $x = -2$ and 3 because $f'(x)$ switches from – to + .

8b. $f(x)$ has at points of inflection at $x = -4.5, -1,$ and 1.5 where $f'(x)$ switches from increasing to decreasing or vice versa.

9a. max pt. at $x = 4$ because $f'(x)$ switches from + to –; min pt. at $x = -7$ because the function starts and $f'(x)$ is positive , and at $x = 7$ because $f'(x)$ is negative and the function stops.
9b. $f(x)$ has at points of inflection at $x = -4$, 0, and 5.5 where $f'(x)$ switches from increasing to decreasing or vice versa.

3.5 Multiple Choice KEY

1. D 2. D 3. A 4. D 5. C 6. D

3.6 Free Response Answers

1a. $t = 2$ and $t = 5$ b. left c. 49 units right of the origin
1d. –6 e. Speeding up

2a. $t = 2$ b. right c. 14 units right of the origin
d. 6 e. Speeding up

3a. $t = -4$, $t = \frac{4}{3}$, and $t = 3$ b. neither; it is stopped
c. 141 units above of the origin d. 420 e. Neither

4a. $t = -3$, $t = -1$, $t = 2$, and $t = 3$ b. neither; it is stopped
c. 1431 units above of the origin d. 1440 e. Neither

5a. $t = 0$, $t = \pm\sqrt{3}$ b. left c. $s = -108$
d. $a(3) = -450$ e. Speeding up

6a. $t = 0$, $t = \pm\sqrt{2.5}$ b. Right c. $s = 0$
d. $a(3) = 98$ e. Speeding up

7. $h(2) = 89$ feet

8. $h(30) = 4420$ meters

9. $t = 7.473$ sec
$t = 1.215$ sec

10. $x(2.5) = -2.25$

11. $x(-3) = 112;\ x(7) = -388$

12. $x(0) = 12;\ x(2) = -4$
$x(0.894) = -19.090;\ x(-0.894) = 23.890$

13. $x(3.5) = 42.5;\ v(3.5) = -13.5$

14. $x(2) = 13;\ v(2) = 0$

15. $x(-2) = -1984;\ v(-2) = 1200,$
$x\left(\frac{20}{9}\right) = 222.398;$
$v\left(\frac{20}{9}\right) = -154.864$

16a. $t \in (-\infty, -2) \cup (6, \infty)$ b. 2 c. $x(t) = \frac{1}{3}t^3 - 2t - 12t + \frac{41}{3}$

17a. $t \in (-4, 0) \cup (4, \infty)$ b. -27 c. $x(t) = \frac{1}{6}t^6 - 4t^4 + 2$

18a. $t \in (0, 3)$ b. dne c. $x(t) = -\frac{1}{3}(9 - t^2)^{3/2} + \frac{14}{3}$

19a. $t \in \left(\frac{\pi}{2}, \pi\right)$ b. 704.595 c. $x(t) = -\cot(1) + \cot(1)$

20a. Always b. -0.107 c. $x(t) = \frac{3}{2}\tan^{-1}\frac{t}{2} + 4 - \frac{3\pi}{8}$

21a. $t \in (0, \infty)$ b. -0.030 c. $x(t) = \frac{1}{2}\ln(t^2 + 4) - \ln 2$

22a) $-5\ ft/\text{min}^2$ 22b) The car traveled 962 feet in these 18 minutes.

22c) Twice 22d) $54.35\ ft/\text{min}$

23a) $-0.08\ mi/hr^2$ 23b) Decreasing

23c) 6.672 miles. 23d) Yes

24a) The team ran approximately 7.65 miles in these 60 minutes.

24b) The result would equal the time, on average, it would take to complete one mile.

24c) $\frac{2}{15000}\ mi/\text{min}^2$ 24d) Yes

25a) $-\frac{1}{6}\ m/\text{sec}^2$ 25b) Slowing down 25c) 13,190 meters

3.6 Multiple Choice Key

1. B 2. C 3. C 4. E 5. A 6. D

7. A 8. B 9. D

3.7 Free Response Answers

1. $\frac{3}{200\pi}$ ft/min 2. $-\frac{120}{7}$ ft/sec 3. $\frac{\pi}{2}$ in^2/min

4. $-\frac{3}{\pi}$ ft/min 5. $\frac{96}{25}\ ft/\text{sec}$ 6. $\frac{25}{3}\ ft/\text{min}$

7. $\frac{dz}{dt} = 65\, mi/h$ 8. 6 ft^2/min 9. –80 in^3/min

10. $-43.393\, in^3/\text{sec}$ 11. $-\frac{1250}{\pi} m^3/\text{min}$ m^3/min 12. $-\frac{\sqrt{3}}{5}$ ft/min

13. $6.4 \frac{ft}{\text{sec}}$ 14. $143.002\, in^2/\text{sec}$ 15. $-\frac{55}{6}$

16a. $2\pi\sqrt{2}$ in/sec 16b. $8\pi - 16$ in^2/sec

3.7 Multiple Choice Answers

1. C 2. D 3. D 4. E 5. D 6. D

7. C 8. C 9. C 10. E 11. D

3.7 Set B Free Response Answers

1. $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

2. 2 cm/sec

3. $250\sqrt{3}$

4. $\frac{-1}{20\pi}$ cm/min

5. $\frac{25}{3}$ ft/min

6. 65 mph

7. -1.6 cm/min

Derivative Applications Test Answers

1.	B	2.	D	3.	E	4.	C	5.	D	6.	B
7.	C	8.	B	9.	B	10.	A	11.	C	12.	D
13.	C										

Free Response:

1a. $v(t) > 0 \rightarrow t \in [0, 1.465] \cup [1.845, 2]$

1b. $a(t) = 3t^4 \cos t^3 + 2t \sin t^3$

1c. $y(t) = -\frac{1}{3}\cos t^3 + \frac{10}{3}$

1d. $y(1.465) = \frac{11}{3}$

2a. $x = -7$ *and* 0.4. $x = -7$ because it is the left endpoint and $f' > 0$ after $x = -7$. $x = 0.4$ because f' switches from positive to negative.

b. $x = \pm 4, -2, 0, 1$ because f' switches from increasing to decreasing or vice versa.

c. $x \in (-5.25, -4)$ and $x \in (0,\ 0.4)$ because f' is negative and increasing is negative and increasing switches from increasing to decreasing or vice versa.

d. $x \in (-5.25, -4)$ and $x \in (0,\ 0.4)$ because f' is negative and increasing

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3a. $\frac{d}{dx}\left[x^2 + 4xy + y^2 = -12\right] \rightarrow 2x + 4x\frac{dy}{dx} + y(4) + 2y\frac{dy}{dx} = 0$

$$(4x+2y)\frac{dy}{dx}=-2x-4y \rightarrow \frac{dy}{dx}=-\frac{x+2y}{2x+y}$$

3b. $(4,-2)$ *and* $(-4,2)$

3c. $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\frac{x+2y}{2x+y}\right]=\frac{(2x+y)\left(-1-2\frac{dy}{dx}\right)-(-x-2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2}$$

$(4,-2)$ is at a minimum because $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$

$(-4,2)$ is at a minimum because $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}<0$
