Calculus'21-22
Fall Final Exam
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Name Solution Key

Score 95 +5

1a. $\frac{d}{dx}\left[x^3-4\sqrt[4]{x^5}+\pi^3-\frac{1}{\sqrt[8]{x^3}}+\frac{1}{x}\right]$

$$=\frac{d}{dx}\left[x^3-4x^{5/4}+\pi^3-x^{-3/8}+x^{-1}\right]$$

$$=3x^2-5x^{1/4}+\frac{3}{8}x^{-11/8}-x^{-2}$$

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b. $\frac{d}{dx}\left(\sqrt[5]{24+5x^3}\right)=\frac{d}{dx}\left(24+5x^3\right)^{1/5}$

$$=\frac{1}{5}\left(24+5x^3\right)^{-4/5}\left(15x^2\right)$$

$$=\frac{3x^2}{\left(24+5x^3\right)^{4/5}}$$

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c. $\frac{d}{dx}\left(\overset{u}{3x^4}\,\overset{v}{\ln(5x^2)}\right)$

$$u\,Dv + v\,Du$$

$$= 3x^4\left(\frac{2}{x}\right) + \ln(5x^2)\,(12x^3$$

$$= 6x^3 + \ln 5x^2\,(12x^3)$$

$$= 6x^3\left(1 + 2\ln(5x^2)\right)$$

$$u = 3x^4$$

$$\frac{du}{dx} = 12x^3$$

$$v = \ln 5x^2$$

$$\frac{dv}{dx} = \frac{1}{5x^2}(10x) = \frac{2}{x}$$

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d. $\frac{d}{dx}\left(\frac{7x}{6-x^3}\right)$

$$= \frac{(6-x^3)(7)-(7x)(-3x^2)}{(6-x^3)^2}$$

$$= \frac{42-7x^3+21x^3}{(6-x^3)^2}$$

$$= \frac{42+14x^3}{(6-x^3)^2}$$

2a. $\int\left(13x^6+13^x-\frac{1}{\sqrt[13]{x^6}}+\frac{1}{13x^6}\right)dx$

$= \int \left(13x^6 + 13^x - x^{-6/13} + \frac{1}{13}x^{-6}\right)dx$

$= \frac{13}{7}x^7 + \frac{13^x}{\ln 13} - \frac{x^{7/13}}{7/13} + \frac{1}{13}\frac{x^{-5}}{-5} + c$

$= \frac{13}{7}x^7 + \frac{13^x}{\ln 13} - \frac{13}{7}x^{7/13} - \frac{1}{65}x^{-5} + c$

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b. $\frac{1}{3}\int\frac{3\,t^2dt}{\sqrt{t^3+5}}$

$u = t^3+5$

$du = 3t^2dt$

$= \frac{1}{3}\int u^{-1/2}\,du$

$= \frac{1}{3}\,\frac{u^{1/2}}{1/2} + c$

$= \frac{2}{3}(t^3+5)^{1/2} + c$

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c. $\int\left(6\sqrt{x^3}+\frac{x^3}{e^{3x^4}}-\tan(3x)\right)dx$

$= \int 6x^{3/2}\,dx + \frac{-1}{12}\int -12x^3 e^{-3x^4}\,dx - \frac{1}{3}\int \tan 3x\,(3)\,dx$

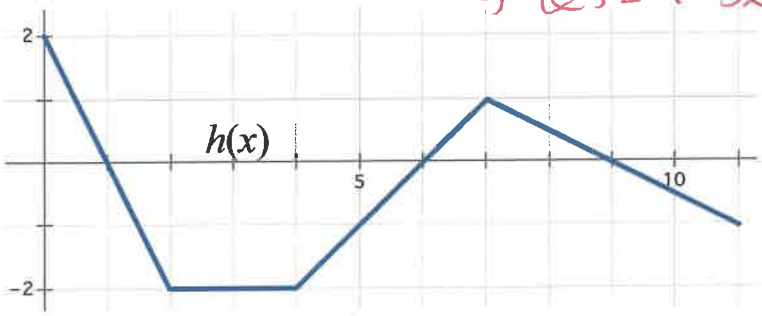
$= \frac{6x^{5/2}}{5/2} - \frac{1}{12}\int e^{u_1}\,du_1 - \frac{1}{3}\int \tan u_2\,du_2$

$= \frac{12}{5}x^{5/2} - \frac{1}{12}e^{u_1} - \frac{1}{3}\ln|\sec u_2| + c$

$= \frac{12}{5}x^{5/2} - \frac{1}{12}e^{-3x^4} - \frac{1}{3}\ln|\sec 3x| + c$

$$f(x) = 4x - x^3$$

$f'(x) = 4 - 3x^2$



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

3. Let $f(x)$ be the function defined by the equation above, let $h(x)$ be the function whose graph is given above, and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given on the table above.

(a) Find the equation of the line tangent to $g(x)$ at $x = 4$.

$y - 3 = 6(x - 4)$

(b) Let K be the function defined by $K(x) = h(f(x))$. Find $K'(1)$.

$K'(x) = h'(f(x)) \cdot f'(x)$

$f(1) = 3$

$K'(1) = h'(3) \cdot f'(1)$

$= 0 \cdot (1)$

$= 0$

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(6)$.

$$M(x) = g'(x) \cdot f(x) + f'(x)\, g(x)$$

$$m'(6) = g'(6)\,(f(6) + f'(6)\, g(6)$$

$$= 12 \cdot (0) + 1\,(6)$$

$$= 6$$

(d) Let J be the function defined by $J(x) = \dfrac{g(x)}{h(2x)}$. Find $J'(4)$.

$$J'(4) = \frac{h(8) \cdot g'(4) - g(4)\, h'(8) \cdot (2)}{[h(8)]^2}$$

$$= \frac{\frac{1}{2}(6) - 3\,(-\frac{1}{2})\,(2)}{(\frac{1}{2})^2}$$

$$= \frac{\cancel{3 + 3/2}}{\cancel{1/4}} = \overset{24}{\cancel{48}}$$

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4. The velocity of an object in rectilinear motion is given by $v(t)=t\sqrt{t^2+3}$ with $x(1)=2$.

a) Find the particular position equation of the object.

$$x(t)=\int t\,(t^2+3)^{1/2}\,dt = \frac{1}{2}\int u^{1/2}\,du$$
$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + c$$
$$= \frac{1}{3}(t^2+3)^{3/2} + c$$

$x(1)=2 \Rightarrow 2=\frac{1}{3}(8)+c \Rightarrow c=-2/3 \qquad x(t)=\frac{1}{3}(t^2+3)^{3/2}-2/3$

b) Find the acceleration equation of the object. What is the acceleration at $t=1$?

$$a=\frac{dv}{dt}=t\left(\tfrac{1}{\cancel{2}}(t^2+3)^{-1/2}(\cancel{2}t)\right)+(t^2+3)^{1/2}(1)$$
$$=\frac{t^2}{(t^2+3)^{1/2}}+(t^2+3)^{1/2} = \frac{t^2+t^2+3}{(t^2+3)^{1/2}} = \frac{2t^2+3}{(t^2+3)^{1/2}}$$

$a(1)=5/2$

c) Find the total distance traveled by the object between $t=-1$ and $t=2$. Show the set up, but solve by calculator.

$$\text{DISTANCE} = \int_{-1}^{2} |v(t)|\,dt$$
$$= -\int_{-1}^{0} v(t)\,dt + \int_{0}^{2} v(t)\,dt$$
$$= 5.376$$

5. Consider the differential equation $\frac{dy}{dx} = \frac{2x+1}{y^2}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $(-1, 3)$. The function $y = f(x)$ is defined for all real numbers.

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a) Find the equation of the line tangent to $y = f(x)$ at $f(1) = 2$

$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3}{4}$

$y - 2 = \frac{3}{4}(x - 1)$

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b) Use your answer in part a) to approximate $f(0.9)$.

$f(.9) \approx y = 2 + \frac{3}{4}(.9 - 1)$

$= 2 - (.075)$

$= 1.925$

c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(-1) = 3$.

$$\int y^2\,dy = \int (2x+1)\,dx$$

$$\frac{y^3}{3} = x^2 + x + C$$

$$(-1,3) \rightarrow 9 = (-1)^2 + 1 + C$$
$$9 = C$$

$$\frac{y^3}{3} = x^2 + x + 9$$

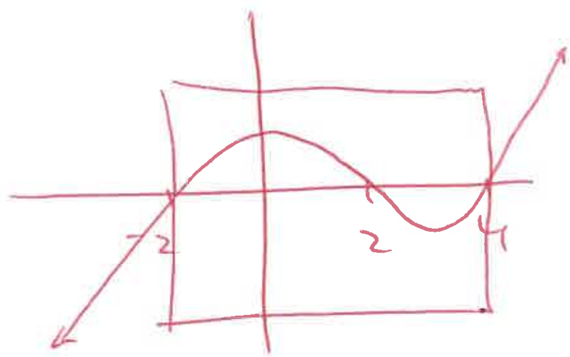
$$y^3 = 3x^2 + 3x + 27$$

$$y = \sqrt[3]{3x^2 + 3x + 27}$$

6. Consider the function $f(x) = x^3 - 4x^2 - 4x + 16$.

$x^2(x-4) - 4(x-4) = (x-2)(x+2)(x-4)$

a) Draw the graph of $f(x)$ on $x \in [-2, 4]$.



b) Prove that the zero of $f(x)$ is at $x = 2$.

$f(2) = 2^3 - 4(2)^2 - 4(2) + 16$

$= 0$

c) Find the exact value of $\int_{-1}^{4} f(x)\,dx$. Show the antiderivative and boundary insertion steps.

$$\int_{-1}^{4} (x^3 - 4x^2 - 4x + 16)\,dx$$

$$= \frac{x^4}{4} - \frac{4x^3}{3} - \frac{4x^2}{2} + 16x \Big]_{-1}^{4}$$

$$= \left(64 + \frac{128}{3} - 32 + 64\right) - \left(\frac{1}{4} + \frac{4}{3} - 2 - 16\right)$$

$$= 27.083$$

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d) Find the exact area between the x-axis and $f(x)$ on $x \in [-2, 4]$. Show the antiderivative and boundary insertion steps.

$$\int_{-2}^{2} f(x)\,dx - \int_{2}^{4} f(x)\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 - 2x^2 + 16x\right]_{-2}^{2} - \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 - 2x^2 + 16x\right]_{+2}^{4}$$

$$= \left(4 - \frac{32}{3} - 8 + 32\right) - \left(4 + \frac{32}{3} - 8 - 32\right) - \left(64 - \frac{128}{3} - 32 + 64\right) - \left(4 - \frac{32}{3} - 8 + 32\right)$$

$$= 49.333$$

t in days	1	3	4	7	11	15	21
$S(t)$ in inches per day	3	7	6	2	7	2	4
$M(t)$ in inches per day	0.8	0.4	0.2	0	0	0.4	0.8

7. The snowfall in Squaw Valley is tracked by the US Weather Service. For the month of March, 2021, the values on the table above were gathered. $S(t)$ represents the rate of snowfall in inches per day, $M(t)$ represents the rate at which the snow melts in inches per day, and t is measured in days.

a) Use a right-hand Reimann Sum to approximate $\int_1^{21} S(t)dt$.

$$\int^{21} S(t)\,dt \approx 2(7)+1(6)+3(2)+4(7)+4(2)+6(4)$$

$= 86$ INCHES

b) Using the data on the table, approximate $M'(19)$.

$$M'(19) \approx \frac{.8-.4}{\cancel{4}-\cancel{22}} = .067 \text{ IN/DAY}^2$$

$21-15$

EC) Using the correct units, explain the answers to a) and b) above in the context of the problem.

a) 86 INCHES OF SNOW FELL IN SQUAW VALLEY DURING THESE 21 DAYS

b) THE RATE AT WHICH THE SNOW IS MELTING IS INCREASING APPROXIMATELY BY .2 INCHES PER DAY PER DAY ON DAY 19.