Chapter 6 Overview: Continuity and Piece-wise Defined Functions

One of the major and foundational issues in Calculus is whether a function is continuous. Visually, continuous simply means that there is not break in the graph. One can draw the graph from left to right without lifting the pencil.

All the algebraic families of functions considered in the previous chapters and the transcendental families of functions in the upcoming chapters are continuous in their domain. In order to investigate continuity, another kind of function, namely a piece-wise defined function, needs to be examined. As the name implies, a piece-wise defined function is one which is defined in pieces. In other words, there are different equations for different parts of the domain. For example:

\[
y = \begin{cases} 
  x + 3, & \text{if } -4 \leq x \leq -2 \\
  1, & \text{if } -2 \leq x \leq 0 \\
  1 - 2x, & \text{if } 0 \leq x \leq 2 
\end{cases}
\]

or

\[
f(x) = \begin{cases} 
  x^2 + 1, & \text{if } x \geq -1 \\
  3 - x, & \text{if } x < -1 
\end{cases}
\]

In order to analyze these functions, a different kind of limit needs to be considered—a limit from one side or the other of the number that separates the domain into parts.

**Note:** All the algebraic and transcendental functions are continuous in their domain.
6-1: One–Sided Limits

In a previous chapter, the \( \lim_{x \to a} f(x) \) was basically defined as what the \( y \)-value should be when \( x = a \), even if \( a \) is not in the domain. For all of the families of functions this was enough. But what if that \( y \)-value might be two different numbers? In a graph of a piece-wise defined function like this,

![Graph of a piece-wise defined function]

It is not clear whether the \( y \)-value for \( x = 1 \) is 1 or 2. The table of values indicates \( x \)-values less than 1 have \( y \)-values that approach 1 while \( x \)-values greater than 1 have \( y \)-values that approach 2:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td>0.7225</td>
</tr>
<tr>
<td>0.9</td>
<td>0.81</td>
</tr>
<tr>
<td>0.95</td>
<td>0.9025</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1.05</td>
<td>1.95</td>
</tr>
<tr>
<td>1.1</td>
<td>1.9</td>
</tr>
<tr>
<td>1.15</td>
<td>1.85</td>
</tr>
</tbody>
</table>

The algebraic ways to describe these differences are one-sided limits. The symbols used are:

\[
\lim_{x \to a^-} f(x) \quad \text{or} \quad \lim_{x \to a^+} f(x)
\]
\[
\lim_{x \to a^-} f(x) \text{ reads “the limit as } x \text{ approaches } a \text{ from the left,” while } \lim_{x \to a^+} f(x) \text{ reads “the limit as } x \text{ approaches } a \text{ from the right.”}
\]

In this example, \( \lim_{x \to 1^-} f(x) = 1 \) and \( \lim_{x \to 1^+} f(x) = 2 \). The \( \lim_{x \to 1} f(x) \) does not exist, because the one-sided limits do not equal each other. “The \( \lim_{x \to 1} f(x) \) does not exist,” means there is not a single REAL number that the limit equals.

**LEARNING OUTCOMES**

Evaluate one-sided limits graphically, numerically, and algebraically.
Interpret vertical asymptotes in terms of one-sided limits.
Evaluate two-sided limits in terms of one-sided limits.

**EX 1** Does \( \lim_{x \to 1^-} f(x) \) exist for \( f(x) = \begin{cases} 
  x^2 + 1, & \text{if } x \geq -1 \\
  3 - x, & \text{if } x < -1
\end{cases} \)?

For \( \lim_{x \to 1^-} f(x) \) to exist, \( \lim_{x \to 1^-} f(x) \) must equal \( \lim_{x \to 1^+} f(x) \). The domain states that any \( x \)-value less than \(-1\) goes into \( 3 - x \). Therefore,

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3 - x) = 3 - (-1) = 4
\]

Similarly, \( x \)-values greater than \(-1\) go into \( x^2 + 1 \), and

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 1) = (-1)^2 + 1 = 2
\]

The two one-sided limits are not equal, therefore, \( \lim_{x \to 1} f(x) \) does not exist.
In the graph, it can be seen that the two pieces do not come together:

The other situation where one-sided limits come into play is at vertical asymptotes. Here, the $y$-value goes to infinity (or negative infinity), which is why these limits are also called **Infinite Limits**.

EX 2 Does $\lim_{x \to -1} f(x)$ exist for $f(x) = \begin{cases} x^2 + 1 & x \neq -1 \\ 4 & x = -1 \end{cases}$

For $\lim_{x \to -1} f(x)$ to exist, $\lim_{x \to -1^-} f(x)$ must equal $\lim_{x \to -1^+} f(x)$. The domain states that any $x$-value less than $-1$ goes into $x^2 + 1$. Therefore,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1} (x^2 + 1)$$

$$= (-1)^2 + 1$$

$$= 2$$

Similarly, $x$–values greater than $-1$ also go into $x^2 + 1$, and

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1} (x^2 + 1)$$

$$= (-1)^2 + 1$$

$$= 2$$

The two one-sided limits are equal, therefore, $\lim_{x \to -1} f(x) = 2$. 

328
EX 2 Evaluate a) \( \lim_{x \to 2} \frac{2}{x-2} \) and b) \( \lim_{x \to 2} \frac{1-x}{2-x} \)

In both these cases, there is a vertical asymptote to consider. The limits, being \( y \) values, would be either positive or negative infinity, depending on if the curve went up or down on that side of the asymptote. Looking at the limit algebraically (as opposed to graphically) though:

a) \( \lim_{x \to 2} \frac{2}{x-2} = \frac{2}{0^-} = -\infty \)

Note that the 0\(^-\) is not “from the left” because the 2 is from the left, but rather that \( x - 2 \) is negative for any \( x \) values less than 2.

b) \( \lim_{x \to 2^+} \frac{1-x}{2-x} = \frac{-1}{0^-} = +\infty \)

Note in this case that the numerator’s sign affects the outcome (two negatives make a positive).

Note: It is debatable whether the infinite limits exist or not. It depends on whether “exist” is defined as equal to a real number or not. Some books would say that \( \lim_{x \to 2^2} \frac{2}{x-2} \) exists because there is one answer. It just happens to be a transfinite number. Other books would say \( \lim_{x \to 2} \frac{2}{x-2} \) does not exist (DNE) because the answer is not a real number.

There are certain infinite limits that must be known:

\[
\begin{align*}
\lim_{x \to 0^+} \frac{1}{x} &= \infty \\
\lim_{x \to 0^-} \frac{1}{x} &= -\infty \\
\lim_{x \to 0^+} (\ln x) &= -\infty
\end{align*}
\]
6-1 Multiple Choice Homework

1. For which of the following does \( \lim_{x \to 4} f(x) \) exist?

   ![Graph of f](image1)
   ![Graph of f](image2)
   ![Graph of f](image3)

   (a) I only  (b) II only  (c) III only  (d) I and II only  (e) I and III only

2. The figure below shows the graph of a function \( f \) with domain \( 0 \leq x \leq 4 \). Which of the following statements are true?

   ![Graph of f](image4)

   I. \( \lim_{x \to 2^-} f(x) \) exists
   II. \( \lim_{x \to 2^+} f(x) \) exists
   III. \( \lim_{x \to 2} f(x) \) exists

   (a) I only  (b) II only  (c) I and II only  (d) I and III only  (e) I, II, and III
3. The function $f$ is defined on the interval $[-5, 5]$ and its graph is shown below. Which of the following statements are true?

- I. $\lim_{x \to 1} f(x) = -1$
- II. $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 2$
- III. $\lim_{x \to -1} f(x) = f(-3)$

(a) I only (b) II only (c) I and II only (d) II and III only (e) I, II, and III

4. Let $f$ be a function with $f(2) = 4$ and derivative $f'(x) = \sqrt{x^3} + 1$. Using a tangent line approximation to the graph of $f$ at $x = 2$, estimate $f(2.2)$.

(a) 4.0 (b) 4.2 (c) 4.4 (d) 4.6 (e) 4.8

5. Which of the following statements are true?

- I. $\lim_{x \to 1} f(x)$ does not exist
- II. $\lim_{x \to -1} f(x) = 1$
- III. $\lim_{x \to 1} f(x) = 3$

(a) I only (b) I and II (c) II and III (d) I, II, and III (e) III only
For problems 1-4, use the figure above and assume there is a vertical asymptote at $x = 5$.

1 a) $\lim_{x \to -10^-} f(x)$
   b) $\lim_{x \to -10^+} f(x)$
   c) $\lim_{x \to -10} f(x)$
   d) $f(-10)$

2 a) $\lim_{x \to 5^-} f(x)$
   b) $\lim_{x \to 5^+} f(x)$
   c) $\lim_{x \to 5} f(x)$
   d) $f(-5)$

3 a) $\lim_{x \to 5^-} f(x)$
   b) $\lim_{x \to 5^+} f(x)$
   c) $\lim_{x \to 5} f(x)$
   d) $f(5)$

4 a) $\lim_{x \to 10^-} f(x)$
   b) $\lim_{x \to 10^+} f(x)$
   c) $\lim_{x \to 10} f(x)$
   d) $f(10)$

Evaluate the following limits.

5. $\lim_{x \to 1^-} \frac{2x}{x^2 - 1}$

6. $\lim_{x \to 3^+} \frac{(x+4)(x-3)}{(x-2)(x+3)(x-1)}$

7. $\lim_{x \to 2^-} \frac{1-2x}{x^2 - 4x + 4}$

8. $\lim_{x \to 2^+} \frac{7x}{x^2 - 4}$
6-2: Continuity and Discontinuity

One of the main topics early in Calculus (which usually confuses everyone) is CONTINUITY. It really is a simple concept, which, like the limit, is made very complicated by its mathematical definition. Let us take a look at the formal definition of continuous:

Vocabulary:
1. **Continuous** – a function $f(x)$ is continuous at $x = a$ if and only if:
   
   i) $f(a)$ exists,
   
   ii) $\lim_{x \to a} f(x)$ exists,
   
   and iii) $\lim_{x \to a} f(x) = f(a)$

A closer, more mathematical look at the definition shows it means:

i) "$f(a)$ exists" means $a$ must be in the domain. (Actual $y$ value)

ii) "$\lim_{x \to a} f(x)$ exists" means $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$. (Limiting $y$ value)

iii) "$\lim_{x \to a} f(x) = f(a)$" should be self explanatory.

Part ii) is the least familiar part of the formal definition, but $\lim_{x \to a^-} f(x)$ reads “the limit as $x$ approaches $a$ from the left,” while $\lim_{x \to a^+} f(x)$ reads “the limit as $x$ approaches $a$ from the right.” These are called **one-sided limits** as discussed in a previous section.
Continuous means the curve has no breaks in it. Breaks are called discontinuities and there are two general kinds:

2. **Removable Discontinuity** – Defn: at \( x = a \), \( \lim_{x \to a} f(x) \) exists, but \( f(a) \) does not exist or \( \lim_{x \to a} f(x) \neq f(a) \)

   Means: there is a “hole” in the curve

There are two kinds of Removable Discontinuity:

1a. **Point of Exclusion (POE):** This is where the \( x \)-value is not in the domain because it would yield \( y = \frac{0}{0} \). It is “removable” because what the \( y \)-value should be is known—namely, the limiting \( y \)-value (explored it in a previous chapter). Removable discontinuities are points; therefore, they have both \( x \) and \( y \)-coordinates.

![Graph showing a removable discontinuity at (2, 3)](image)

Note the missing point at (2, 3).
1b. **Point of Displacement (POD):** This happens in a piecewise-defined function where the $x$-value would be a point of exclusion, but a specific $y$-value—*not equal to the limiting $y$-value*—is defined.

Here, $2$ is in the domain, but actual $y$-value $f(2) = 5$, does not equal the limiting $y$-value, $\lim_{x \to 2} f(x) = 3$.

3. **Essential Discontinuity** – Defn: at $x = a$, $\lim_{x \to a} f(x)$ does not exist

   Means: the two parts of the curve do not come together

There are two kinds of Essential Discontinuity:

2a. **Vertical Asymptotes:** These are vertical boundary line (drawn—not graphed—as a dotted line) on the graph at the $x$-value where the denominator of the function would equal zero, but the numerator would not equal zero (the actual $y$-value approaches infinity). The two parts of the curve cannot come together without crossing the boundary line.
2b. **Jump Discontinuity**: This occurs in a piecewise defined function when the two pieces of the equation approach different $y$-values, i.e. there are two limiting $y$-values, one from the left and one from the right, hence the limit does not exist.

![Graph showing jump discontinuity]

**LEARNING OUTCOME**

Prove continuity or discontinuity of a given function.
EX 1  Is \( f(x) = \begin{cases} 
  x^2 + 1, & \text{if } x \geq -1 \\
  3 - x, & \text{if } x < -1
\end{cases} \) continuous at \( x = -1 \)?

To answer this question, check each part of the definition.

i)  Does \( f(-1) \) exist? Yes, the top line says that \(-1\) is in the domain and that \( y = 2 \) if \( x = -1 \).

So, \( f(-1) = 2 \).

ii)  Does the \( \lim_{x \to -1} f(x) \) exist? Answer this by looking at the two one-sided limits.

\[
\lim_{x \to -1^{-}} f(x) = \lim_{x \to -1^{-}} (3 - x) = 4 \\
\lim_{x \to -1^{+}} f(x) = \lim_{x \to -1^{+}} (x^2 + 1) = 2
\]

Since \( \lim_{x \to -1^{-}} f(x) \neq \lim_{x \to -1^{+}} f(x) \) then \( \lim_{x \to -1} f(x) \) DNE.

There is no need to check the third part of the definition. Once one part of the definition is not met, discontinuity is proven.

\[
\text{So } f(x) \text{ is not continuous at } x = -1.
\]

The graph shows that the two pieces do not come together:
EX 2 Let \( g(x) = \begin{cases} x^2 - 2x + 1, & \text{if } x > -1 \\ 2, & \text{if } x = -1 \\ 3 - x, & \text{if } x < -1 \end{cases} \). Is \( g(x) \) continuous at \( x = -1 \)?

To answer this question, check each part of the definition.

i) Does \( g(-1) \) exist? Yes, the middle line says that \(-1\) is in the domain and that \( y = 2 \) if \( x = -1 \).

So, \( g(-1) = 2 \).

ii) Does the \( \lim_{x \to -1} g(x) \) exist? Answer this by looking at the two one-sided limits.

\[
\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} 3 - x = 4 \quad \lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} x^2 - 2x + 1 = 4
\]

Since \( \lim_{x \to -1^-} g(x) = \lim_{x \to -1^+} g(x) \) then \( \lim_{x \to -1} g(x) = 4 \).

iii) Does \( \lim_{x \to -1} g(x) = g(-1) \)? No. \( \lim_{x \to -1} g(x) = 4 \), while \( g(-1) = 2 \)

So \( g(x) \) is not continuous at \( x = -1 \).
EX 3 Let \( F(x) = \begin{cases} x^2 - 5, & \text{if } x > 0 \\ x + 2, & \text{if } x < 0 \end{cases} \). Is \( F(x) \) continuous at \( x = 0 \)? Why not?

\( F(x) \) is not continuous at \( x = 0 \) because 0 is not in the domain. Notice neither inequality includes an equal sign.

Summary of Continuity:

\begin{align*}
\text{Removable Discontinuity} & \quad \lim_{x \to a} f(x) \text{ does exist} \\
\text{Essential Discontinuity} & \quad \lim_{x \to a} f(x) \text{ does not exist}
\end{align*}

\( f(a) \) does not exist

\begin{align*}
f(a) & \text{ exists} \\
\text{Point of Exclusion (POE)} & \quad \text{Vertical asymptotes} \\
\text{Point of Displacement (POD)} & \quad \text{Jump Discontinuity}
\end{align*}
6-2 Multiple Choice Homework

1. Which of the following is true about the function \( f \) if \( f(x) = \frac{(x - 1)^2}{2x^2 - 5x + 3} \)?

   I. \( f \) is continuous at \( x = 1 \)
   II. The graph of \( f \) has a vertical asymptote at \( x = 1 \)
   III. The graph of \( f \) has a horizontal asymptote at \( y = \frac{1}{2} \)

   (a) I only
   (b) II only
   (c) III only
   (d) II and III only
   (e) I, II, and III

2. Which function is **not** continuous everywhere?

   (a) \( y = |x| \)
   (b) \( y = x^{\frac{2}{3}} \)
   (c) \( y = \sqrt{x^2 + 1} \)
   (d) \( y = \frac{x}{x^2 + 1} \)
   (e) \( y = \frac{4x}{(x+1)^2} \)

3. If \( f \) is continuous at \( x = 1 \), and if \( f(x) = \begin{cases} \frac{\sqrt{x+3} - \sqrt{3x+1}}{x-1} & \text{for } x \neq 1 \\ k & \text{for } x = 1 \end{cases} \), then \( k = \)

   (a) 0
   (b) 1
   (c) \( \frac{1}{2} \)
   (d) \( -\frac{1}{2} \)
   (e) None of the above
4. If \( \lim_{x \to 2} \frac{f(x)}{x - 2} = f'(2) = 0 \), which of the following must be true?

I. \( f(2) = 0 \)
II. \( f(x) \) is continuous at \( x = 2 \)
III. \( f(x) \) has a horizontal tangent line at \( x = 2 \)

(a) I only 
(b) II only 
(c) I and II only 
(d) II and III only 
(e) I, II, and III

5. The figure below shows the graph of a function \( f(x) \) which has horizontal asymptotes of \( y = 3 \) and \( y = -3 \). Which of the following statements are true?

\[ f'(x) < 0 \] for all \( x \geq 0 \)
\[ \lim_{x \to +\infty} f'(x) = 0 \]
\[ \lim_{x \to -\infty} f'(x) = 3 \]

(a) I only 
(b) II only 
(c) III only 
(d) I and II only 
(e) I, II, and III
6-2 Free Response Homework

Is the given function continuous at \( x = a \) for each of the following functions? Is the discontinuity (if any) removable? What type of discontinuity (if any) is it?

1. \( f(x) = \begin{cases} 
  x^2 - 1, & \text{if } x > -1 \\
  0, & \text{if } x = -1 \\
  4 - x, & \text{if } x < -1 
\end{cases} \); \( a = -1 \)

2. \( k(x) = \begin{cases} 
  2x + 1, & \text{if } x \leq -1 \\
  3 - x^2, & \text{if } x > -1 
\end{cases} \); \( a = -1 \)

3. \( g(x) = \frac{x^2 - 4x - 5}{x^2 - 1} \); \( a = -1 \)

4. \( F(x) = \begin{cases} 
  2x + 1, & \text{if } x \leq -1 \\
  -x^2, & \text{if } x > -1 
\end{cases} \); \( a = -1 \)

5. \( h(x) = \begin{cases} 
  \frac{x^2 - 4}{x - 2}, & \text{if } x \neq 2 \\
  4, & \text{if } x = 2 
\end{cases} \); \( a = 2 \)

6. \( h(x) = \begin{cases} 
  2x + 1, & \text{if } x < -1 \\
  0, & \text{if } x = -1 \\
  -x^2, & \text{if } x > -1 
\end{cases} \); \( a = -1 \)

7. \( k(x) = \begin{cases} 
  2x + 1, & \text{if } x \leq -1 \\
  3 - x^2, & \text{if } x > -1 
\end{cases} \); \( a = 3 \)
6-3: Differentiability and Smoothness

The second important underlying concept to Calculus, after continuity, is differentiability. Almost all theorems in Calculus begin with “If a function is continuous and differentiable…” It is actually a very simple idea.

**Vocabulary:**

1. **Differentiable** – the derivative exists. $F(x)$ is differentiable at $a$ if and only if $F'(a)$ exists and $F(x)$ is differentiable on $[a, b]$ if and only $F(x)$ is differentiable at every point in the interval. For $F'(a)$ to exist, the derivative from the left and the derivative from the right must be equal one another.

**LEARNING OUTCOMES**

Determine if a function is differentiable or not.
Demonstrate understanding of the connections and differences between differentiability and continuity.

There are two ways that a function would not be differentiable:

1) The tangent line could be vertical, causing the slope to be infinite.

**EX 1** Is $y = \sqrt[3]{x}$ differentiable at $x = 0$?

If $y = \sqrt[3]{x}$, then $\frac{dy}{dx} = \frac{1}{3x^{2/3}}$. At $x = 0$, $\frac{dy}{dx} = \frac{1}{0} = \text{DNE}$. 

![Graph of function near x = 0](image_url)
2) Just as the limit does not exist if the two one-sided limits are not equal, a derivative would not exist if the two one-sided derivatives are not equal. That is, the slopes of the tangent lines to the left of a point are not equal to the tangent slopes to the right. A curve like this is called non-smooth.

EX 2 Is the function represented by this curve differentiable at \( x = 1 \)?

As can be seen, the slope to the left of \( x = 1 \) is 1, while the slope to the right of \( x = 1 \) is 0. So this function is not differentiable at \( x = 1 \). This is an example of a curve that is not smooth.

\[
F(x) \text{ is differentiable at } x = a \text{ if and only if:}
\]

i) \( F(x) \) is continuous at \( x = a \),

ii) \( \lim_{x \to a^-} F'(x) \) and \( \lim_{x \to a^+} F'(x) \) both exist,

and  iii) \( \lim_{x \to a^-} F'(x) = \lim_{x \to a^+} F'(x) \)
EX 3 Is $G(x) = \begin{cases} 
  x^3, & \text{if } x \leq 0 \\
  x^2 - 2x, & \text{if } x > 0 
\end{cases}$ differentiable at $x = 0$?

i) a) $G(0)$ exists

b) $\lim_{x \to 0^-} G(x) = 0^3 = 0$
$\lim_{x \to 0^+} G(x) = \lim_{x \to 0^+} 0^2 - 2(0) = 0$
$\lim_{x \to 0^-} G(x) = \lim_{x \to 0^-} G(x)$, so
$\lim_{x \to 0} G(x)$ exists

c) $G(0) = \lim_{x \to 0} G(x)$

So $G(x)$ is continuous at $x = 0$.

$G'(x) = \begin{cases} 
  3x^2, & \text{if } x < 0 \\
  2x - 2, & \text{if } x > 0 
\end{cases}$

ii) $\lim_{x \to 0^-} G'(x) = \lim_{x \to 0} 3x^2 = 0$ so $\lim_{x \to 0^-} G'(x)$ exists
$\lim_{x \to 0^+} G'(x) = \lim_{x \to 0^+} (2x - 2) = -2$ so $\lim_{x \to 0^+} G'(x)$ exists

iii) These two one-sided derivatives are not equal.

Therefore, $G(x)$ is not differentiable at $x = 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\end{figure}
All three of these examples show functions that are continuous, but not differentiable. Continuity does not insure differentiability. But the converse is true.

\[
\text{If } f(x) \text{ is differentiable, it is continuous.}
\]

**EX 4** Given that \( f(x) = \begin{cases} k\sqrt{x+1} & \text{if } x \leq 3 \\ 2+mx & \text{if } x > 3 \end{cases} \) is differentiable at \( x = 3 \), find \( m \) and \( k \).

\[
f'(x) = \begin{cases} k\frac{1}{2\sqrt{x+1}} & \text{if } x \leq 3 \\ m & \text{if } x > 3 \end{cases}
\]

If \( f(x) \) is differentiable at \( x = 3 \), then \( k\frac{1}{2\sqrt{3+1}} = m \Rightarrow \frac{1}{4}k = m \).

If \( f(x) \) is differentiable at \( x = 3 \), it is also continuous.

Therefore, at \( x = 3 \),
\[
k\sqrt{3+1} = 2 + mx \\
k\sqrt{3+1} = 2 + m \cdot 3 \\
2k = 3m + 2
\]

Since both \( \frac{1}{4}k = m \) and \( 2k = 3m + 2 \) must be true, use linear combination or substitution to solve for \( k \) and \( m \).

\[
2k = 3\left(\frac{1}{4}k\right) + 2 \\
2k = \frac{3}{4}k + 2 \\
\frac{5}{4}k = 2 \\
k = \frac{8}{5} \Rightarrow m = \frac{2}{5}
\]
6-3 Multiple Choice Homework

1. The graph of the function \( f \) is shown below. At which value of \( x \) is \( f \) continuous, but not differentiable?

   \[ f(x) = \begin{cases} 
   x + 2 & \text{if } x \leq 3 \\
   4x - 7 & \text{if } x > 3 
   \end{cases} \]

   (a) \( a \)  
   (b) \( b \)  
   (c) \( c \)  
   (d) \( d \)  
   (e) \( e \)

2. Let \( f \) be the function given below. Which of the following statements are true about \( f \)?

   I. \( \lim_{{x \to 3}} f(x) \) exists
   II. \( f \) is continuous at \( x = 3 \)
   III. \( f \) is differentiable at \( x = 3 \)

   (a) None  
   (b) I only  
   (c) II only  
   (d) I and II only  
   (e) I, II, and III

3. Let \( f \) be the function defined below, where \( c \) and \( d \) are constants. If \( f \) is differentiable at \( x = 2 \), what is the value of \( c + d \)?

   \[ f(x) = \begin{cases} 
   cx + d & \text{if } x \leq 2 \\
   x^2 - cx & \text{if } x > 2 
   \end{cases} \]

   (a) \(-4\)  
   (b) \(-2\)  
   (c) \(0\)  
   (d) \(2\)  
   (e) \(4\)
4. Let $P(x)$ and $Q(x)$ be polynomials. Find $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$ if the degree of $P(x)$ is 5 and the degree of $Q(x)$ is 9.

(a) $\frac{5}{9}$
(b) $\frac{9}{5}$
(c) 0
(d) DNE
(e) There is not enough information to answer the question

5. Let the function $f$ be differentiable on the interval $[0, 2.5]$ and define $g$ by $g(x) = f(f(x))$. Use the table to estimate $g'(1)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1.7</td>
<td>1.8</td>
<td>2.0</td>
<td>2.4</td>
<td>3.1</td>
<td>4.4</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>0.1</td>
<td>0.3</td>
<td>0.6</td>
<td>1.1</td>
<td>2.0</td>
<td>2.2</td>
</tr>
</tbody>
</table>

(a) 0.8
(b) 1.2
(c) 1.6
(d) 2.0
(e) 2.4

6. Use the graph of $f$ below to select the correct answer from the choices below.

(a) $f$ has a relative maximum at $x = 3$
(b) $f$ is continuous at $x = 3$
(c) $f$ is differentiable at $x = 3$
(d) $f'(3) < f(3)$
(e) None of the above
7. Use the graph of \( f \) below to select the correct answer from the choices below.

(a) \( f \) has no extreme points
(b) \( f \) is continuous at \( x = 2 \)
(c) \( f \) is differentiable for \( x \in (0, 4) \)
(d) \( f \) has a relative maximum at \( x = 2 \)
(e) \( f \) is decreasing for \( x \in (0, 4) \)

6-3 Free Response Homework

1. Given that \( f(x) = \begin{cases} 
  mx + 2, & \text{if } x \leq 1 \\
  k(x - 1), & \text{if } x > 1 
\end{cases} \) is differentiable at \( x = 1 \), find \( m \) and \( k \).

2. Given that \( f(x) = \begin{cases} 
  mx - 5, & \text{if } x \leq -2 \\
  kx^2 + 1, & \text{if } x > -2 
\end{cases} \) is differentiable at \( x = -2 \), find \( m \) and \( k \).

3. Given that \( f(x) = \begin{cases} 
  k + x, & \text{if } x \leq 0 \\
  3 - mx, & \text{if } x > 0 
\end{cases} \) is differentiable at \( x = 0 \), find \( m \) and \( k \).

4. Given that \( f(x) = \begin{cases} 
  mx - 2, & \text{if } x \leq 2 \\
  k\sqrt{x^2 - 3}, & \text{if } x > 2 
\end{cases} \) is differentiable at \( x = 2 \), find \( m \) and \( k \).

For problems 5-7, consider the function: \( f(x) = \begin{cases} 
  x - 2, & \text{if } x \leq -2 \\
  \sqrt{3x^2 + 4}, & \text{if } -2 < x < 0 \\
  2, & \text{if } 0 \leq x < 3 \\
  3x - 7, & \text{if } x \geq 3 
\end{cases} \)

5. Is \( f(x) \) differentiable at \( x = -2 \)? Show why or why not.

6. Is \( f(x) \) differentiable at \( x = 0 \)? Show why or why not.
7. Is $f(x)$ differentiable at $x = 3$? Show why or why not.
REMEMBER: Traits

1. Domain
2. y-intercept
3. Zeros
4. Vertical Asymptotes
5. Points of Exclusion
6. End Behavior
7. Extreme Points
8. Range

LEARNING OUTCOME

Find all the traits and sketch a fairly accurate polynomial curve algebraically.

EX 1 Find all the traits and sketch \( y = \begin{cases} x^2 + 1, & \text{if } x \geq 0 \\ x^3 - 3x, & \text{if } x < 0 \end{cases} \).

1. Domain: \( x = \) all Real Numbers
2. y-int: \( x = 0 \) gives \( y = 1 \), so \( 0, 1 \)
3. Zeros: \( y = \begin{cases} x^2 + 1 = 0 \rightarrow \text{no answers} \\ x^3 - 3x = 0 \rightarrow x = 0, \ \pm \sqrt{3} \end{cases} \) but only \( -\sqrt{3} \) is in the domain of this line
   \( \therefore \left(-\sqrt{3}, \ 0\right) \)
4. End Behavior: The domain of this function is all real numbers. Since both \( \pm \infty \) are in the domain, consider the limit as \( y \) goes to \( \pm \infty \).

\[
\lim_{x \to \pm \infty} x^2 + 1 = \infty \quad \text{and} \quad \lim_{x \to \pm \infty} x^3 - 3x = -\infty.
\]
The left end goes down and the right end goes up.

5. Extreme Points:

\[
\frac{dy}{dx} = \begin{cases} 
2x = 0 \rightarrow x = 0 \\
3x^2 - 3 = 0 \rightarrow x = \pm 1 
\end{cases}
\]
but only \(-1\) is in the domain of this line.

Critical Values: \( x = 0, -1 \)
Extreme Values: \( y = 1, 2 \)

Extreme Points: \((-1, 2)\) and \((0, 1)\)

6. Range: Sketching the curve determines that the range is all Real Numbers.

\[
y = \begin{cases} 
x^2 + 1, & \text{if } x \geq 0 \\
x^3 - 3x, & \text{if } x < 0
\end{cases}
\]
EX 2  Find all the traits and sketch \( y=\begin{cases} 
  x+3, & \text{if } -4 \leq x \leq -2 \\
  1, & \text{if } -2 \leq x \leq 0 \\
  1-2x, & \text{if } 0 \leq x \leq 2
\end{cases} \).

1. Domain: \( x \in [-4, 2] \)

2. \( y \)-int: \( x = 0 \) gives \( y = 1 \), so \((0, 1)\).

3. Zeros: \( y = \begin{cases} 
  x+3=0, & \rightarrow x = -3 \\
  1=0 & \rightarrow \text{no answers} \\
  1-2x=0, & \rightarrow x = \frac{1}{2}
\end{cases} \)

\[ \therefore (-3, 0) \text{ and } \left( \frac{1}{2}, 0 \right) \]

4. End Behavior: There is no EB as our domain has been reduced to \( x \in [-4, 2] \), i.e. there is no \( \pm \infty \) in our domain, hence no EB.

5. Extreme Points: \( \frac{dy}{dx} = \begin{cases} 
  1, & \text{if } -4 \leq x < -2 \\
  0, & \text{if } -2 < x < 0 \\
  -2, & \text{if } 0 < x < 2
\end{cases} \)

Critical values: \( x = \{-4, 2\} \cup [-2, 0] \)

Extreme Values: \( y = -1, 1, -3 \)

The first and third parts of \( \frac{dy}{dx} \neq 0 \), so the only extremes points are the endpoints of the domains: \((-4, -1), (-2, 1), (0, 1)\) and \((2, -3)\). Since the middle part always equals 0, all its points are extreme points.

6. Range: Sketching the curve determines that the range is \( y \in [-3, 1] \).
EX 3 Find all the traits and sketch \( h(x) = \begin{cases} 
2x + 1, & \text{if } x < -1 \\
0, & \text{if } x = -1 \\
-x^2, & \text{if } x > -1 
\end{cases} \).

1. Domain: All Real Numbers

2. \( y \)-int: \( x = 0 \) gives \( y = 0 \), so \((0, 0)\)

3. Zeros: \( h(x) = \begin{cases} 
2x + 1 = 0 \Rightarrow x = -\frac{1}{2} \\
0, & \text{if } x = -1 \\
-x^2 = 0, \Rightarrow x = 0 
\end{cases} \)

\[ \frac{1}{2} \text{ is not in the domain for the piece that yielded } -\frac{1}{2} \]

\[ \therefore (-1, 0) \text{ and } (0, 0) \]

4. End Behavior: Both ends go down
5. **Extreme Points:** \( h'(x) = \begin{cases} 
2, & \text{if } x < -1 \\
-2x, & \text{if } x > -1 
\end{cases} \)

Critical values: \( x = -1, 0 \)

Extreme values: \( y = 0 \)

Extreme Points: \((-1, 0)\) and \((0, 0)\)

6. **Range:** Sketching the curve determines that the range is \( y \in (-\infty, 0] \).

\[ h(x) = \begin{cases} 
2x + 1, & \text{if } x < -1 \\
0, & \text{if } x = -1 \\
-x^2, & \text{if } x > -1 
\end{cases} \]
6-4 Multiple Choice Homework

1. At \( x = 0 \), which of the following statements is TRUE of the function \( f \) defined by \( f(x) = \sqrt{x^2 + 0.0001} \).

I. \( f \) is discontinuous  
II. \( f \) has a horizontal tangent  
III. \( f' \) is undefined

(a) I only  (b) II only  (c) III only  (d) I and III only  (e) I, II, and III

2. Let \( f \) be the function defined below. Which of the following statements about \( f \) are true?

\[ f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\
1 & \text{if } x = 2
\end{cases} \]

I. \( f \) is differentiable at \( x = 2 \)
II. \( f \) has a limit at \( x = 2 \)
III. \( f \) is continuous at \( x = 2 \)

(a) I only  (b) II only  (c) III only  (d) I and II only  (e) I, II, and III

3. Find the equation of the tangent line to \( y = \frac{\sqrt{x}}{2x+2} \) at \((4, 0.2)\).

(a) \( y = \frac{-19}{50}x + \frac{9}{50} \)
(b) \( y = \frac{37}{50}x + \frac{129}{130} \)
(c) \( y = \frac{-3}{200}x + \frac{13}{50} \)
(d) \( y = \frac{109}{120}x + \frac{7}{90} \)
(e) \( y = \frac{107}{120}x + \frac{13}{90} \)
4. Which of the following statements are true about function \( f \), if its derivative \( f' \), is defined by \( f'(x) = x(x-a)^3 \), \( a > 0 \).

I. The graph of \( f \) is increasing at \( x = 2a \)  
II. The function \( f \) has a local maximum at \( x = 0 \)  
III. The function \( f \) has a local minimum at \( x = a \)

(a) I only (b) I and II only (c) I and III only (d) II and III only (e) I, II and III

5. The graphs of the functions \( f \) and \( g \) are shown below. If \( h(x) = f(g(x)) \), which of the following statements are true about the function \( h \)?

I. \( h(2) = 5 \)  
II. \( h \) is increasing at \( x = 4 \)  
III. The graph of \( h \) has a horizontal tangent at \( x = 1 \)

(a) I only (b) II only (c) III only (d) II and III only (e) I, II, and III

6-4 Free Response Homework

List the traits and sketch.

1. \( y = \begin{cases} x^2 - x, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases} \)

2. \( y = \begin{cases} x^2 - x, & \text{if } x < 1 \\ 2x + 1, & \text{if } x \geq 1 \end{cases} \)
3. \( y = \begin{cases} 
2x - 3, & \text{if } x < 1 \\
-x - 2x^2, & \text{if } x \geq 1 
\end{cases} \)

4. \( h(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2}, & \text{if } x \neq 2 \\
4, & \text{if } x = 2 
\end{cases} \)

5. \( f(x) = \begin{cases} 
x^2 - 1, & \text{if } x > -1 \\
0, & \text{if } x = -1 \\
4 - x, & \text{if } x < -1 
\end{cases} \)

6. \( G(x) = \begin{cases} 
x^3, & \text{if } x \leq 0 \\
x^2 - 2x, & \text{if } x > 0 
\end{cases} \)

7. \( k(x) = \begin{cases} 
2x + 1, & \text{if } x \leq -1 \\
3 - x^2, & \text{if } x > -1 
\end{cases} \)
Piece-wise Defined Functions Practice Test
Part 1: CALCULATOR REQUIRED
Round to 3 decimal places. Show all work.

Multiple Choice (3 pts. each)

1. If \( f \) is continuous at \( x = 2 \), and if
   \[
   f(x) = \begin{cases} 
   \frac{\sqrt{x+2} - \sqrt{2x}}{x-2} & \text{for } x \neq 2 \\
   k & \text{for } x = 2
   \end{cases}
   \]
   then \( k = \)
   
   (a) \(-\frac{1}{2}\)  
   (b) \(-\frac{1}{4}\)  
   (c) 0  
   (d) \frac{1}{4}  
   (e) \frac{1}{2}

2. When the area of an expanding square, in square units, is increasing three times as fast as its side is increasing, in linear units, the side is

   (a) \(\frac{2}{3}\)  
   (b) \(\frac{3}{2}\)  
   (c) 3  
   (d) 2  
   (e) 1

3. The figure below shows a road running in the shape of a parabola from the bottom of a hill at A to point B. At B it changes to a line and continues on to C. The equation of the road is

   \[ R(x) = \begin{cases} 
   ax^2 & \text{from A to B} \\
   bx + c & \text{from B to C}
   \end{cases} \]

   B is 1000 feet horizontally from A and 100 feet higher. Since the road is smooth, \( R'(x) \) is continuous. What is the value of \( b \)?

   (a) 0.2  
   (b) 0.02  
   (c) 0.002  
   (d) 0.0002  
   (e) 0.00002

359
4. The function \( f \) defined on all reals such that \( f(x) = \begin{cases} x^2 + kx - 3 & \text{for } x \leq 1 \\ 3x + b & \text{for } x > 1 \end{cases} \). For which of the following values of \( k \) and \( b \) will the function \( f \) be both continuous and differentiable on its entire domain?

(a) \( k = -1, \ b = -3 \)
(b) \( k = 1, \ b = 3 \)
(c) \( k = 1, \ b = 4 \)
(d) \( k = 1, \ b = -4 \)
(e) \( k = -1, \ b = 6 \)

**Free Response** (10 pts. each)

1. Is the function \( f \), given by \( f(x) = \begin{cases} x^2 - \pi x + 1 & x \geq \pi \\ \sqrt{\frac{x^2}{x + 2}} & x < \pi \end{cases} \), continuous at \( x = \pi \)? If not, is the discontinuity removable?
2. Given the graph of \( f(x) \) below, find the values of the following:

\[
\begin{align*}
  \lim_{x \to -3} f(x) & \quad \lim_{x \to 1} f(x) & \quad \lim_{x \to 3} f(x) \\
  \lim_{x \to -3} f(x) & \quad \lim_{x \to 1}^+ f(x) & \quad \lim_{x \to 1} f(x) & \quad \lim_{x \to 3} f(x) \\
  \lim_{x \to -3} f(x) & \quad \lim_{x \to -1} f(x) & \quad \lim_{x \to 1} f(x) & \quad \lim_{x \to 1} f(x) \\
  f(-3) & \quad f(-1) & \quad f(1) & \quad f(3)
\end{align*}
\]
Multiple Choice (3 pts. each)
5. Let \( f \) be a function with first derivative given by \( f'(x) = x^2(x - 3)(x - 6) \). What are the \( x \)-coordinate(s) of the minimum?

(a) 0 only
(b) 6 only
(c) 0 and 6 only
(d) 3 and 6 only
(e) 0, 3, and 6

6. A function \( f(x) \) has a vertical asymptote at \( x = 2 \). The derivative of \( f(x) \) is positive for all \( x \neq 2 \). Which of the following statements are true?

I. \( \lim_{x \to 2} f(x) = +\infty \)
II. \( \lim_{x \to 2} f(x) = +\infty \)
III. \( \lim_{x \to 2^-} f(x) = +\infty \)

(a) I only (b) II only (c) III only (d) I and II only (e) I, II, and III

7. The function \( f(x) = \begin{cases} 4 - x^2 & \text{for } x \leq 1 \\ mx + b & \text{for } x > 1 \end{cases} \) is continuous and differentiable for all real numbers. The values of \( m \) and \( b \) are

(a) \( m = 2, \ b = 1 \)
(b) \( m = 2, \ b = 5 \)
(c) \( m = -2, \ b = 1 \)
(d) \( m = -2, \ b = 5 \)
(e) None of these
8. The graph of the function \( f(x) \) is shown below. Which of the following statements must be false?

(a) \( f(a) \) exists
(b) \( f(x) \) is defined for \( 0 < x < a \)
(c) \( f(x) \) is not continuous at \( x = a \)
(d) \( \lim_{x \to a} f(x) \) exists
(e) \( \lim_{x \to a} f'(x) \) exists

**Free Response (10 pts. each)**

3. Is the function \( f, \) given by \( f(x) = \begin{cases} 2x + 3 & \text{if } x < -3 \\ -3 & \text{if } x = -3 \\ x^2 + 4x & \text{if } x > -3 \end{cases} \), continuous at \( x = -3? \)

differentiable at \( x = -3? \)
4. List all traits and sketch \( f(x) = \begin{cases} 
2x + 3 & \text{if } x < -3 \\
-3 & \text{if } x = -3 \\
x^2 + 4x & \text{if } x > -3 
\end{cases} \).

Domain:

Zeros:

\( y \)-int:

VAs:

POEs:

End Behavior:

Extreme Points:

Range:
### Piece Wise Defined Functions Homework Answer Key

#### 6-1 Multiple Choice Homework
1. D  
2. C  
3. D  
4. D  
5. A

#### 6-1 Free Response Homework
1a. 4  
b. 0  
c. DNE  
d. 0

2a. 5  
b. 5  
c. 5  
d. 2

3a. $+\infty$  
b. $-\infty$  
c. DNE  
d. DNE

4a. 0  
b. 0  
c. 0  
d. 0

5. $-\infty$

6. $-\infty$

7. $-\infty$

8. $+\infty$

#### 6-2 Multiple Choice Homework
1. C  
2. E  
3. D  
4. E  
5. D

#### 6-2 Free Response Homework
1. No, $\lim_{x \to -1} f(x) \neq \lim_{x \to 1} f(x)$  
   Non-removable jump discontinuity.

2. No, $\lim_{x \to -1} f(x) \neq \lim_{x \to 1} f(x)$  
   Non-removable jump discontinuity.

3. No, $g(-1)$ does not exist.  
   Removable POE.

4. Yes

5. Yes

6. No, $\lim_{x \to -1} h(x) \neq h(-1)$  
   Removable POD.

7. Yes
6-3 Multiple Choice Homework
1. A
2. D
3. B
4. C
5. B
6. A
7. B

6-3 Free Response Homework
1. \( m = -2, k = -2 \)
2. \( m = -6, k = \frac{3}{2} \)
3. \( m = -1, k = 3 \)
4. \( m = \frac{4}{3}, k = \frac{2}{3} \)
5. Not differentiable at -2 because it’s not continuous there.
   \[ \lim_{x \to -2} f(x) = -2 - 2 = -4 \]
   \[ \lim_{x \to -2} f(x) = \sqrt{3(-2)^2} + 4 = 4 \]
6. Differentiable at 0.
7. Not differentiable at 3 because
   \[ \lim_{x \to 3} f'(x) = \frac{dy}{dx}(2) = 0 \]
   \[ \lim_{x \to 3} f'(x) = \frac{dy}{dx}(3x - 7) = 3. \]

6-4 Free Response Homework
1. Domain: All Reals
   Range: \( y \in (0, \infty) \)
   Zeros: None
   \( y \)-int: \((0, 1)\)
   VAs: None
   POEs: None
   EB: Both ends up
   Extreme Point: \((0, 1)\)
2. Domain: All Reals
   Range: \( y \in [-1, \infty) \)
   Zeros: \((\pm 1, 0)\)
   \( y \)-int: \((0, -1)\)
   VAs: None
   POEs: None
   EB: Both ends up
   Extreme Points: \((-1, 0), (0, -1)\)

6-4 Multiple Choice Homework
1. B
2. B
3. C
3. Domain: All Reals
   Range: \( y \in \left[ -\frac{1}{4}, \infty \right) \)
   Zeros: \((0, 0)\)
   \(y\)-int: \((0, 0)\)
   VAs: None
   POEs: None
   EB: Both ends up
   Extreme Points: \(\left( \frac{1}{2}, -\frac{1}{4} \right), (1, 3)\)

4. Domain: All Reals
   Range: All Reals
   \(y\)-int: \((0, 0)\)
   Zeros: \((0, 0), (2, 0)\)
   VAs: None

5. Domain: All Reals
   Range: \( y \in (-\infty, -1] \)
   Zeros: None
   \(y\)-int: \((0, -3)\)
   VAs: None
   POEs: None
   EB: Both ends down
   Extreme Point: \((1, -1)\)

6. Domain: All Reals
   Range: \( y \in (-\infty, 3] \)
   Zeros: \((\sqrt{3}, 0)\)
   \(y\)-int: \((0, 3)\)
7. Domain: All Reals
   Range: All Reals
   Zeros: (−2, 0)

   y-int: (0, 2)
   VAs: None
   POEs: None
   EB: Left end down, right end up
   Extreme Points: None
Piece Wise Defined Functions Practice Test Answer Key

Multiple Choice
1. B
2. B
3. A
4. D
5. B
6. B
7. D
8. E

Free Response
1. \( f \) is not continuous at \( x = \pi \). This discontinuity is not removable.

2a. \( \infty \)  
2e. 2  
2i. 2  
2m. \(-1\)

b. \( \infty \)  
f. 0  
j. 2  
n. \(-1\)

c. \( \infty \)  
g. DNE  
k. 2  
o. \(-1\)

d. DNE  
h. 0  
l. 3  
p. DNE

3. \( f \) is continuous at \( x = -3 \). \( f \) is not differentiable at \( x = -3 \).

4. Domain: \( x \in \text{All Reals} \)
   Zeros: \((0, 0)\)
   y-int: \((0, 0)\)
   VAs: None
   POEs: None
   EB: Left end down, right end up
   Extreme Points: \((-2, -4), (-3, -3)\)
Range: $y \in \text{All Reals}$