Chapter 5:

Limits and Derivatives

Chapter 5 Overview: Introduction to Limits and Derivatives

In a later chapter, maximum and minimum points of a curve will be found both by calculator and algebraically. While the algebra of this process is relatively easy, the concepts underlying it are among the major stumbling blocks to the first time Calculus student. Therefore, the underlying operations have been separated from the process.

The process of finding critical values, extreme values, or extreme points relies on two things—the slope of a tangent line and how to deal with the indeterminate form of a number $\frac{0}{0}$. At a high or low point of a polynomial curve, the tangent line will be horizontal and, therefore, its slope will be zero. The slope formula from Algebra 1 requires two points, while a tangent line has only one point associated with the curve. If one point is used in both places in the slope formula, the result is $\frac{0}{0}$*, which is not a real number (because division by zero is undefined).*

In this chapter, there are two basic applications of the derivative:

- The equation of the tangent line and its use for approximations.
- Derivatives as rate of change and their applications to motion.

5-1: The Indeterminate Forms and Limits

Vocabulary:

1. ***Indeterminate Form of a Number*** – Defn: a number for which further analysis is necessary to determine its value

   Means: the number equals $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, or some other strange thing

Since all previous rules from algebra and arithmetic cannot address $\frac{0}{0}$, a new rule/process is needed. It is called a "limit," and the formal definition is fairly bizarre and unintelligible to most students. The good news is that we will not be working with this definition.

Defn: $\lim_{x \to a} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

$\lim_{x \to a} f(x)$ is read "the limit, as *x* approaches *a*, of *f* of *x*." What the definition means is, if *x* is almost equal to *a*, $f(x)$ is almost equal to *L*. In fact, they are so close, they could be rounded off and considered equal. In practice, this means $\lim_{x \to a} f(x) = f(a)$—in other words, *y* of $x = a$, as long as $y \neq \frac{0}{0}$. If $y = \frac{0}{0}$, then factor and cancel the terms that gave the zeros. No matter how small the factors get, they cancel to 1 as long as they do not quite equal $\frac{0}{0}$.

LEARNING OUTCOME

Evaluate limits.

Steps toward finding a limit:

1. Find $f(a)$. If $f(a) \neq \frac{0}{0}$, the problem is done.

2. If $f(a) = \frac{0}{0}$, factor, multiply by a conjugate, or add fractions with a common denominator, cancel, and plug *a* in for *x*.

EX 1 Find $\lim_{x \to 5} (x+2)$, $\lim_{x \to -4} (x^2+3)$, $\lim_{x \to 5} \frac{x^2-25}{x-4}$, and $\lim_{x \to 5} \frac{x^2-24}{x-5}$

$\lim_{x \to 5} (x+2) = 5+2 = 7$

$\lim_{x \to -4} (x^2+3) = (-4)^2 + 3 = 19$

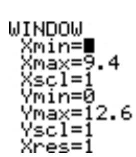
$\lim_{x \to 5} \frac{x^2-25}{x-4} = \frac{5^2-25}{5-4} = \frac{0}{1} = 0$

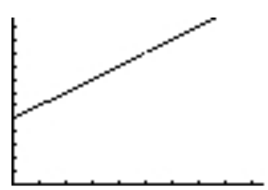
$\lim_{x \to 5} \frac{x^2-24}{x-5} = \frac{5^2-24}{5-5} = \frac{1}{0} = dne$

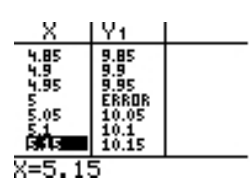
EX 2 Find $\lim_{x \to 5} \frac{x^2-25}{x-5}$

If $x = 5$ here, $\frac{x^2-25}{x-5}$, which is $\frac{(x-5)(x+5)}{x-5}$, would $= \frac{0}{0}$. But with a limit, *x* is only *almost* equal to 5, and, therefore, $\frac{x-5}{x-5} = 1$. So,

$$\begin{aligned} \lim_{x \to 5} \frac{x^2-25}{x-5} &= \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} \\ &= \lim_{x \to 5} (x+5) \\ &= 5+5 \\ &= 10 \end{aligned}$$







Both the graph (in the given window) and the table show that, while no *y*-value exists for $x = 5$, the *y*-values of the points on either side of $x = 5$ show *y* should be 10.

Basically, the limit allows you to factor and cancel before substituting the number "a" for x.

EX 3 Find $\lim_{x\to 2}\frac{x^2+4x-12}{x^2+3x-10}$

If $x=2$, then $\frac{x^2+4x-12}{x^2+3x-10}=\frac{0}{0}$, so this fraction can be factored and cancelled. In fact, since $x=2$, one of the factors in both the numerator and denominator must be $(x-2)$, otherwise the fraction would not yield zeros. So,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2+3x-10}=\lim_{x\to 2}\frac{(x-2)(x+6)}{(x-2)(x+5)}$$
$$=\lim_{x\to 2}\frac{x+6}{x+5}$$
$$=\frac{8}{7}$$

EX 4 Find $\lim_{x\to -3}\frac{2x^3+x^2-13x+6}{x^2+x-6}$

Note that what *x* is approaching is the number to use in the synthetic division:

–3	2	1	–13	6
		–6	15	–6
	2	–5	2	0

$$\lim_{x\to -3}\frac{2x^3+x^2-13x+6}{x^2+x-6}=\lim_{x\to -3}\frac{(x+3)(2x^2-5x+2)}{(x+3)(x-2)}$$
$$=\lim_{x\to -3}\frac{(2x^2-5x+2)}{(x-2)}$$
$$=\frac{35}{-5}$$
$$=-7$$

EX 5 Find $\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2}$

Unlike the previous examples, this fraction does not factor. Yet it must simplify, somehow, to eliminate the indeterminate number. Multiply by conjugates to eliminate the radicals from the numerator:

$$\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2} = \lim_{x\to 2} \frac{\left(\sqrt{4-x}-\sqrt{2}\right)\left(\sqrt{4-x}+\sqrt{2}\right)}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$

$$= \lim_{x\to 2} \frac{4-x-2}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$

$$= \lim_{x\to 2} \frac{2-x}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$

$$= \lim_{x\to 2} \frac{-1}{\sqrt{4-x}+\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}+\sqrt{2}}$$

$$= \frac{-1}{2\sqrt{2}} \text{ or } \frac{-\sqrt{2}}{4}$$

5-1 Free Response Homework

Evaluate the limit.

1. $\lim_{x \to 3} \frac{x^2 + x - 12}{x^2 - 9}$

2. $\lim_{x \to -1} \frac{x^2 + 4x + 3}{x^2 - 3x - 4}$

3. $\lim_{x \to 2} \frac{x^2 + 4x - 12}{3x - 1}$

4. $\lim_{x \to 4} \frac{4x - 12}{x^2 + 3x - 10}$

5. $\lim_{x \to -2} \frac{4x + 8}{x^2 + 6x + 8}$

6. $\lim_{x \to \frac{1}{3}} \frac{6x^2 + x - 1}{3x^2 - 16x + 5}$

7. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 + 3x - 4}$

8. $\lim_{x \to -3} \frac{x^3 + 3x^2 - 4x - 12}{x^2 - x - 12}$

9. $\lim_{x \to \frac{3}{2}} \frac{2x^3 + x^2 - 8x + 3}{6x^3 - 7x^2 - 7x + 6}$

10. $\lim_{x \to 1} \frac{2x^3 + x^2 - x + 3}{x^3 - x^2 - 5x - 3}$

11. $\lim_{x \to \sqrt{2}} \frac{x^4 + 4x^2 - 12}{x^3 + x^2 - 2x - 2}$

12. $\lim_{x \to 2} \frac{x^2 + 4x - 1}{x^3 + 3x}$

13. $\lim_{x \to 0} \frac{\sqrt{5 - x} - \sqrt{5}}{x}$

14. $\lim_{x \to 4} \frac{x^2 - 4x}{\sqrt{x} - 2}$

15. $\lim_{x \to 1} \frac{1 - x^2}{x^3 - 3x + 2}$

16. $\lim_{x \to 9} \frac{3 - \sqrt{x}}{x^2 - 81}$

17. $\lim_{x \to -1} \frac{\left(1 - x^2\right)^2}{x^3 - 2x^2 - 3x}$

18. $\lim_{x \to 4} \frac{\sqrt{x - 3} - 1}{x - 4}$

19. $\lim_{x \to -1} \frac{x^4 - 1}{x^4 + x^3 + x^2 + 2x + 1}$

5-1 Multiple Choice Homework

1. If $a \neq 0$, the $\lim_{x\to a} \frac{x^2 - a^2}{x^4 - a^4}$ is

a) $\frac{1}{a^2}$ b) $\frac{1}{2a^2}$ c) $\frac{1}{6a^2}$ d) 0 e) Nonexistent

---

2. $\lim_{x\to 1} \frac{x^2 + 2x - 3}{x^2 - 1} =$

a) –2 b) –1 c) 10 d) 1 e) 2

---

3. What is $\lim_{x\to 0} \frac{\frac{1}{x-1}+1}{x}$?

a) –1 b) 0 c) 1 d) 2 e) DNE

---

4. $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x}$

a) 0.5 b) 0.25 c) 0 d) –0.25 e) –0.5

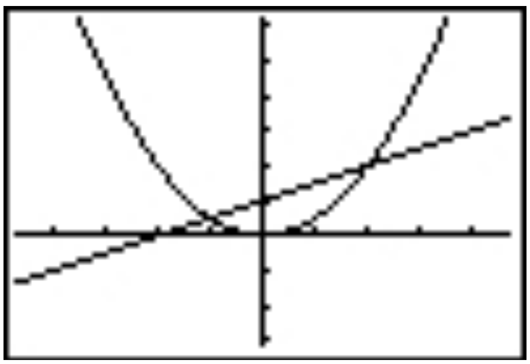
---

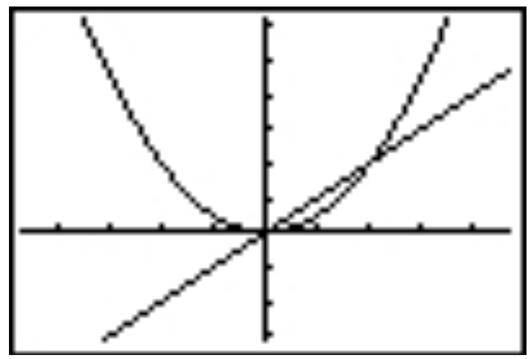
5. $\lim_{x\to 8} \frac{x^2 - 64}{x - 8}$

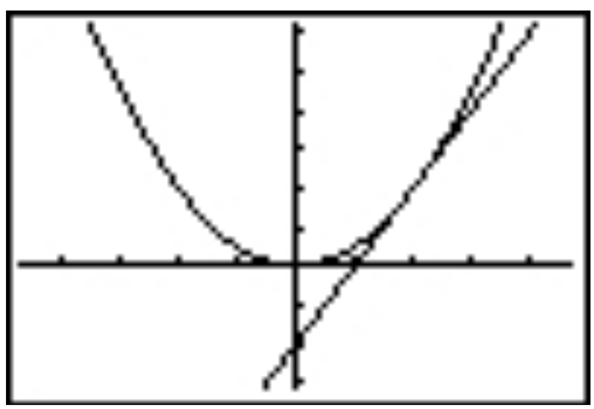
a) 0 b) 8 c) 16 d) –8 e) DNE

5-2: Slope of a Tangent Line

In geometry, a tangent line was defined as a line that touches a curve in exactly one point. But that is only true when the curve is a circle or parabola. In other fields of math, the line might hit the curve again, further down or up, and still be considered a tangent line. A tangent line is the limit of a secant line as one secant point moves towards the other.







Vocabulary:

1. ***Derivative*** – a function that determines the slope of the line tangent to a curve at any point
2. ***Numerical Derivative*** – a function that determines the slope of the line tangent to a curve at a specific point, a.k.a. the ***instantaneous rate of change***

The symbols for the numerical derivative are:

$$f'(a),\ y'(a), \text{ and } \left.\frac{dy}{dx}\right|_{x=a}$$

There are two forms of the equation for the numerical derivative:

The Numerical Derivative:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

These equations arise from the slope equation from algebra:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

With a tangent line, the slope formula will equal $\frac{0}{0}$ because there is only one point to put in for both (x_1, y_1) and (x_2, y_2). The limit in the numerical derivative formulas accounts for the two points sliding together to become one.

The first equation is the common slope equation with the moving point being $(x, f(x))$ and the stationary point being $(a, f(a))$. The second equation uses h to represent the horizontal distance between the points, making the two points $(a+h, f(a+h))$ and $(a, f(a))$, respectively.

LEARNING OUTCOMES

Find the slope of the tangent line to a given curve at a given point.
Find the numerical derivative.
Use the limit definition to find the derivative.

EX 1 Find the slope of the line tangent to $f(x)=\sqrt{x}$ at $x=3$.

$$f'(3)=\lim_{h\to 0}\frac{\left(\sqrt{3+h}-\sqrt{3}\right)\left(\sqrt{3+h}+\sqrt{3}\right)}{h\left(\sqrt{3+h}+\sqrt{3}\right)}$$
$$=\lim_{h\to 0}\frac{3+h-3}{h\left(\sqrt{3+h}+\sqrt{3}\right)}$$
$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{3+h}+\sqrt{3}\right)}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{3+h}+\sqrt{3}}$$
$$=\frac{1}{\sqrt{3+0}+\sqrt{3}}$$
$$=\frac{1}{\sqrt{3}+\sqrt{3}}$$
$$=\frac{1}{2\sqrt{3}}$$

EX 2 If $f(x)=\sqrt{4-x}$, find $f'(2)$

$$f'(2)=\lim_{x\to 2}\frac{\sqrt{4-x}-\sqrt{2}}{x-2}\quad=\lim_{x\to 2}\frac{\left(\sqrt{4-x}-\sqrt{2}\right)\left(\sqrt{4-x}+\sqrt{2}\right)}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$
$$=\lim_{x\to 2}\frac{4-x-2}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$
$$=\lim_{x\to 2}\frac{2-x}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$
$$=\lim_{x\to 2}\frac{-1}{\sqrt{4-x}+\sqrt{2}}$$
$$=\frac{-1}{\sqrt{2}+\sqrt{2}}$$
$$=\frac{-1}{2\sqrt{2}}\text{ or }\frac{-\sqrt{2}}{4}$$

What is commonly referred to as "the derivative," is a generalization of the second numerical derivative equation.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ is the derivative.}$$

There are several different symbols for the derivative:

$\frac{dy}{dx}$ = "*d*-*y*-*d*-*x*", where the *d* is delta, or change, and means $\frac{\text{rise}}{\text{run}}$ as in slope

$f'(x)$ = "*f* prime of *x*"

y' = "*y* prime"

$\frac{d}{dx}$ = "*d*-*d*-*x*"

D_x = "*d* sub *x*"

These last two are the forms used to represent the operation of taking a derivative, as opposed to the derivative function itself.

The derivative is a function, that is, an equation and not a single number. The functions up until now yielded the *y*-coordinate for any given *x*-value, whereas the derivative yields the value of the slope of the tangent line at the point of tangency.

EX 3 Find the derivative of $y=3x^2+5x+1$

$f(x)=3x^2+5x+1 \qquad f(x+h)=3(x+h)^2+5(x+h)+1$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{\left[3(x+h)^2+5(x+h)+1\right]-\left(3x^2+5x+1\right)}{h}\\
&= \lim_{h\to 0}\frac{\left[3\left(x^2+2xh+h^2\right)+5(x+h)+1\right]-\left(3x^2+5x+1\right)}{h}\\
&= \lim_{h\to 0}\frac{3x^2+6xh+3h^2+5x+5h+1-3x^2-5x-1}{h}\\
&= \lim_{h\to 0}\frac{6xh+3h^2+5h}{h}\\
&= \lim_{h\to 0}\left(6x+3h+5\right)\\
&= 6x+0+5
\end{aligned}$$

$$\frac{dy}{dx}=6x+5$$

This means that the x-coordinate of any point on the parabola $y=3x^2+5x+1$ can be plugged into $6x+5$, and the answer will be slope of the tangent line there.

EX 4 Find $D_x\left[x^5\right]$

$D_x\left[x^5\right]=\lim_{h\to 0}\frac{(x+h)^5-x^5}{h}$ use Pascal's triangle to expand:

$$\begin{aligned}
&= \lim_{h\to 0}\frac{x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5-x^5}{h}\\
&= \lim_{h\to 0}\frac{5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5}{h}\\
&= \lim_{h\to 0} 5x^4+10x^3h+10x^2h^2+5xh^3+h^4\\
&= 5x^4+10x^3(0)+10x^2(0)^2+5x(0)^3+(0)^4\\
&= 5x^4
\end{aligned}$$

EX 5 Find $\frac{dy}{dx}$ if $y=\frac{1}{x^2}$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$
$$=\lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}$$
$$=\lim_{h\to 0}\frac{x^2-x^2-2xh-h^2}{h(x+h)^2x^2}$$
$$=\lim_{h\to 0}\frac{-2xh-h^2}{h(x+h)^2x^2}$$
$$=\lim_{h\to 0}\frac{h(-2x-h)}{h(x+h)^2x^2}$$
$$=\lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2}$$
$$=\frac{-2x-0}{(x+0)^2x^2}$$
$$=\frac{-2x}{x^4}$$
$$=\frac{-2}{x^3}$$

5-2 Free Response Homework

Use the Definition of a derivative to evaluate.

1. $D_x\left[x^2+5x+13\right]$

2. $\frac{d}{dx}\left[2x^2-x+3\right]$

3. $\frac{d}{dx}\left[x^3+4x\right]$

4. $D_x\left[\sqrt{x+1}\right]$

5. $\frac{d}{dx}\left[x^6\right]$

6. $\frac{d}{dx}\left[x^5+5x^3-3x\right]$

7. $D_x\left[\sqrt{x^2-1}\right]$

8. $\frac{d}{dx}\left[\frac{1}{\sqrt{x-5}}\right]$

Find the following numerical derivatives.

9. If $f(x)=x^3$, find $f'(3)$

10. If $f(x)=x^2+4x-3$, find $f'(2)$

11. If $f(x)=x^3-1$, find $f'(1)$

12. If $f(x)=3x^2+5x+1$, find $f'(4)$

Use the Numerical Derivative Definition to find $f'(2)$ for the following:

13. $f(x)=x^2+4x+3$

14. $f(x)=x^4+2x^2-3$

15. $f(x)=x^3-8$

16. $f(x)=\sqrt{x^2+4}$

5-2 Multiple Choice Homework

1. The limit $\lim_{h\to 0}\frac{8(2+h)^3-8(2)^3}{h}$ is the derivative of what function at what x-value?

a) $f(x)=8x^3$, $x=2$ b) $f(x)=2x^3$, $x=8$ c) $f(x)=8x^3$, $x=-2$ d) $f(x)=2x^3$, $x=-8$

---

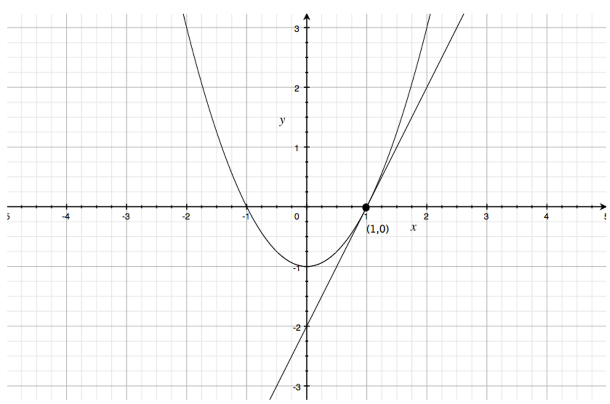
2. The limit $\lim_{x\to 8}\frac{x^2-64}{x-8}$ is the derivative of what function at what x-value?

a) $f(x)=x^2$, $x=8$ b) $f(x)=2x$, $x=8$ c) $f(x)=x^2$, $x=-8$ d) $f(x)=x^2$, $x=64$

---

3. Use the graph of the function to estimate $f'(1)$.

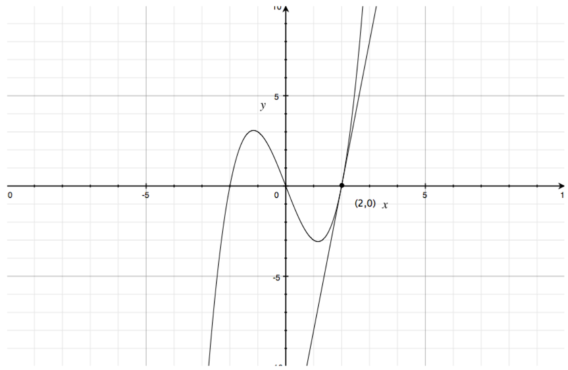
a) 0
b) 1
c) 2
d) –3
e) 4



---

4. Use the graph of the function to estimate $f'(2)$.

a) 0
b) 1
c) –2
d) 8
e) –8



---

5. What is the instantaneous rate of change at $x = 2$ of the function *f* given by $f(x) = x^2 - 2$?

a) –2 b) 2 c) $\frac{1}{2}$ d) $\frac{1}{6}$ e) 4

5-3: The Power Rule

Analysis of several problems like those in the previous section shows there is a short cut to finding certain derivatives. It is called the Power Rule, because it has to do with the exponents.

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

LEARNING OUTCOME

Find the derivative of a polynomial quickly.

EX 1 $\frac{d}{dx}\left[x^5\right] = 5x^{5-1} = 5x^4$

Note that this gives the same answer as EX 4 in the previous section.

EX 2 $\frac{d}{dx}\left[x^{11}\right] = 11x^{11-1} = 11x^{10}$

EX 3 $\frac{d}{dx}\left[6\right] = \frac{d}{dx}\left[6x^0\right] = 6(0)x^{0-1} = 0$

Other Rules:

$$D_x\left[c\right]=0\text{, where } c \text{ is a constant}$$

$$D_x\left[cx^n\right]=(cn)x^{n-1}$$

$$D_x\left[f(x)+g(x)\right]=D_x\left[f(x)\right]+D_x\left[g(x)\right]$$

These rules allow us to easily differentiate a polynomial—term-by-term.

EX 4 $y=3x^2+5x+1$; find $\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left[3x^2+5x+1\right]\\&=3(2)x^{2-1}+5(1)x^{1-1}+1(0)\\&=6x+5\end{aligned}$$

EX 5 $\dfrac{d}{dx}\left[\dfrac{1}{x^2}\right]$

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{x^2}\right]&=\frac{d}{dx}\left[x^{-2}\right]\\&=-2x^{-2-1}\\&=-2x^{-3}\\&=\frac{-2}{x^3}\end{aligned}$$

EX 6 $\frac{d}{dx}\left[\sqrt[3]{x}\right]$

$$\frac{d}{dx}\left[\sqrt[3]{x}\right]=\frac{d}{dx}\left[x^{1/3}\right]$$
$$=\frac{1}{3}x^{1/3-1}$$
$$=\frac{1}{3}x^{-2/3}$$
$$=\frac{1}{3x^{2/3}}$$

Vocabulary:
1. ***Second Derivative*** – the derivative of the derivative

Higher Order Derivative Symbols

Liebnitz: $\frac{d^2y}{dx^2}$= d squared y, d x squared; $\frac{d^3y}{dx^3}$;… $\frac{d^ny}{dx^n}$

Function: $f''(x)$= f double prime of x; $f'''(x)$**;** $f^{IV}(x)$;… $f^{n}(x)$

Combination: y'' = y double prime

EX 7 Find $f''(x)$, if $y = 16x^4 - 9x^3 + 17x^2 + 3$

To find $f''(x)$, first find $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(16x^4 - 9x^3 + 17x^2 + 3\right) \\ &= 4\left(16x^{4-1}\right) - 3\left(9x^{3-1}\right) + 2\left(17x^{2-1}\right) \\ &= 64x^3 - 27x^2 + 34x \end{aligned}$$

Now, find $f''(x)$ by taking the derivative of $f'(x)$:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(64x^3 - 27x^2 + 34x\right) \\ &= 3\left(64x^{3-1}\right) - 2\left(27x^{2-1}\right) + 1\left(34x^{1-1}\right) \\ &= 192x^2 - 54x + 34 \end{aligned}$$

5-3 Free Response Homework

1. $D_x\left[4x^2-3x+1\right]$

2. $D_x\left[4x^7-2x^6+6x^4+7x^2+1\right]$

3. $\frac{d}{dx}\left[2x^3+16x^2-3x+16\right]$

4. $\frac{d}{dx}\left[x^3+x^2-3x-3\right]$

5. $f(x)=x^2+4x-\pi$; find $f'(x)$

6. $f(x)=\frac{1}{x^2}+4x^4$; find $f'(x)$

7. $y=\frac{1}{x^4}+\frac{4}{x^3}-\frac{5}{x}$; find $\frac{dy}{dx}$

8. $y=\sqrt{x^3}+\frac{4}{\sqrt{x}}-\sqrt[4]{x^3}$; find $\frac{dy}{dx}$

9. $D_x\left[3x^2-4x^3+\frac{2}{x^2}-\frac{1}{\sqrt[3]{x}}+6^4\right]$

10. $\frac{d}{dx}\left[8x^7-3x^2+\frac{1}{x^7}-\frac{2}{\sqrt[4]{x^9}}+\pi^4\right]$

11. $f(x)=\sqrt[3]{x^5}+\frac{9}{x^3}-2\sqrt[5]{x^7}-\pi x$; find $f'(x)$

5-3 Multiple Choice Homework

1. If $f(x)=x^{3/2}$, then $f'(4)=$

a) –6 b) –3 c) 3 d) 6 e) 8

---

2. The derivative of $\sqrt{x}-\dfrac{1}{x\sqrt[3]{x}}$ is

a) $\frac{1}{2}x^{-1/2}-x^{-4/3}$

b) $\frac{1}{2}x^{-1/2}+\frac{4}{3}x^{-7/3}$

c) $\frac{1}{2}x^{-1/2}-\frac{4}{3}x^{-1/3}$

d) $-\frac{1}{2}x^{-1/2}+\frac{4}{3}x^{-7/3}$

e) $-\frac{1}{2}x^{-1/2}-\frac{4}{3}x^{-1/3}$

---

3. Given $f(x)=\dfrac{1}{2x}+\dfrac{1}{x^2}$, find $f'(x)$.

a) $-\dfrac{1}{2x^2}-\dfrac{2}{x^3}$

b) $-\dfrac{2}{x^2}-\dfrac{2}{x^3}$

c) $\dfrac{2}{x^2}-\dfrac{2}{x^3}$

d) $-\dfrac{1}{2x^2}+\dfrac{2}{x^3}$

e) $\dfrac{1}{2x^2}-\dfrac{2}{x^3}$

---

4. Find $f''(x)$ if $f(x) = x^4 + 4x^3 - \frac{2}{x} + 9$.

a) $4x^3 + 12x^2 + \frac{2}{x^2}$

b) $4x^3 + 12x^2 - 2$

c) $12x^2 + 24x - \frac{4}{x^3}$

d) $12x^2 + 24x$

e) $12x^2 + 24x + \frac{4}{x^3}$

---

5. Find $\frac{d^3y}{dx^3}$ if $y = 3x^3 + 6x^2 - 4x$.

a) $9x^2 + 12x - 4$

b) 18

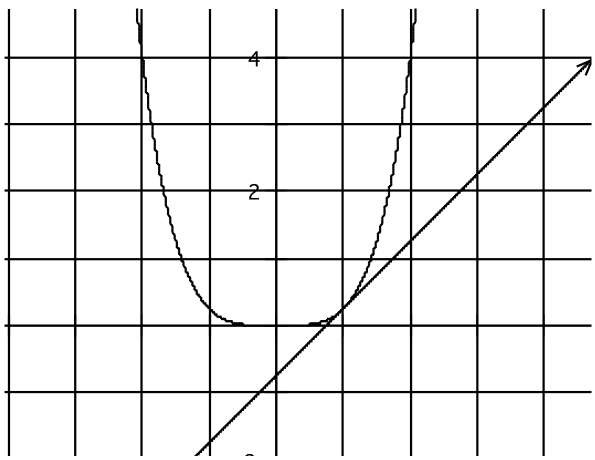
c) $18x + 12$

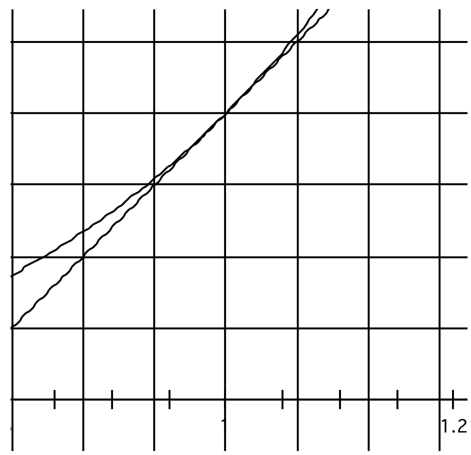
d) $9x + 18$

e) 9

5-4: Local Linearity and Approximations

Before calculators, one of the most valuable uses of the derivative was to find approximate function values from a tangent line. Since the tangent line only shares one point on the function, y-values on the line are very close to y-values on the function. This idea is called local linearity—near the point of tangency, the function curve appears to be a line. This can be easily demonstrated with the graphing calculator by zooming in on the point of tangency. Consider the graphs of $y = 0.25x^4$ and its tangent line at $x = 1$, $y = x - 0.75$.





Zoom in closer and the line and the curve become one. The y-values on the line are good approximations of the y-values on the curve if the x-value is close to the point of tangency. For a good animation of this concept, see

http://www.ima.umn.edu/~arnold/tangent/tangent.mpg

Since it is easier to find the y-value of a line arithmetically than for other functions—especially transcendental functions—the tangent line approximation is useful if a calculator is not permitted.

Vocabulary:
1. ***Normal line*** – the line perpendicular to the tangent line of a curve at the point of tangency

LEARNING OUTCOMES

Find the equations of tangent and normal lines to a given curve at a given point.
Use the equation of a tangent line to approximate function values.

EX 1 Find the tangent line equation to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

The slope of the tangent line will be $f'(-1)$

$$f'(x) = 4x^3 - 3x^2 - 4x$$
$$f'(-1) = -3$$

Note that this value can also be found using the nDeriv function on the calculator. This function may be found using MATH 8. In this example, nDeriv$\left(x^4 - x^3 - 2x^2 + 1, x, -1\right)$, followed by the <ENTER> key, is –3.

$f(-1) = 1$, so the tangent line will be

$$y - 1 = -3(x + 1)$$
or
$$y = -3x - 2$$

Though not as practically useful as the tangent lines, another context for the derivative is in finding the equation of the normal line.

EX 2 Find the equation of the line normal to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

In EX 1, the slope of the tangent line was $m_{\tan} = f'(-1) = -3$. Therefore, $m_{normal} = \frac{1}{f'(-1)} = \frac{1}{3}$, and the equation of the line is

$$y - 1 = \frac{1}{3}(x + 1)$$

or

$$y = \frac{1}{3}x - \frac{4}{3}$$

EX 3 Find the equations of the lines tangent and normal to $y = x^3 - 2x$ at $x = -2$.

For the equation of any line, a point and the slope are needed. The x-coordinate of the point is given, so $y = (-2)^3 - 2(-2) = -4$.

$$\frac{dy}{dx} = 3x^2 - 2$$

$$\left.\frac{dy}{dx}\right|_{x=-2} = 3(-2)^2 - 2 = 10$$

so the tangent line is

$$y + 4 = 10(x + 2)$$

and the normal line is

$$y + 4 = -\frac{1}{10}(x + 2)$$

Note that there are three unknown in these problems: x, y, and $\frac{dy}{dx}$. Given any one of them, the other two can be found.

EX 4 Find an equation of the tangent line to the curve $f(x) = 4x^3 - 3x - 1$ at the point in the first quadrant where $\frac{dy}{dx} = 45.$

$$\frac{dy}{dx} = 12x^2 - 3 = 45$$
$$x = \pm 2$$

$$f(2) = 4(2)^3 - 3(2) - 1 = 25$$
$$f(-2) = 4(-2)^3 - 3(-2) - 1 = -27$$

Since there are two points where $\frac{dy}{dx} = 45$, there are two tangent lines that meet the criteria:

$$y - 25 = 45(x - 2) \quad \text{and} \quad y + 27 = 45(x + 2)$$

EX 5 Use the equation of the line tangent to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$ to approximate value of $f(-0.9)$.

In EX 1, the tangent line was $y = -3x - 2$. So if $x = -0.9$ on the tangent line, then

$$f(-0.9) \approx y(-0.9) = -3(-0.9) - 2 = 0.7$$

This last example might seem somewhat trivial in that $x = -0.9$ could be plugged into $f(x) = x^4 - x^3 - 2x^2 + 1$ to find the exact value even without a calculator. But if the function were more complicated—for example, if there where a radical or an exponential in the function—it would have been very difficult. The tangent line approximation will help in those situations, which will arise in a later chapter.

Summary of Tangent Line Approximations

NB. a. Use $y - y_1 = m(x - x_1)$ as the general equation of the tangent line.

b. There are three unknowns— $x_1, y_1,$ *and m*. One of them will have a given value.

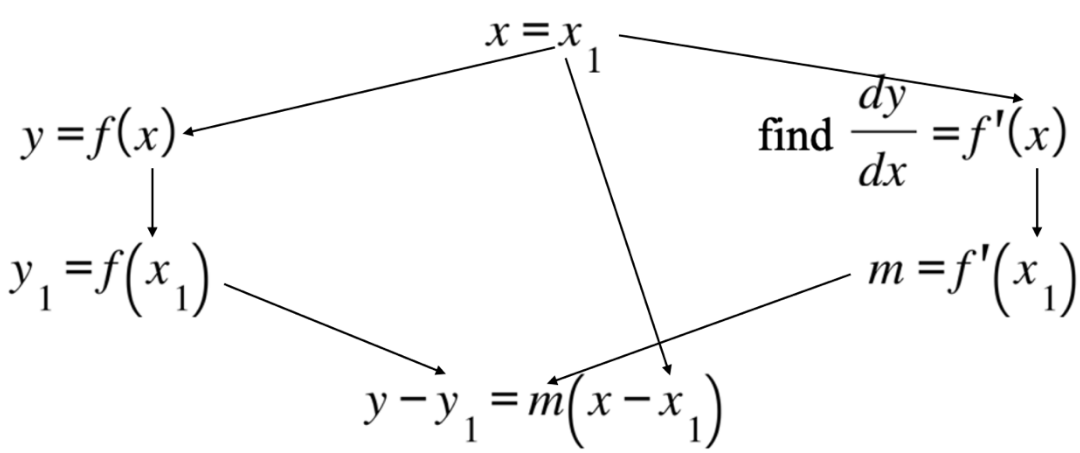
Steps

1. Find values for the other two unknowns.

2. Set up the specific tangent line equation.

3. Approximate $f(a)$ by substituting *a* for *x* in the tangent line equation and simplifying.

Summary of Tangent Line Approximations



$$f(a) \approx y = m(a - x_1) + y_1$$

5-4 Free Response Homework

1. Find the equation of the tangent line to $f(x) = x^5 - 5x + 1$ at $x = -2$.

2. Find the tangent line equation to $F(x) = \frac{5}{x^2} - \sqrt{x}$ at $x = 1$.

3. Find the equation of the tangent line to $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$ at $x = 0$.

4. Find the equation of the tangent line to $f(x) = \frac{1}{2}x^4 + \frac{1}{3}x^3 + 2x^2 + 2x - 7$ at $x = -1$.

Find the equations of the lines tangent and normal to the given functions at the given x-value.

5. $g(x) = x^3 - x^2 + 4x - 4$ at $x = -3$

6. $y = 2x^3 - x^2 + 3x - 4$ at $x = -1$

7. $y = 3x^3 - 4x^2 + 2x - 4$ at $x = 0$

8. $y = 8x^3 - x^2 + x - 4$ at $x = 1$

9. Use the equation of the tangent line to $f(x) = x^5 - 5x + 1$ at $x = -2$ to approximate the value of $f(-1.9)$.

10. Use the tangent line equation to $g(x) = x^3 - x^2 + 4x - 4$ at $x = -3$ to approximate the value of $g(-2.9)$

11. Use the equation of the tangent line to $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$ at $x = 0$ to approximate $f(-0.1)$.

12. Use the equation of the tangent line to $f(x)=\frac{1}{2}x^4+\frac{1}{3}x^3+2x^2+2x-7$ at $x=-1$ to approximate the value of $f(-0.9)$.

13. At what point on the graph of $y=x^2-3x-4$ is the tangent parallel to the line $3x-y=3$?

14. At what point on the graph of $y=\frac{1}{2}x^2$ is the tangent parallel to the line $2x-4y=3$?

15. Find the equation of the line tangent to $f(x)=2x^3-9x^2-12x$ where $f'(x)=12$.

16. Find the equation of the line tangent to $f(x)=\frac{1}{5}x^5+\frac{2}{3}x^3-8x$ where $f'(x)=1$.

5-4 Multiple Choice Homework

1. An equation of the line tangent to the graph of $y=x^3+3x^2+2$ at $x=-1$ is

a) $y=-3x+1$ b) $y=-3x-7$ c) $y=x+5$

d) $y=3x+1$ e) $y=3x+7$

---

2. Which of the following is an equation of the line tangent to the graph of $f(x)=x^4+2x^2$ at the point where $f'(x)=1$?

a) $y=8x-5$ b) $y=x+7$ c) $y=x+0.763$

d) $y=x-0.122$ e) $y=x-2.146$

---

3. Let *f* be a differentiable function such that $f(3)=2$ and $f'(3)=5$. If the tangent line to the graph of *f* at $x=3$ is used to find an approximation to a zero of *f*, that approximation is

a) 0.4 b) 0.5 c) 2.6 d) 3.4 e) 5.5

---

4. Find an equation of the line tangent to $f(x)=x^2-4x$ at the point $(3, -3)$.

a) $2x-y=9$ b) $x-2y=9$ c) $y-2x=9$ d) $2y-x=9$ e) $x-y=9$

---

5. Find an equation of the line tangent to the curve $y=x+\frac{1}{x}$ at the point $\left(5, \frac{26}{5}\right)$

a) $24x-25y=-10$
b) $24x-25y=10$
c) $25x-24y=10$
d) $25x-24y=-10$
e) $24x-24y=10$

---

6. If the line tangent to $y=2x^2+x+k$ is the line $3x+y=1$, then $k=$

a) 1 b) 2 c) 3 d) 4 e) 5

5-5: Rectilinear Motion

Derivatives are used extensively in physics to describe particle motion (or any other linear motion). If the variables represent time and distance, the derivative will be a rate or velocity of the particle. Because of their meaning, the letters *t* and either *s* or *x* are used to represent time and distance, respectively.

Vocabulary:

1. ***Rectilinear Motion*** **–** movement that occurs in a straight line
2. ***Velocity*** – Defn: directed speed
   Means: how fast something it is going and whether it is moving right or left, up or down
3. ***Average Velocity*** – Defn: distance traveled divided by time or $\frac{x_2 - x_1}{t_2 - t_1}$
   Means: the average rate, as used in algebra
4. ***Instantaneous Velocity*** – Defn: velocity at a particular time *t*
   Means: $\frac{ds}{dt}$, $\frac{dx}{dt}$, or $\frac{dy}{dt}$, or the rate at any given instant
5. ***Acceleration*** – the rate of change of the velocity or $\frac{dv}{dt}$

Three things are implied in the definitions:

1. The derivative of a position or distance equation is the velocity equation.

2. The derivative of velocity is acceleration. Therefore, acceleration is known as the second derivative of the position or distance equation.

   The second derivative is denoted by $x''(t)$ or $\frac{d^2x}{dt^2}$.

3. The sign of the velocity determines the direction of the movement:

   Velocity > 0 means the movement is to the right (or up)
   Velocity < 0 means the movement is to the left (or down)
   Velocity = 0 means the movement is stopped.

LEARNING OUTCOMES

Given a position function of an object in rectilinear motion, find the velocity and acceleration functions.
Use the velocity function to describe the rectilinear motion of an object.
Interpret sign patterns in context of rectilinear motion.
Given the height function of an object in projectile motion, find the velocity and acceleration functions.

We now have three equations which describe different aspects of one situation. The biggest problem is determining which equation to use when.

Key Phrases:

1. The Answer to "When" questions will be time.
2. "Where" questions are answered by the position equation.
3. "Which direction" questions are answered by the velocity equation.
4. "Speeding up/Slowing down" questions are answered a combination of the velocity and acceleration equations.

EX 1 The position of a particle is described by $x(t)=t^3-3t^2-24t+3$.

a) Where is the particle at $t=3$?
b) When the particle is stopped?
c) Which direction it is moving at $t=3$ seconds?
d) Find $a(3)$.

a) "Where it is at $t=3$" means find $x(3)$

$$x(3)=3^3-3(3)^2-24(3)+3=-69$$

b) "When the particle is stopped" means "at what time is the velocity zero?"

$$v(t)=x'(t)=3t^2-6t-24=0$$

$$\frac{3t^2-6t-24=0}{3}$$

$$t^2 - 2t - 8 = 0$$
$$(t+2)(t-4) = 0$$
$$T = -2 \text{ or } 4$$

c) “Which direction it is moving at $t = 3$ seconds” means what is the sign of the velocity?”

$$v(3) = 3(3)^2 - 6(3) - 24 = -15$$

The particle is moving left because the velocity is negative.

d) $a(3)$ means plug 3 into the acceleration equation.

$$a(t) = v'(t) = 6t - 6$$
$$a(3) = v'(t) = 6(3) - 6 = 12$$

The sign pattern of the velocity function is used to answer questions about an object moving left/right/up/down or at rest.

EX 2 A particle's distance $x(t)$ from the origin at time $t \geq 0$ is described by $x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$. Where is it when it stops to switch directions?

$$x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$$
$$v(t) = 4t^3 - 6t^2 - 22t + 12 = 0$$
$$= 2t^3 - 3t^2 - 11t + 6 = 0$$
$$= (2t-1)(t+2)(t-3) = 0$$
$$t = \frac{1}{2}, -2, \text{ and } 3$$

But the problem specifies that $t \geq 0$, so $t = \frac{1}{2}$ and 3.

$$x\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 11\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) + 1 = 4.063$$
$$x(3) = (3)^4 - 2(3)^3 - 11(3)^2 + 12(3) + 1 = -35$$

Therefore, this particle stops 4.063 units to the right of the origin $\left(\text{at } \frac{1}{2} \text{ seconds}\right)$ and 35 units left of the origin (at 3 seconds).

EX 3 A particle's distance $x(t)$ from the origin at time $t \geq 0$ is described by $x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$. What is the acceleration when the particle stops to switch directions?

The particle stops when the velocity is zero. As seen in the previous example:

$$
\begin{aligned}
x(t) &= t^4 - 2t^3 - 11t^2 + 12t + 1 \\
x'(t) &= v(t) = 4t^3 - 6t^2 - 22t + 12 = 0 \\
&= 2t^3 - 3t^2 - 11t + 6 = 0 \\
&= (2t-1)(t+2)(t-3) = 0 \\
t &= \frac{1}{2}, -2, \text{ and } 3
\end{aligned}
$$

and $t = \frac{1}{2}$ and 3, because the time $t \geq 0$. Now substitute these times into the acceleration equation.

$$v'(t) = a(t) = 12t^2 - 12t - 22$$

$$a(3) = 12(3)^2 - 12(3) - 22 = 50$$

$$a\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) - 22 = -25$$

$$a = 50 \text{ and } a = -25$$

Note that the interpretation of a negative acceleration is not, like velocity, the direction it is moving, but which way the acceleration is affecting the particle. The acceleration and velocity's directions together determine if the particle is slowing down or speeding up.

EX 4 If an object's position is described by $x(t)=2t^3-3t^2-12t+1$, when is it moving left?

$$x(t)=2t^3-3t^2-12t+1$$
$$v(t)=6t^2-6t-12$$

Since the particle stops at $v=0$,

$$v(t)=6t^2-6t-12=0$$
$$v(t)=t^2-t-2=0$$
$$v(t)=(t+1)(t-2)=0$$
$$t=-1 \text{ or } 2$$

The sign pattern is $\begin{matrix} v & \longleftarrow & + & 0 & - & 0 & + & \longrightarrow \\ t & & & -1 & & 2 & & \end{matrix}$ and "moving left" means the velocity is negative. Therefore,

$$t\in(-1,\ 2)$$

EX 5 Find the position of the object described by $x(t)=2t^3-3t^2-12t+1$, when the particle is at rest.

The question asks to find the position when the particle is at rest. Therefore, the only pertinent information from the previous example is

$$v(t)=0\rightarrow t=-1 \text{ or } 2$$

The positions at which the object is at rest are $x(-1)=8$ and $x(2)=-19$.

EX 6 A gun is fired up in the air from a 1600-foot tall building at 240 ft/second. How fast is the bullet going when it hits the ground?

Any object launched from height h_0 feet with initial velocity v_0 ft/sec follows the equation:

$$h(t) = -\frac{1}{2}at^2 + v_0 t + h_0$$
$$h(t) = -16t^2 + 240t + 1600 = 0$$
$$t^2 - 15t - 100 = 0$$
$$(t-20)(t+5) = 0$$

$t = 20$ seconds is when it hits the ground.

$$v(t) = -32t + 240$$
$$v(20) = -32(20) + 240 = -400$$

The velocity is negative because the bullet is coming down. The speed it is going is 400 ft/sec.

EX 7 What is the acceleration due to gravity of any falling object?

$$h = -16t^2 + v_0 t + h_0$$
$$v = h' = -32t + v_0$$

$$a = v' = -32 \text{ ft/sec}^2$$

This number is known as the Gravitational Constant.

5-5 Free Response Homework

The motion of a particle is described by the following distance equations. For each, find:

a) when the particle is stopped,

b) which direction it is moving at $t = 3$ seconds,

c) where it is at $t = 3$, and

d) $a(3)$

1. $x(t) = 2t^3 - 21t^2 + 60t + 4$

2. $x(t) = t^3 - 6t^2 + 12t + 5$

3. $x(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48$

4. $x(t) = 12t^5 - 15t^4 - 220t^3 + 270t^2 + 1080t$

5. $y(t) = 5t^3 - t^5$

6. $y(t) = t^4 - 5t^2 - 36$

Find $x(t)$ when $v = 0$.

7. $x(t) = t^2 - 5t + 4$

8. $x(t) = t^3 - 6t^2 - 63t + 4$

9. $x(t) = 6t^5 - 15t^4 - 8t^3 + 24t^2 + 12$

Find $x(t)$ and $v(t)$ when $a(t) = 0$.

10. $x(t) = 2t^3 - 21t^2 + 60t + 4$

11. $x(t) = t^3 - 6t^2 + 12t + 5$

12. $x(t)=9t^4-4t^3-240t^2+576t-48$

Find maximum height of a projectile launched vertically from the given height and with the given initial velocity.

13. $h_0=25$ feet, $v_0=64$ ft/sec

14. $h_0=10$ meters, $v_0=294$ m/sec

15. The equations for free fall at the surfaces of Mars and Jupiter (*s* in meters, *t* in seconds) are $s=1.86t^2$ on Mars, $s=11.44t^2$ on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec on each planet?

5-5 Multiple Choice Homework

1. A particle moves along a straight line with equation of motion $s=t^3+t^2$. Find the value of *t* at which the acceleration is zero.

a) $-\frac{2}{3}$ b) $-\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{1}{3}$ e) $-\frac{1}{2}$

---

2. A particle moves on the *x*-axis with velocity given by $v(t)=3t^4-11t^2+9t-2$ for $-3\le t\le 3$. How many times does the particle change direction as *t* increases from –3 to 3?

a) 0 b) 1 c) 2 d) 3 e) 4

---

3. A particle moves on the *x*-axis so that its position is given by $x(t)=t^2-6t+5$. For what value of *t* is the velocity of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of *t*

4. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the acceleration of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

---

5. Find the acceleration at time $t = 9$ seconds if the position (in cm.) of a particle moving along a line is $s(t) = 6t^3 - 7t^2 - 9t + 2$.

a) 310 cm/sec^2 b) 310 cm/sec c) 1323 cm/sec^2

d) 1323 cm/sec e) -1323 cm/sec

5-6: Parametric Motion

If x and y are both dependent on some other variable, then each value of that variable will give a value for x and y. Most often this other variable is represented as a t, though, later it will be seen as θ. It does not necessarily represent time, but in many cases it does. In these cases, both x and y are dependent variables and t is the independent variable. In fact, parametric motion problems look like pairs of problems from the last section, which simultaneously describe the motion of a particle.

Vocabulary:
1. ***Parameter*** – dummy variable that determines x- and y-coordinates independent of one another
2. ***Parametric Motion*** – movement that occurs in a plane
3. ***Vector*** – a directed line segment. As such, it can be used to represent anything that has magnitude and direction (e.g., force, velocity, etc.)
4. ***Magnitude*** – the length/size of a vector
5. ***Direction of a Vector*** – the standard position angle when a vector is placed with its tail at the origin

The rectilinear motion formulas can be generalized to parametric motion formulas:

$$\text{Position } = \left\langle x(t),\, y(t) \right\rangle$$

$$\text{Velocity } = \left\langle x'(t),\, y'(t) \right\rangle$$

$$\text{Acceleration } = \left\langle x''(t),\, y''(t) \right\rangle$$

$$\text{Speed } = \left| v(t) \right| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$$

LEARNING OUTCOMES

Given the position vector of an object in parametric motion, find the speed of the object, and find the velocity and acceleration functions.
Interpret the sign pattern of the velocity functions of an object in parametric motion.

EX 1 Find the velocity vector for the particle whose position is described by $\langle t^2 - 2t,\ t^2 + 1\rangle$

Since this is a position vector:

$$x(t) = t^2 - 2t \text{ and } y(t) = t^2 + 1$$

Taking the derivatives:

$$\frac{dx}{dt} = 2t - 2 \text{ and } \frac{dy}{dt} = 2t$$

Since the derivatives of x and y are actually velocities, write the velocity vector using chevrons.

$$\text{Velocity} = \langle 2t-2,\ 2t\rangle$$

EX 2 At $t = 3$, find the speed of the particle whose velocity vector is $\langle 2t-2,\ 2t\rangle$.

$$\text{speed} = |v(t)| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$$

For this particular velocity vector, $\text{speed} = \sqrt{(2t-2)^2 + (2t)^2}$

at $t = 3$,

$$\text{speed} = \sqrt{\left(2(3)-2\right)^2 + \left(2(3)\right)^2} = \sqrt{52}$$

EX 3 If the position of an object is described by $\left\langle 4t^2-3t-5,\ 6-t^2\right\rangle$, when is the object moving left and down?

Velocity = $\left\langle 8t-3,\ -2t\right\rangle$

Parametric motion velocity has two parts, therefore there are two sign patterns—one for x and one for y:

x': $-$ 0 $+$
t: $\frac{3}{8}$

y': $+$ 0 $-$
t: 0

The question, though, is when is the object moving BOTH left and down? That is, when are the two velocity components both negative? It might be easier to answer this question if the sign patterns aligned vertically:

x': $-$ 0 $+$
t: $\frac{3}{8}$

y': $+$ 0 $-$
t: 0

$$t \in \left(0,\ \frac{3}{8}\right)$$

An example of parametric motion has been encountered in Physics class, though it is seldom described as parametric motion. Two-dimensional projectile motion is an example of parametric motion. Notice that, though the first equation is similar, this is not the kind of projectile motion problem from the last chapter. In that case, the object went straight up and straight down. In this situation, the object is traveling horizontally as well as vertically.

EX 4 A baseball is thrown into the air. Its height is described by the equation $y(t) = -4.9t^2 + 14t + 1$ and its horizontal distance is described by the equation $x(t) = 10t$ for values of $t \in [0, 2.927]$. Write the position, velocity, and acceleration vectors for the ball.

Since the position of the ball was changing, the equations describing its movement are a vector:

$$\langle 10t,\ -4.9t^2 + 14t + 1 \rangle$$

The velocity and acceleration vectors require the derivative of position once or twice, respectively.

$$\text{Velocity} = \langle 10,\ -9.8t + 14 \rangle$$

$$\text{Acceleration} = \langle 0,\ -9.8 \rangle$$

5-6 Free Response Homework:

Find a) the velocity vector, b) acceleration vector, and c) the speed at $t = 2$ for each of the following parametric equations.

1. $x(t) = t^2 - 5t,\ y(t) = -4t^2 + 1$

2. $x(t) = t^3 - 6t^2 - 63t + 4,\ y(t) = 4t$

3. $x(t) = 2t^3 - 21t^2 + 60t + 4,\ y(t) = t^2$

4. $x(t) = 2t^2 + 6t - 5,\ y(t) = 2t^2 + 6t - 5$

5. $x(t) = t^3 + 5t - 6,\ y(t) = 5t^3 - t^5$

6. $x(t) = 3t^2 + 6\sqrt{t} - 15,\ y(t) = t^4 - 5t^2 - 36$

7. A particle's position $\langle x(t),\ y(t)\rangle$ is described by the parametric equations $x(t) = 4t^2 + 3$ and $y(t) = \dfrac{2}{t} - t^2$.
   a) What is the particle's velocity at $t = 2$?
   b) What is the particle's acceleration at $t = 2$?

8. A particle's position $\langle x(t),\ y(t)\rangle$ is described by the parametric equations $x(t) = t^3 - 6t^2 + 3$ and $y(t) = t^2 - 8t - 1$. When is the particle at rest?

9. Two particles move in the *xy*-axis plane. For $t \geq 0$, the position of Particle A is given by $\langle t-2, (t-2)^2\rangle$ and the position of Particle B is given by $\left\langle \dfrac{3t}{2} - 4, \dfrac{3t}{2} - 2\right\rangle$. Determine the exact time when the particles collide (i.e., they are in the same position at the same time).

5-6 Multiple Choice Homework

1. At time $t \geq 0$, a particle moving in the *xy*-plane has velocity vector given by $v(t) = \langle t^2,\ 5t \rangle$. What is the acceleration vector of the particle at time $t = 3$?

a) $\left\langle 9,\ \frac{45}{2} \right\rangle$ b) $\langle 6,\ 5 \rangle$ c) $\langle 2,\ 0 \rangle$

d) $\sqrt{306}$ e) $\sqrt{61}$

---

2. At time $t \geq 0$, a particle moving in the *xy*-plane has velocity vector given by $v(t) = \langle t^2,\ 5t \rangle$. What is the speed of the particle at time $t = 3$?

a) $\left\langle 9,\ \frac{45}{2} \right\rangle$ b) $\langle 6,\ 5 \rangle$ c) $\langle 2,\ 0 \rangle$

d) $\sqrt{306}$ e) $\sqrt{61}$

---

3. A particle moves on a plane curve so that at any time $t > 0$ its velocity vector is given by the equations $v_x = t^2 - t$ and $v_y = 2t - 1$. The acceleration vector of the particle at $t = 1$ is

a) $\langle 0,\ 1 \rangle$ b) $\langle 0,\ -1 \rangle$ c) $\langle 2,\ 2 \rangle$

d) $\langle -1,\ 1 \rangle$ e) $\langle 1,\ 2 \rangle$

---

4. A particle moving in the *xy*-plane with its *x* coordinate given by $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+\frac{1}{2}t^2-1$ and its *y* coordinate given by $y(t)=\frac{1}{2}t^2-t+1$. When the particle is moving up it is also

a) moving right
b) moving left
c) at rest
d) cannot be determined
e) does not exist

---

5. A particle moves on a plane so that its position vector is $p(t)=\left\langle \frac{1}{3}t^3+\frac{1}{2}t^2-2t+7, \frac{2}{3}t^3+\frac{3}{2}t^2-2t+\pi^6 \right\rangle$ is at rest when

a) $t=1$ only b) $t=\frac{1}{2}$ only c) $t=-2$ only

d) $t=1, \frac{1}{2}$ e) $t=1, \frac{1}{2}, -2$

Limits and Derivatives Practice Test
Part 1: CALCULATOR REQUIRED

Round to 3 decimal places. Show all work.

Multiple Choice (3 points each)

1. Find the y-intercept of the tangent line to the curve $y = x^3$ at the point $(2,\ 8)$.

a) –4
b) 4
c) –16
d) 12
e) –8

2. Let $f(x) = 2\sqrt{x}$. If $f(c) = f'(c)$, then c equals

a) 0
b) 0.82
c) 1.2
d) 0.5
e) 2.1

3. If $f(x) = x^2 - 1$, then $\lim_{x \to 1} \frac{f(x+1) - f(2)}{x^2 - 1}$ is

a) 0
b) 1
c) 2
d) 3
(e) nonexistent

4. An equation of the line normal to the graph of $y = 7x^4 + 2x^3 + x^2 + 2x + 5$ at the point where $x = 0$ is

a) $x + 2y = 10$
b) $2x + y = 10$
c) $5x + 5y = 2$
d) $2x - y = -5$
e) $2x + y = -10$

5. A particle moves along a straight line with the equation of motion $s = t^2 - 2t$. Find the instantaneous velocity of the particle at time $t = 1$.

a) 1
b) 4
c) 0
d) 3
e) 8

6. If $f(x) = e^x$, which of the following is equal to $f'(e)$?

a) $\lim\limits_{h \to 0} \dfrac{e^{x+h}}{h}$
b) $\lim\limits_{h \to 0} \dfrac{e^{x+h} - e^e}{h}$
c) $\lim\limits_{h \to 0} \dfrac{e^{e+h} - e}{h}$
d) $\lim\limits_{h \to 0} \dfrac{e^{x+h} - 1}{h}$
e) $\lim\limits_{h \to 0} \dfrac{e^{e+h} - e^e}{h}$

Limits and Derivatives Practice Test
Part 2: CALCULATOR ALLOWED

Round to 3 decimal places. Show all work.

1. A particle's position $\langle x(t), y(t) \rangle$ at time t is described by the parametric equations $x(t) = 2t^3 - t^2 - 18t + 9$, $y(t) = t^2 - 4t - 5$. When is the particle moving right and down?

2. The motion of a particle is described by $x(t) = 5t^4 + t^3 - 9t^2 + 4t - 7$.

a) When is the particle stopped?
b) Which direction is it moving at $t = 7$?
c) <u>Where</u> is it when $t = 7$?
d) Find $a(7)$.

3. Evaluate the following limits:

a) $$\lim_{x\to 1}\frac{x^2+2x-3}{x^2-1}=$$

b) $$\lim_{x\to 3}\frac{x^3-27}{x^4-14x^2+45}$$

c) $$\lim_{x\to -1}\frac{3x^2+2x-1}{x^3+x^2+4x+4}=$$

Limits and Derivatives Practice Test
Part 3: NO CALCULATOR ALLOWED

Show all work.

4. Set up, but do not solve, the limit definition of the derivative for $f(x) = 5x^7 + 2x^4 - 7\pi$.

5. Use the Power Rule to find:

a) $\frac{dy}{dx}$ if $y = 6x^7 - 19x^4 + 3x^2 - 12x - 13$

b) $D_x\left[\sqrt[4]{x^7} - \frac{6}{x^5} - \sqrt[3]{x} + \pi^2 - x\right]$

c) $\frac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^3 - \frac{1}{\sqrt[7]{x^4}} + \frac{1}{5x}\right]$

d) Let $y = 4x^{17} + 6x^3 - 2x + 17 + \pi + \sqrt[3]{x^8} - \frac{2}{x^3}$. Find y'.

Limits and Derivatives Homework Answer Key

5-1 Free Response

1.	$\frac{7}{6}$	2.	$-\frac{2}{5}$	3.	0	4.	$\frac{2}{9}$
5.	2	6.	$-\frac{5}{14}$	7.	$\frac{3}{5}$	8.	$-\frac{5}{7}$
9.	$\frac{17}{25}$	10.	$-\frac{5}{8}$	11.	$\frac{8}{1+\sqrt{2}}$	12.	$\frac{11}{14}$
13.	$-\frac{1}{2\sqrt{5}}$	14.	16	15.	DNE	16.	$-\frac{1}{108}$
17.	0	18.	$\frac{1}{2}$	19.	4		

5-1 Multiple Choice

1. B 2. E 3. A 4. D 5. C

5-2 Free Response

1.	$2x+5$	2.	$4x-1$	3.	$3x^2+4$
4.	$\frac{1}{2\sqrt{x+1}}$	5.	$6x^5$	6.	$5x^4+15x^2-3$
7.	$\frac{x}{\sqrt{x^2-1}}$	8.	$\frac{-1}{2(x-5)\sqrt{x-5}}$	9.	27
10.	8	11.	3	12.	29
13.	$f'(2)=8$	14.	$f'(2)=40$	15.	$f'(2)=12$

16. $f'(2)=\frac{1}{\sqrt{2}}$

5-2 Multiple Choice

1. A 2. A 3. C 4. D 5. E

5-3 Free Response

1. $8x-3$

2. $28x^6-12x^5+24x^3+14x$

3. $6x^2+32x-3$

4. $3x^2+2x-3$

5. $f'(x)=2x+4$

6. $f'(x)=-2x^{-3}+16x^3$

7. $\frac{dy}{dx}=\frac{-4}{x^5}-\frac{12}{x^4}+\frac{5}{x^2}$

8. $\frac{dy}{dx}=\frac{3}{2}x^{1/2}-\frac{1}{2x^{3/2}}-\frac{3}{4x^{1/4}}$

9. $6x-12x^2-\frac{4}{x^3}+\frac{1}{3x^{4/3}}$

10. $56x^6-6x-\frac{7}{x^8}+\frac{9}{2x^{13/4}}$

11. $\frac{5}{3}\sqrt[3]{x^2}-\frac{27}{x^4}-\frac{14}{5}\sqrt[5]{x^2}-\pi$

5-3 Multiple Choice

1. C 2. B 3. A 4. C 5. B

5-4 Free Response

1. $y+21=75(x+2)$

2. $y-4=-\frac{21}{2}(x-1)$

3. $y=5$

4. $y+\frac{41}{6}=-3(x+1)$

5. Tangent: $y+52=37(x+3)$; Normal: $y+52=-\frac{1}{37}(x+3)$

6. Tangent: $y+10=11(x+1)$; Normal: $y+10=-\frac{1}{11}(x+1)$

7. Tangent: $y+4=2(x-0)$; Normal: $y+4=-\frac{1}{2}(x-0)$

8. Tangent: $y-4=23(x-1)$; Normal: $y-4=-\frac{1}{23}(x-1)$

9. $f(-1.9)\approx -13.5$

10. $g(-2.9)\approx -48.3$

11. $f(-0.1)\approx 5$

12. $f(-0.9)\approx -7\frac{2}{15}$

13. $(3, -4)$

14. $\left(\frac{1}{2}, \frac{1}{8}\right)$

15. $y+64=12(x-4)$ and $y-1=12(x+1)$

16. $y+14.933=x-2$ and $y-14.933=x+2$

5-4 Multiple Choice

1. A 2. D 3. C 4. A 5. A 6. C

5-5 Free Response

1a. $t=2$ and $t=5$
b. left
c. 49 units right of the origin
d. –6

2a. $t=2$
b. right
c. 14 units right of the origin
d. 6

3a. $t=-4$, $t=\frac{4}{3}$, and $t=3$
b. neither; it is stopped
c. 141 units above of the origin
d. 420

4a. $t=-3$, $t=-1$, $t=2$, and $t=3$
b. neither; it is stopped

c. 1431 units above of the origin d. 1440

5a. $t=0,\ t=\pm\sqrt{3}$ b. left

c. $y=-108$ d. $a(3)=-450$

6a. $t=0,\ t=\pm\sqrt{2.5}$ b. Right c. $y=0$ d. $a(3)=98$

7. $x(2.5)=-2.25$

8. $x(-3)=112;\ x(7)=-388$

9. $x(0)=12;\ x(2)=-4;\ x(0.894)=-19.090;\ x(-0.894)=23.890$

10. $x(3.5)=42.5;\ v(3.5)=-13.5$

11. $x(2)=13;\ v(2)=0$

12. $x(-2)=-1984;\ v(-2)=1200,\ x\left(\frac{20}{9}\right)=222.398;\ v\left(\frac{20}{9}\right)=-154.864$

13. $h(2)=89$ feet

14. $h(30)=4420$ meters

15. $t=7.473$ sec; $t=1.215$ sec

5-5 Multiple Choice

1. B 2. C 3. C 4. E 5. A

5-6 Free Response

1a. $\langle 2t-5,\ -8t\rangle$ b. $\langle 2,\ -8\rangle$ c. 16.031

2a. $\langle 3t^2-12t-63,\ 4\rangle$ b. $\langle 6t-12,\ 0\rangle$ c. 75.107

3a. $\langle 6t^2-42t+60,\ 2t\rangle$ b. $\langle 12t-42,\ 2\rangle$ c. 4

4a. $\langle 4t+6,\ 4t+6\rangle$ b. $\langle 4,\ 4\rangle$ c. 19.799

5a. $\langle 3t^2+5,\ 15t^2-5t^4\rangle$ b. $\langle 6t,\ 30t^2-20t^3\rangle$ c. 26.249

6a. $\left\langle 6t+\frac{3}{\sqrt{t}},\ 4t^3-10t\right\rangle$ b. $\left\langle 6-\frac{3}{3\sqrt{t^3}},\ 12t^2-10\right\rangle$ c. 18.531

7a. $\langle 16,\ -4.5\rangle$ b. $\langle 8,\ -1.5\rangle$ 8. $t=4$

9. $t=4$

<u>5-6 Multiple Choice</u>

1. B 2. D 3. E 4. A 5. C

<u>Polynomial Practice Test Answer Key</u>

<u>Multiple Choice</u>

1. C 2. D 3. C 4. A 5. C 6. E

<u>Free Response</u>

1. a) 2 b) $\frac{9}{8}$ c) $-\frac{4}{5}$

2. $(\pm 2,0)$ *and* $\left(\frac{1}{3},0\right)$

3. $(1.271,-6.708)$ *and* $(-1.049,12.025)$

4. $(\pm 2,0)$ *and* $(\pm 5,0)$

5. $(\pm 3.807,-110.250)$ *and* $(0,100)$

6. $\lim_{h\to 0} \frac{5(x+h)^7+2(x+h)^4-7\pi-5x^7-2x^4+7\pi}{h}$

7a) $\frac{dy}{dx}=42x^6-76x^3+6x-12$

7b) $\frac{7}{4}x^{3/4}+\frac{30}{x^6}-\frac{1}{3x^{2/3}}-1$

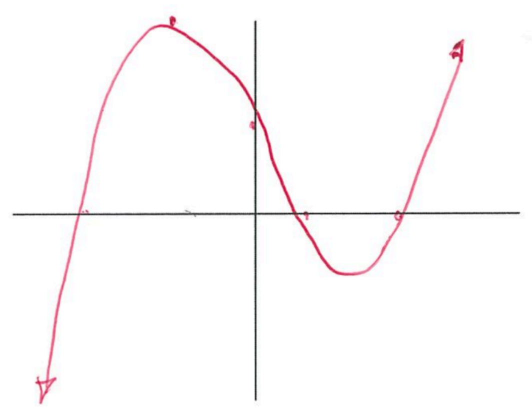
7c) $7x^6-\frac{7}{2x^{1/8}}+\frac{4}{7x^{11/7}}-\frac{1}{5x^2}$

8. Domain: $x \in$ All Reals Range: $y \in$ All Reals

Zeros: $(\pm 2,0)$ *and* $\left(\frac{1}{3},0\right)$ $Y-$Int: $(0,4)$

End Behavior (Left): Down End Behavior (Left): Up

Extreme Points: $(1.271,-6.708)$ *and* $(-1.049,12.025)$



9. Domain: $x \in$ All Reals Range: $y \in [-110.250,\infty)$

Zeros: $(\pm 2,0)$ *and* $(\pm 5,0)$ $Y-$Int: $(0,100)$

End Behavior (Left): Up End Behavior (Left): Up

Extreme Points: $(\pm 3.807, -110.250)\ and\ (0, 100)$