Chapter 6:

Polynomial Functions

Chapter 6 Overview: Polynomial Functions

In a previous chapter, limits and derivatives were introduced as the core operations of differential calculus. In this chapter, the main application of these operations—finding critical values, extreme values, and extreme points of a function—will be explored. In a previous chapter, maximum and minimum points of a curve were found graphically, but one of the main goals of calculus is finding these points algebraically. In this chapter, at the high and low points on a curve, the tangent line will be horizontal, that is, the slope of the tangent line will equal zero.

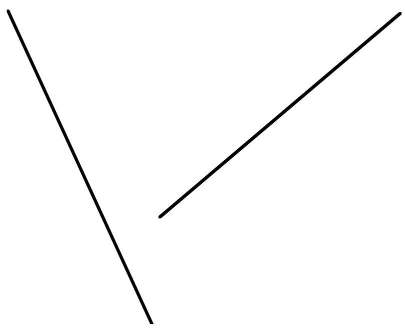
This chapter will also:

- revisit sign patterns in the context of the signs of the derivatives,
- apply derivatives to mathematical modeling, and
- put all the traits together to graph polynomial functions.

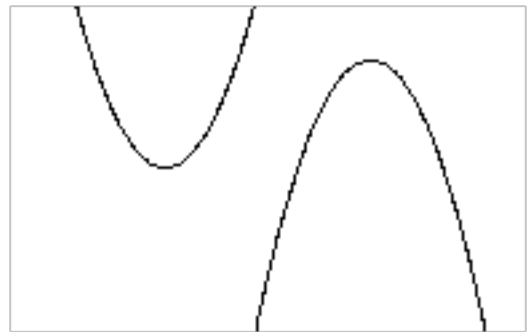
The traits of a polynomial function are:

- Domain
- *y*-intercept
- Zeros
- End Behavior
- Extreme Points
- Range

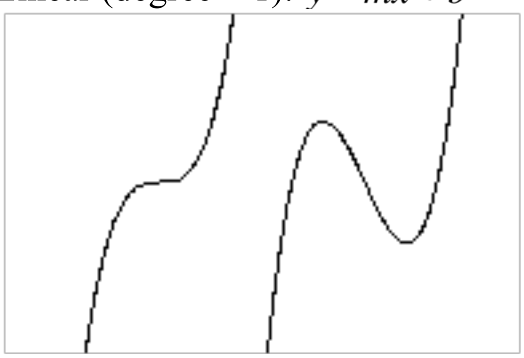
In terms of graphs of polynomials, remember that polynomials have the general equation $y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ and are named for—and the general shape of their graph is generally determined by—their degree.



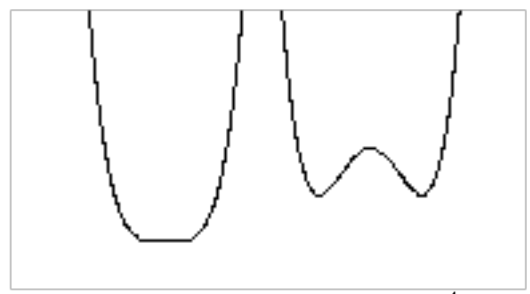
Linear (degree = 1): $y = mx + b$



Quadratic (degree = 2): $y = ax^2 + bx + c$



Cubic (degree = 3): $y = a_3 x^3 + a_2 x^2 + a_1 x + a_0$



Quartic (degree = 4): $y = a_4 x^4 + \ldots + a_0$

Quintic (degree = 5): $y = a_5 x^5 + \ldots + a_0$

etc.

6-1: Critical Values and Extreme Points of a Polynomial Function

Vocabulary:

1. ***Extreme Points*** – the collective word for maximum and minimum points
2. ***Maximum Value*** – the y-coordinate of a high point
3. ***Minimum Value*** – the y-coordinate of a low point
4. ***Relative Extreme Values*** – the highest or lowest y-values in any section of the curve
5. ***Absolute Extreme Values*** – the highest or lowest y-values of the entire function

Note: The word "value" indicates an x-value OR a y-value, not both. An extreme POINT, on the other hand, indicates a coordinate point, and must be written as such.

Definition: A **critical value** is where

i) $\frac{dy}{dx} = 0$ at that value;

ii) $\frac{dy}{dx}$ does not exist at that value;

or iii) an endpoint of a domain restriction.

Key Idea:

The x-coordinate of an extreme point must be a critical value.

Note that this does not mean all critical values are an x-coordinate of an extreme point

LEARNING OUTCOMES
Use the derivative to find the extreme points of a polynomial. Use the derivative to find the range of a polynomial.

EX 1 Find the critical values of $y = x^3 - 4x^2 - 35x + 2$.

$$\frac{dy}{dx} = 3x^2 - 8x - 35$$

i) $\frac{dy}{dx} = 0$: $\frac{dy}{dx} = 3x^2 - 8x - 35 = (3x+7)(x-5) = 0$

$x = 5 \text{ or } -\frac{7}{3}$

ii) $\frac{dy}{dx}$ dne: None because $\frac{dy}{dx}$ for a polynomial always exists

or iii) Endpoints of a domain restriction: None given

$$x = 5 \text{ or } -\frac{7}{3}$$

*Note that the problem did not ask for the y-values (i.e. the extreme values).

EX 2 Find the critical values algebraically of $y = 3x^4 + 2x^3 - 39x^2 + 36x - 4$ on $x \in [-2,6]$.

$$\frac{dy}{dx} = 12x^3 + 6x^2 - 78x + 36$$

i) $\frac{dy}{dx} = 0$: $\frac{dy}{dx} = 12x^3 + 6x^2 - 78x + 36 = 0$

$$\frac{dy}{dx} = 2x^3 + x^2 - 13x + 6 = (2x-1)(x-2)(x+3) = 0$$

$x = \frac{1}{2}$, 2, or -3, but –3 is not in the restricted domain.

ii) $\frac{dy}{dx}$ dne: None because $\frac{dy}{dx}$ for a polynomial always exists

or iii) Endpoints of a domain restriction: $x = 2$and 6

Therefore, $x = 2, \frac{1}{2}, -2$, and 6 are the critical values.

EX 3 Find the extreme values of $y = x^3 - 9x$.

$$\frac{dy}{dx} = 3x^2 - 9$$

i) $\frac{dy}{dx} = 0$: $\frac{dy}{dx} = 3x^2 - 9 = 0$

$$x^2 = 3 \rightarrow x = \pm\sqrt{3}$$

ii) $\frac{dy}{dx}$ dne: None because $\frac{dy}{dx}$ for a polynomial always exists

or iii) Endpoints of a domain restriction: None given

So, the critical values are $x = \pm\sqrt{3}$, but the question asked for the extreme values. We need to find the y-value for each critical value:

$$y\left(\sqrt{3}\right) = \left(\sqrt{3}\right)^3 - 9\sqrt{3} = -6\sqrt{3}$$
$$y\left(-\sqrt{3}\right) = \left(-\sqrt{3}\right)^3 - 9\left(-\sqrt{3}\right) = 6\sqrt{3}$$

$y = \pm 6\sqrt{3}$ are the extreme values. A quick sketch will show that the first is a relative minimum value and the second is a relative maximum value.

EX 4 Find the range of $y = x^4 - x^3 - 2x^2 + 1$.

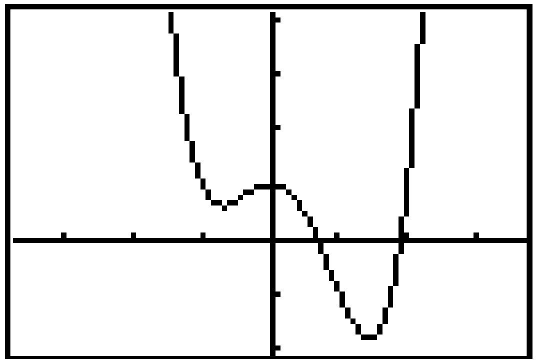
For an even degree function, finding the range requires finding the extreme values. Remember, with an odd degree function, the range is all real numbers.

$$\frac{dy}{dx} = 4x^3 - 3x^2 - 4x = 0$$
$$x\left(4x^2 - 3x - 4\right) = 0$$
$$x = 0 \text{ or } \frac{3 \pm \sqrt{3^2 - 4(4)(-4)}}{2(4)}$$
$$x = 0 \text{ or } \frac{3 \pm \sqrt{73}}{8} \approx -0.693 \text{ or } 1.443$$

$$y(0) = 1$$
$$y(-0.693) = 0.602$$
$$y(1.443) = -1.833$$

Therefore, the range is $y \in [-1.833, \infty)$.

This is seen in the graph of the function as well.



$y = x^4 - x^3 - 2x^2 + 1$

SUMMARY:

<u>Steps to Finding the Extreme Points of a Polynomial Function</u>

1. Find $\frac{dy}{dx}$ using the Power Rule.

2. Find the Critical Values:
   i. $\frac{dy}{dx} = 0 \rightarrow$ solve for x.
   ii. $\frac{dy}{dx}$ $dne \rightarrow$ not applicable to polynomials.
   iii. The x-coordinates of any domain restriction.

3. Find the y-coordinates for each of the critical values.

6-1: <u>Free Response Homework</u>

Find the critical values.

1. $y = x^2 + 4x - 3$

2. $y = x^3 + x^2 - 5x - 3$

3. $y = 2x^3 + x^2 - 4x + 8$

4. $y = x^3 + 6x^2 + 12x - 1$

5. $y = x^5 + 5x^4 - 5x^3 - 35x^2 - 40x - 1$

6. $y = 4x^3 - 21x^2 + 48x - 7$

7. $y = x^4 - 32x^2 + 12$

8. $y = 3x^5 - 40x^3 + 180x - 71$

9. $y = x^3 + x^2 - 5x - 3$ on $x \in [-3, 1]$

10. $y = 2x^3 + x^2 - 4x + 8$ on $x \in [0, 7]$

11. $y = -3x^4 + 24x^2 + 5$ on $x \in [-3, 5]$

12. $y = 2x^5 - 35x^3 + 135x - 29$ on $x \in [-2, 6]$

Find the extreme points.

13. $y = x^5 - 16x + 12$

14. $y = 2x^3 + 9x^2 - 168x$

15. $y = x^4 - 6x^2 - 12$ on $x \in [-1, 2]$

16. $y = 3x^3 + 2x^2 - 27x - 18$ on $x \in [-3, 3)$

Find the range.

17. $y = x^2 + 5x - 7$

18. $y = 3x^4 + 2x^3 + 12x^2 + 12x - 42$

19. $y = x^5 + 12x^4 - 4x^3 - x^2 - 8x - 1$

6-1: Multiple Choice Homework

1. Give the value of x where the function $f(x)=x^3-9x^2+24x+4$ has a relative maximum point.

a) 4 b) –2 c) 2 d) –4 e) 3

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2. Give the value of x where the function $f(x)=x^3-\frac{33}{2}x^2+84x-2$ has a relative minimum point.

a) –4 b) –7 c) 4 d) 5 e) 7

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3. Give the approximate location of a relative maximum point for the function $f(x)=3x^3+5x^2-3x$.

a) $(-1.357,5.779)$ b) $(0.2457,-0.3908)$ c) $(-1.357,5.713)$

d) $(0.2457,-0.3216)$ e) $(-1.357,-0.3908)$

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4. A ball dropped from the top of a building has a height of $s(t)=400-16t^2$ meters after t seconds. How long does it take the ball to reach the ground? What is the ball's velocity at the moment of impact?

a) 25 sec, –800 m/sec b) 5 sec, 160 m/sec

c) 5 sec, –160 m/sec d) 10 sec, –80 m/sec

e) None of these

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5. Find the equation of the tangent line to $y = \frac{x^3}{2}$ at $x = 8$.

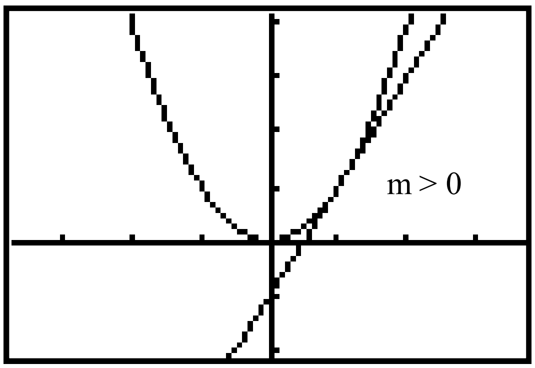
a) $y = 512x + 96$ b) $y = 32x + 512$ c) $y = 32x - 512$

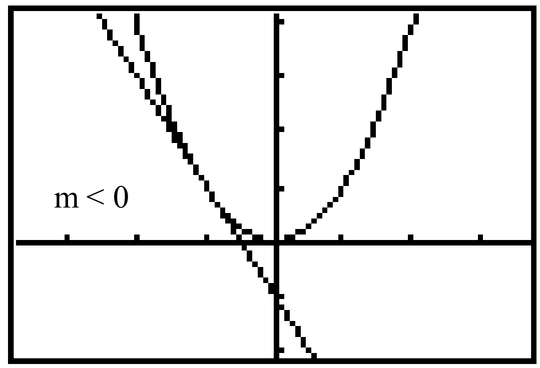
d) $y = 96x - 512$ e) None of these

6-2: Interpreting the Sign Pattern of the First Derivative

In a previous section, the difference between a relative maximum and a relative minimum was distinguished by graphing. There is an algebraic way to do this, which relies on sign patterns.

The derivative represents the slope of the tangent line. If the slope of the tangent line is positive, the curve must be going up (as viewed from left to right). Similarly, if the slope of the tangent line is negative, the curve is going down from left to right.





The point at which the sign of the slope changes is the value of x where $\frac{dy}{dx}=0$, or the critical value. Which way the sign changes determines whether the critical value represents the x-value of a maximum point or minimum point.

Vocabulary:
1. ***Interval of Increasing*** – the interval of x-values for which the curve is rising from left to right
2. ***Interval of Decreasing*** – the interval of x-values for which the curve is dropping from left to right

EX 1 Given this sign pattern for the derivative of $F(x)$, find the interval(s) of increasing.

$$\begin{array}{cccccc} F'(x) & + & 0 & - & 0 & + \\ \hline x & & -2 & & 2 & \end{array}$$

The intervals of increasing are where the sign of the derivative is positive. Therefore,

$$x \in (-\infty,\ -2) \cup (2,\ \infty)$$

EX 2 Find the interval(s) of decreasing of $y = 3x^4 - 4x^3 - 12x^2$.

$$\frac{dy}{dx} = 12x^3 - 12x^2 - 24x = 0$$
$$= 12x(x-2)(x+1) = 0$$

c.v.s: $x = 0,\ 2,\ -1$

$$\begin{array}{c|ccccccc} y' & - & 0 & + & 0 & - & 0 & + \\ x & & -1 & & 0 & & 2 & \end{array}$$

y is decreasing on $x \in (-\infty,\ -1) \cup (0,\ 2)$ because $\frac{dy}{dx} < 0$.

Remember from the previous Chapter:

Definition: A **critical value** is where

i) $\frac{dy}{dx} = 0$ at that value;

ii) $\frac{dy}{dx}$ does not exist at that value;

or iii) an endpoint of a domain restriction.

Key Idea:

The *x*-coordinate of an extreme point must be a critical value, but <u>not all critical values are an *x*-coordinate of an extreme point</u>.

So how does one know if a critical value is the x-value of an extreme or not?

The First Derivative Test*

As the sign pattern of the 1st Derivative is viewed left to right, the critical value represents the x-value of a

- relative maximum point if the sign of $\frac{dy}{dx}$ changes from + to –
- relative minimum point if the sign of $\frac{dy}{dx}$ changes from – to +
- non-extreme if the sign does not change.

LEARNING OUTCOMES

Use sign patterns to determine the intervals where a function is increasing or decreasing.
Use the First Derivative Test to identify the type of extreme point represented by a particular critical value.

EX 3 Determine at which values of x will $y = x^3 - 4x^2 - 35x + 2$ be at a maximum point and at which values of x will there be a minimum point. Justify your answer.

$$\frac{dy}{dx} = 3x^2 - 8x - 35 = 0$$
$$= (3x+7)(x-5) = 0$$
$$\text{c.v.s: } x = 5,\ -\frac{7}{3}$$

$$\begin{array}{l|ccccc} \frac{dy}{dx} & + & 0 & - & 0 & + \\ \hline x & & -\frac{7}{3} & & 5 & \end{array}$$

At $x = -\frac{7}{3}$, the sign of the first derivative changes from + to –, therefore $x = -\frac{7}{3}$ is at a maximum point. At $x = 5$, the sign of the first derivative changes from – to +, therefore $x = 5$ is at a minimum point.

NB. The sign pattern is not sufficient justification. A sentence explaining the sign change is the actual answer.

EX 4 Determine at which values of *x* will $y = x^4 - 2x^3$ be at a maximum point and at which values of *x* will there be a minimum point.

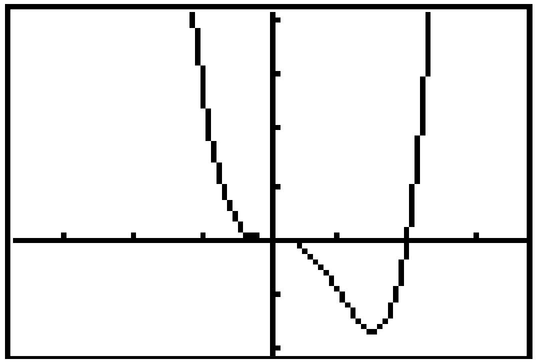
$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 6x^2 = 0 \\ &= 2x^2(2x-3) = 0 \\ x &= 0,\ \frac{3}{2} \end{aligned}$$

$$\begin{array}{c|ccccc} y' & - & 0 & - & 0 & + \\ \hline x & & 0 & & \frac{3}{2} & \end{array}$$

$x = 0$ is not an *x*-value for an extreme point because the signs do not change on either side of $x = 0$.

$x = \frac{3}{2}$ is the critical value for a minimum point because the signs change from – to +.

The solution is consistent with the graph of the function:



$y = x^4 - 2x^3$

Process for the First Derivative Test:

1. Differentiate the equation.
2. Find the critical values.
3. Sketch a sign pattern **without** any endpoints of a given domain.
4. Add the endpoints to the sign pattern
5. Determine which critical values are at maximum points vs. minimum points from the sign change.

The First Derivative Test is clear when the critical values came from the derivative equals zero, but what about at the end points? There are no signs beyond the endpoints, so we cannot talk about a sign change.

The First Derivative Test Corollary for Endpoints

The Left Endpoint is at a:

i) maximum if the critical value is followed by a – on the $\frac{dy}{dx}$ sign pattern

ii) minimum if the critical value is followed by a + on the $\frac{dy}{dx}$ sign pattern

The Right Endpoint is at a:

i) minimum if the critical value is preceded by a – on the $\frac{dy}{dx}$ sign pattern

ii) maximum if the critical value is preceded by a + on the $\frac{dy}{dx}$ sign pattern

EX 5 Determine the *x*-coordinates of the maximums for:

$$y = 3x^4 + 2x^3 - 39x^2 + 36x - 4 \text{ on } x \in [-2,\ 6].$$

In the last section, we found that $\frac{dy}{dx} = 0$ led to $x = \frac{1}{2}$, 2, or -3 , but –3 was not in the stated domain and that $x = -2$ and 6 are the endpoints of the stated domain.

Therefore, $x = 2,\ \frac{1}{2},\ -2,$ and 6 are the critical values.

$$\begin{array}{ccccccccc} y' & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & 1/2 & & 2 & \end{array}$$

Adding the domain to the sign pattern gives us this:

$$
\begin{array}{c|ccccc|}
y' & + & 0 & - & 0 & + \\
x & & \tfrac{1}{2} & & 2 & \\
\multicolumn{1}{c}{-2} & & & & & \multicolumn{1}{c}{6}
\end{array}
$$

The maximums are at $x = \dfrac{1}{2}$ because the signs of the first derivative change from + to $-$, and at $x = 6$ because it is the right endpoint and is preceded by a + on the derivative sign pattern.

What if the domain of Ex 5 were different?

EX 5 again Determine the x-coordinates of the maximums for:

$$y = 3x^4 + 2x^3 - 39x^2 + 36x - 4 \text{ on } x \in [-4, 1].$$

The work would remain the same and would lead to the same sign pattern

$$
\begin{array}{ccccccccc}
y' & - & 0 & + & 0 & - & 0 & + \\
x & & -3 & & \tfrac{1}{2} & & 2 &
\end{array}
$$

Adding the domain to the sign pattern gives us this:

$$
\begin{array}{c|ccccc|}
y' & - & 0 & + & 0 & - \\
x & & -3 & & \tfrac{1}{2} & \\
\multicolumn{1}{c}{-4} & & & & & \multicolumn{1}{c}{1}
\end{array}
$$

Noe maximum is still at $x = \dfrac{1}{2}$ because the signs of the first derivative change from + to $-$. The other maximum is at $x = -4$ because it is the left endpoint and is followed by a $-$ on the derivative sign pattern.

6-2 Free Response Homework

1. Given this sign pattern for the derivative of $F(x)$, what are the intervals of increasing?

$$\begin{array}{c|ccccccc} F'(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & 0 & & 3 & \end{array}$$

2. Given this sign pattern for the derivative of $G(x)$, what are the intervals of decreasing?

$$\begin{array}{c|ccccccc} G'(x) & + & 0 & + & 0 & - & 0 & - \\ \hline x & & -3 & & \frac{2}{3} & & 2 & \end{array}$$

3. The sign pattern for the derivative of $H(x)$ is given. a) Is $x=-4$ at a maximum point, a minimum point, or neither? b) Is $x=-1$ at a maximum point, a minimum point, or neither?

$$\begin{array}{c|ccccccc} dH/dx & + & 0 & - & 0 & - & 0 & + \\ \hline x & & -4 & & -1 & & 2 & \end{array}$$

4. Given this sign pattern for the derivative of $F(x)$, for what x-values does $F(x)$ attain a minimum value?

$$\begin{array}{c|ccccccc} F'(x) & - & 0 & + & 0 & - & 0 & - \\ \hline x & & -2 & & 0 & & 3 & \end{array}$$

5. Given this sign pattern for the derivative of $G(x)$, what are the intervals of decreasing?

$$\begin{array}{c|ccccccc} G'(x) & - & 0 & - & 0 & + & 0 & + \\ \hline x & & -3 & & 2/3 & & 2 & \end{array}$$

6. The sign pattern for the derivative of $f(x)$ is given. a) Is $x = 3$ at a maximum point, a minimum point, or neither? b) Is $x = -2$ at a maximum point, a minimum point, or neither?

$$\begin{array}{c|ccccccc} f'(x) & - & 0 & + & 0 & - & 0 & - \\ \hline x & & -2 & & 0 & & 3 & \end{array}$$

Find the interval(s) of increasing.

7. $y = x^3 - 4x^2 + x + 6$

8. $y = x^3 - x^2 - 4x + 4$

9. $y = x^3 - 7x^2 + 11x + 3$

10. $y = 3x^3 - x^2 - 40x + 48$

Find the interval(s) of decreasing.

11. $y = -2x^3 + 2x^2 + 7x - 10$

12. $y = x^4 + 4x^3 - 12x^2 - 32x$

13. $y = \frac{1}{4}x^4 - \frac{7}{3}x^3 + \frac{11}{2}x^2 + 3x - 2$

14. $y = \frac{1}{5}x^5 + x^4 - x^3 - 7x^2 - 8x - 5$

Determine if the critical values are at a maximum point, a minimum point, or neither.

15. $y = x^3 - 4x^2 + x + 6$

16. $y = x^3 - x^2 - 4x + 4$

17. $y = x^3 - 7x^2 + 11x + 3$

18. $y = 3x^3 - x^2 - 40x + 48$

19. $y = -2x^3 + 2x^2 + 7x - 10$

20. $y = x^4 + 4x^3 - 12x^2 - 32x$

21. $y = \frac{1}{4}x^4 - \frac{7}{3}x^3 + \frac{11}{2}x^2 + 3x - 2$

22. $y = \frac{1}{5}x^5 + x^4 - x^3 - 7x^2 - 8x - 5$

23. For $t \geq 0$, a particle is moving along the x-axis such that its position is described by $x(t) = 2t^3 - 21t^2 + 60t + 4$.

a) What is the minimum velocity?
b) What is the maximum acceleration?

24. A particle is moving along the x-axis such that its position is described by $x(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48$.

a) What is the relative maximum velocity?
b) What is the maximum acceleration?

25. $y = x^3 + x^2 - 5x - 3$ on $x \in [-3, 1]$

26. $y = 2x^3 + x^2 - 4x + 8$ on $x \in [0, 7]$

27. $y = -3x^4 + 24x^2 + 5$ on $x \in [-3, 5]$

28. $y = 2x^5 - 35x^3 + 135x - 29$ on $x \in [-2, 6]$

29. $y = x^4 - 6x^2 - 12$ on $x \in [-1, 2]$

30. $y = 3x^3 + 2x^2 - 27x - 18$ on $x \in [-3, 3)$

31. $y = x^4 - 4x^3$ on $x \in [-1, 2]$

6-2: Multiple Choice Homework

1. The derivative of a function is given by $h'(w) = 2w^2(w+2)(w+1)(w-3)$. Find the interval(s) where h is increasing.

a) $w \in (-2, -1) \cup (0, 3)$

b) $w \in (-2, -1) \cup (3, \infty)$

c) $w \in (-1, 3)$

d) $w \in (-1, 0) \cup (2, 3)$

e) $w \in (-\infty, -2) \cup (-1, 0) \cup (3, \infty)$

2. Given this sign pattern $\begin{array}{c|ccccccc} f'(x) & - & 0 & + & 0 & - & 0 & - \\ \hline x & & -4 & & -1 & & 2 & \end{array}$, at what value of x does f has a relative minimum point?

a) –4 b) –1 c) 2 d) 1 e) no value

3. Which of the following sign patterns apply to the equation $f(x) = x^4 - 2x^2 - 8$?

I. $\begin{array}{c|ccccccccc} y & + & 0 & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -2 & & -\sqrt{2} & & \sqrt{2} & & 2 & \end{array}$

II. $\begin{array}{c|ccccccc} f'(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -2 & & 0 & & 2 & \end{array}$

III. $\begin{array}{c|ccccc} f(x) & + & 0 & - & 0 & + \\ \hline x & & -2 & & 2 & \end{array}$

a) None of these b) I only c) III only

d) I and III only e) I, II, and III

4. Given this sign pattern $\begin{array}{c|ccccc} f'(x) & - & 0 & + & 0 & - \\ \hline x & & -2 & & 2 & \end{array}$, which of the following might be the sign pattern of $f(x)$?

a) $\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & + & 0 & - \\ \hline x & & -3 & & 0 & & 3 & \end{array}$

b) $\begin{array}{c|ccccccc} f(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & 0 & & 3 & \end{array}$

c) $\begin{array}{c|ccccccc} f(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -2 & & 0 & & 2 & \end{array}$

d) $\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & + & 0 & - \\ \hline x & & -2 & & 0 & & 2 & \end{array}$

e) $\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & - & 0 & + \\ \hline x & & -2 & & 0 & & 2 & \end{array}$

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5. What are all values of x for which the function $f(x) = 2 - 3x - x^2 + \frac{1}{3}x^3$ is decreasing?

a) $-1 < x < 3$

b) $-3 < x < 1$

c) $x < -3$ or $x > 1$

d) $x < -1$ or $x > 3$

e) All real numbers

6-3: Optimization Problems

LEARNING OUTCOME

Solve maximum and minimum polynomial word problems.

In a previous chapter, optimization problems were examined and solved using the graphing calculator.

REMEMBER:

Common Formulas for Optimization Problems:

Pythagorean Theorem

$$x^2 + y^2 = r^2$$

Area Formulas

Circle: $A = \pi r^2$ Rectangle: $A = lw$ Trapezoid: $A = \frac{1}{2}h\left(b_1 + b_2\right)$

Volume Formulas

Sphere: $V = \frac{4}{3}\pi r^3$ Right Prism*: $V = Bh$

Cylinder: $V = \pi r^2 h$ Cone: $V = \frac{1}{3}\pi r^2 h$ Right Pyramid*: $V = \frac{1}{3}Bh$

Surface Area Formulas

Sphere: $S = 4\pi r^2$ Cylinder: $S = 2\pi r^2 + 2\pi rh$

Cone: $S = \pi r^2 + \pi rl$ Right Prism*: $S = 2B + Ph$

REMEMBER: Process for Approaching Optimization Problems:

1. Determine the "primary" equation—the one you want to put in the calculator.
    a. Pay attention to superlatives ("-est" words).
2. If there are more than two variables in the primary equation that will be going into the calculator.
    2a. Draw a picture if one is needed and not provided.
    2b. Label the picture.
        i. Do not use x or y as variables in the diagram.*
    2c. Set up a secondary equation.
    2d. Isolate the variable you want to eliminate from the primary equation.
3. Reduce the number of variables in the primary equation to two with auxiliary work, by substituting the rearranged secondary equation into the primary equation.
4. Check the domain implied in the problem to determine the window to use.
5. Use Math 3 or Math 4 to determine the minimum or maximum.
6. Reread the question and answer the question asked. (Did they ask for the maximum or the dimensions that yield the maximum.)

Now, these same problems will now be explored algebraically, based on what was learned in the last two sections:

REMEMBER: Critical values of a polynomial occur when

i) $\frac{dy}{dx} = 0$ at that value;

ii) $\frac{dy}{dx}$ does not exist at that value;

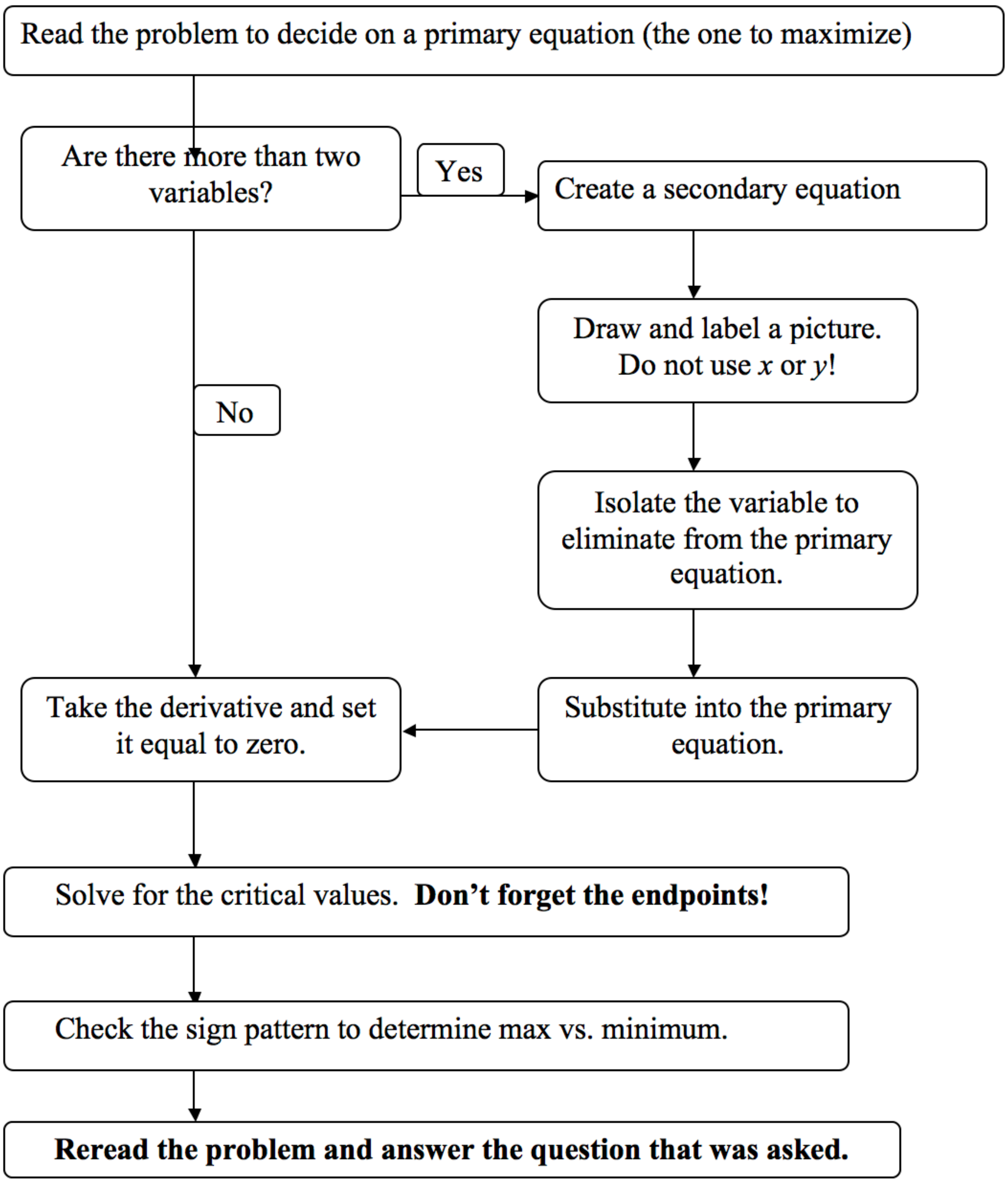
or iii) a value at an endpoint of a domain restriction.

The situation of the problem will determine the polynomial and the solution will be the critical value, extreme value, or the coordinates of the extreme point, depending on how the problem is worded.

Process for Approaching Optimization Problems:

1. Determine the "primary" equation—the one you want to put in the calculator.
    a. Pay attention to superlatives ("-est" words).
2. If there are more than two variables in the primary equation that will be going into the calculator.
    2a. Draw a picture if one is needed and not provided.
    2b. Label the picture.
        i. Do not use *x* or *y* as variables in the diagram.
    2c. Set up a secondary equation.
    2d. Isolate the variable you want to eliminate from the primary equation.
3. Reduce the number of variables in the primary equation to two with auxiliary work, by substituting the rearranged secondary equation into the primary equation.
4. Check the domain implied in the problem to determine the window to use.
5. Find the critical values.
6. Sketch a sign pattern to determine the critical values of maximum points vs. minimum points.
7. Don't forget the context of the situation. Check for a restricted domain.
8. Reread the question and answer the question asked. (Did they ask for the maximum or the dimensions that yield the maximum.)

<u>Strategy for Approaching Optimization Problems with Calculus</u>



EX 1 A water balloon is thrown upwards from a 400' tall building at 48 ft/sec. How high will it go?

Recall from a previous chapter: $h=-\frac{1}{2}at^2+v_0t+h_0$

So the equation for this projectile motion is $h=-16t^2+48t+400$.

$$\frac{dh}{dt}=-32t+48=0$$

$$t=1.5$$

The sign pattern show that this critical value is at a maximum point:

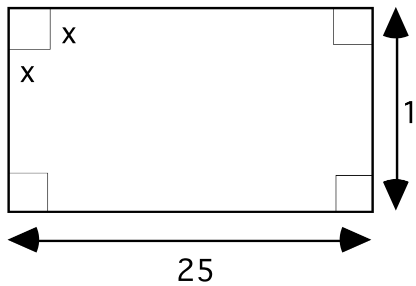
$$\begin{array}{c|ccc} \frac{dh}{dt} & + & 0 & - \\ x & & 1.5 & \end{array}$$

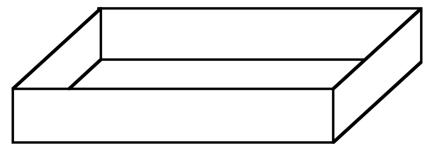
and

$$h(1.5)=436 \text{ feet}$$

The water balloon will reach a maximum height of 436 feet.

EX 2 Find the maximum volume of a box made by cutting squares from the four corners of a 25 in. by 14 in. piece of cardboard and folding the edges up.





$$V = l \bullet w \bullet h = (25-2x)(14-2x)x$$
$$= 4x^3 - 78x^2 + 350x$$

There is an implied domain here. Less than 0 or more than 7 inches cannot be cut from the cardboard, so $x \in [0,\ 7]$. But $v(0)$ and $v(7)$ both are 0. These are the critical values that yield the minimum volume.

$$\frac{dV}{dx} = 12x^2 - 156x + 350 = 0$$

The quadratic formula yields $x = 10.117$ or 2.883 in

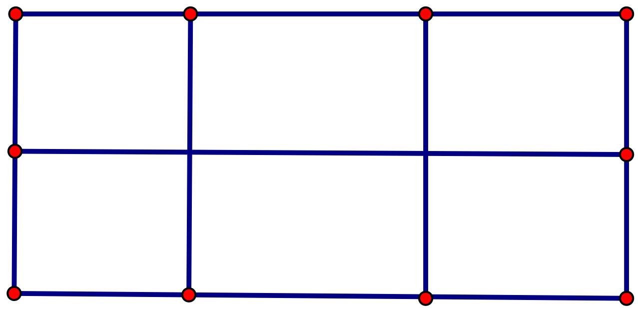
$$\text{c.v.s}: x = 0,\ 2.883,\ 7$$

$\frac{dV}{dx}$	End	+	0	−	End
x	0		2.883		7

$x = 2.883$ yields the maximum volume:

$$V = (25 - 2(2.883))(14 - 2(2.883))(2.883) = 456.589 \text{ in}^3$$

EX 3 A 4000 square-foot field is surrounded and divided into six equal parts by a fence.



Find the minimum amount of fencing.

Consider the large rectangle rather than the six smaller ones. The larger rectangle has three horizontal runs of fence and four verticals. Therefore, the total anount of fence f is

$$f = 3l + 4w$$

$$\text{Area} = l \bullet w = 4000 \to l = \frac{4000}{w}$$

$$f = 3\left(\frac{4000}{w}\right) + 4w = 12000w^{-1} + 4w$$

$$f' = -12000w^{-2} + 4 = 0$$

$$4 = \frac{12000}{w^2}$$

$$w^2 = 3000$$

$$w = \sqrt{3000} = 54.772$$

$$f = 3\left(\frac{4000}{\sqrt{3000}}\right) + 4\sqrt{3000} = \frac{24000}{\sqrt{3000}} = 438.178\,ft$$

EX 4 The volume of a cylindrical cola can is 32π in^3. What is its minimum surface area for such a can?

The problem asks to minimize the surface area, which is determined by:

$$S = 2\pi r^2 + 2\pi rh$$

Either *r or h* need to be eliminated in this formula before differentiating. The volume is $V = \pi r^2 h = 32\pi$, so $h = \dfrac{32}{r^2}$ and

$$S = 2\pi r^2 + 2\pi r\left(\frac{32}{r^2}\right) = 2\pi r^2 + \left(\frac{64\pi}{r}\right)$$

$$S' = 4\pi r - \left(\frac{64\pi}{r^2}\right) = 0$$

$$4\pi r = \frac{64\pi}{r^2}$$

$$r^3 = 16$$

$$r = 2.5198$$

$$\begin{array}{cccc} S' & - & 0 & + \\ r & \longleftarrow & 2.520 & \longrightarrow \end{array}$$

$$S(2.5198...) = 2\pi(2.5198...)^2 + \left(\frac{64\pi}{(2.5198...)}\right)$$

$$= 119.687 \text{ in}^2$$

The minimum surface area of the cola can is 119.687 in^2.

6-3 Free Response Homework

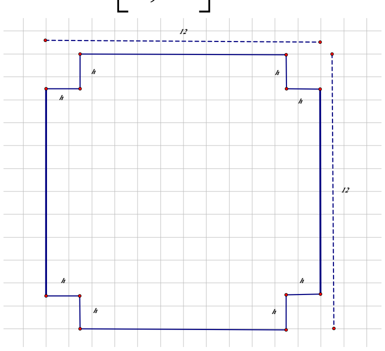
Solve these problems algebraically.

1. Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$. a) Find the average cost per machine of producing the first 100 washing machines. b) Find $\frac{dc}{dx}$ (the marginal cost) when 100 washing machines are produced. c) How many machines would result in the maximum cost?

2. The total profit, P, in thousands of dollars, from the sale of x thousand units of a new prescription drug is given by $P = -x^3 + 3x^2 + 72x$ for $x \in [0, 10]$.
   a) Find the number of units that should be sold to maximize profit.
   b) What is the maximum profit?
   c) For what portion of the domain is the profit decreasing?

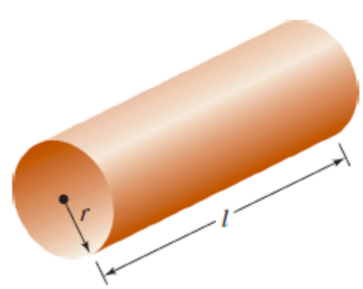
3. Suppose that a manufacturer sells cutlery with the following profit function where x is the number of sets sold, $P(x) = \frac{-x^2}{1000} + 24x - 4000$ on $x \in [0, 17000]$. What is the maximum profit?

4. During a four-week long flu epidemic, the number of people, $P(t)$, infected t days after the epidemic starts is given by the function $P(t) = t^3 - 54t^2 + 672t + 1568$ on $t \in [0, 28]$. At what time will the number of people infected be at a maximum?



5. An open-top box is to be made by cutting small congruent squares from the corners of a 12-by-12 inch sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

6. Postal regulations specify that a parcel sent by parcel post may have a combined length and girth of no more than 108 inches. Find the dimensions of the cylindrical package of greatest volume that may be sent through the mail. What is



the volume of such a package? (Hint: The length plus the girth $= 2\pi r + l$.)

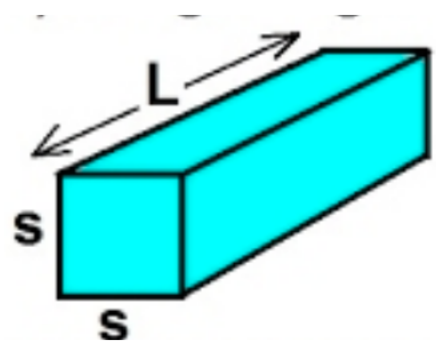
7. A 45-caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t seconds. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ feet after t seconds. How long will the bullet be aloft in each case? How high would the bullet go?

8. What is the smallest perimeter possible for a rectangle whose area is 16 cm^2?

9. Find the number between 0 and 1 whose difference between itself and its cube is maximized.

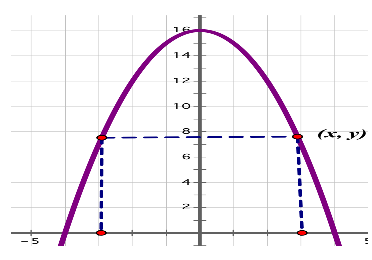
10. A cylindrical can has a volume of 16π in^3. (a). What radius would minimize the surface area? (b) If the top and bottom of the can cost 6 cents/square inch and the sides cost 3 cents/square inch, what radius would minimize the cost?

11. Find the dimensions of a rectangular field with one side against a cliff which can be surrounded by 160' of fence and which has the maximum area.



12. The US Postal Service has a size limit on packages. The length + girth of the box must not exceed 108 inches. If the cross-section of the package is a square, what is the maximum volume?

13. Find the maximum area of a rectangle inscribed between the x-axis and the parabola $y = 16 - x^2$.



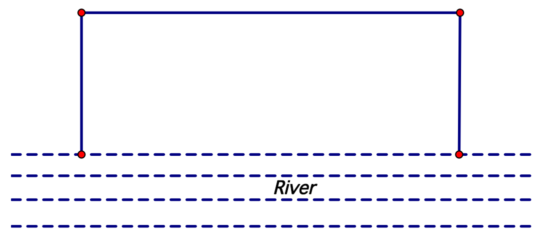
14. Find the maximum area of a rectangle inscribed between the x-axis and the parabola $y = 24 - \frac{1}{2}x^2$.

15. Sensitivity to a specific medicine is described by $R = M^2\left(15 - \frac{M}{3}\right)$, where R is the temperature reaction and M is the amount of medicine. How much medicine M would cause the highest reaction R?

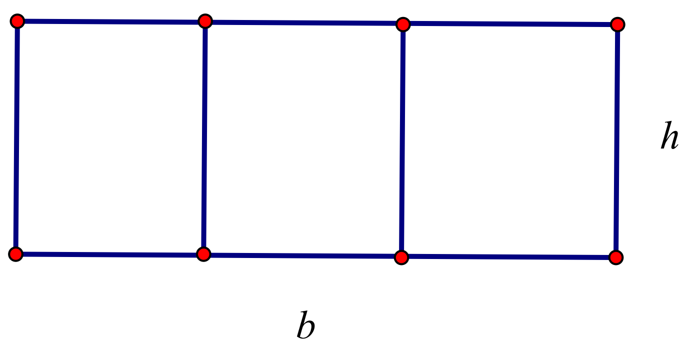
16. A person's sensitivity to medicine can be generalized to $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$, where R is the temperature reaction, M is the amount of medicine and C is a positive constant. How much medicine M would cause the highest reaction R?

17. Suppose you have 120 in^2 to make the surface area of a cylindrical soda can $\left[SA = 2\pi r^2 + 2\pi rh\right]$. What radius would give the largest volume? What is that largest volume?

18. The owner of the Rancho Grande has 3000 yards of fencing material with which to enclose a rectangular piece of grazing land along the straight portion of a river. If fencing is not required along the river, what are the dimensions of the largest area he can enclose? What is the area?

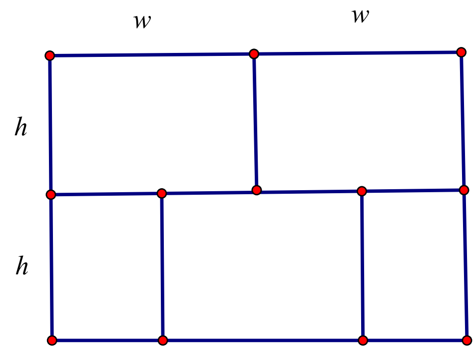


19. You want to enclose a rectangular area of 1200 square meters. You need the area fenced in. You also need to divide the area into three separate regions, also using your fencing material. Determine the minimum amount of fencing material required.



20. Given the situation in problem 17, if the material you use for the outside perimeter costs \$1.50 per linear foot and the inner fencing material costs \$0.90 per linear foot, find the minimum cost.

21. A field with an area equal to 1600 square yards in partitioned as the diagram below.



What is the minimum amount of fencing required for the portioning?

22. A farmer with 1400 feet of fencing is wants to enclose a rectangular area and then divide it into five pens with fencing parallel to the **horizontal** side of the rectangle. What is the largest possible total area of the five pens?

23. At SI, there was an outbreak of the norovirus. The number of members of the school community as well as the numbers of siblings and members of other infected people outside of SI is given by the function $N(t) = \frac{1}{6}t^3 - 20t^2 + 580t + 1$ on $t \in [0,48]$, where N is the number of people infected, and t is the number of hours since the outbreak started.

a) Find all times at which you have a maximum or minimum number of people infected.
b) Find the absolute maximum number of people infected.
c) Find the interval of time when the number of people infected with the norovirus is increasing.

<u>6-3 Multiple Choice Homework</u>

1. Find two positive numbers whose product is a maximum but whose sum is 144.

a) 70, 74 b) 72, 72 c) –16, 160 d) 0, 144 e) 44, 100

---

2. A farmer with 890 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

a) 19,825.5 ft^2 b) 19,802.5 ft^2 c) 19,801.5 ft^2

d) 19,902.5 ft^2 e) 19,791.5 ft^2

---

3. A piece of wire 10 meters long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut for the square so that the total area enclosed is a minimum? Round to the nearest hundredth. (Area of an equilateral triangle: $A = \frac{\sqrt{3}}{4}s^2$, where s is the length of a side of the triangle)

a) 5.35 meters b) 4.40 meters c) 4.35 meters

d) 0 meters e) 3.25 meters

---

4. Find all critical values of $g(x) = x^3 - 3x^2 + 1$.

a) $x = 1$ b) $x = -1,\ 1,\ 3$ c) $x = 0,\ 2$

d) $x = 2$ e) $x = -1,\ 0,\ 1,\ 2,\ 3$

---

5. Find the <u>locations</u> of the **absolute** extreme values of $g(x) = x^3 - 3x^2 + 1$ on the interval $x \in [-1,\ 4]$.

a) Absolute maximum at $x = 0$, Absolute minimum at $x = 2$
b) No absolute maximum, Absolute minimum at $x = 1$
c) Absolute maximum at $x = 4,\ 0$, Absolute minimum at $x = -1,\ 2$
d) Absolute maximum at $x = 4$, Absolute minimum at $x = 0,\ 2$
e) Absolute maximum at $x = 4$, Absolute minimum at $x = 2,\ -1$

6-4: General Polynomial Sketching

REMEMBER: Polynomial Traits

1. Domain—the set of x-values with give a real number for y.
2. y-intercept—the point where $x = 0$.
3. Zeros—the points where $y = 0$.
4. End Behavior—what the curve does as x approaches $\pm\infty$.
5. Extreme Points
6. Range—the set of values of y on the curve.

LEARNING OUTCOME

Find all the traits and sketch a fairly accurate polynomial curve algebraically.

1. Domain: In general, the domain for a polynomial is All Real Numbers. That is, $x \in (-\infty, \infty)$.

There are only three things that change the domain of a function from all real numbers:

i. Dividing by zero,
ii. Taking an even root of a negative, or
iii. Taking the log of a non-positive number.

Since polynomial functions do not have denominators, roots or logs, the domain of any polynomial will be all real numbers, *unless an arbitrary domain is assigned in the problem.*

2. y-intercept: $x = 0$. Do not forget to list this as a point.

3. Zeros: See previous chapters. Again, do not forget to list these as points.

4. End Behavior: As with sign patterns, examining the coefficient of the degree term determines whether the right end goes up or down.

- Right end behavior is determined by the sign of the coefficient of the degree term.
- Left end behavior:
  - Same as right end behavior if the degree is even.
  - Opposite of the right end behavior if the degree is odd.

5. Extreme Points:

If a value is a critical value, then either

i) $\frac{dy}{dx} = 0$ at that value;

ii) $\frac{dy}{dx}$ does not exist at that value;

or iii) a value at an endpoint of any domain restriction.

6. Range:

i. If the degree is odd, $y \in (-\infty, \infty)$

ii. If the degree is even, either $y \in (-\infty, \text{ max value}]$ or $y \in [\text{min value}, \infty)$, depending on the if the degree term's coefficient is positive or negative, respectively.

NB. The range gets more complicated with more complex functions. It is best to sketch the graph first and analyze the range visually in order to determine the numbers involved.

EX 1 Find all the traits and sketch $y = x^3 - x^2 - 4x + 4$.

The traits of a polynomial curve are:

1. Domain: $x \in (-\infty, \infty)$

2. y-intercept: $x = 0$ gives $y = 4$, so $(0, 4)$

3. Zeros:
$$y = x^3 - x^2 - 4x + 4$$
$$y = (x-2)(x+2)(x-1)$$
$$x = 2, -2, \text{ or } 1$$
$$(2, 0), (-2, 0), (1, 0)$$

4. End Behavior:
The right end goes up because x^3 is positive.
The left end goes down. (Since the degree is odd, the left does the opposite from the right.)

5. Extreme Points: $\frac{dy}{dx} = 3x^2 - 2x - 4 = 0$

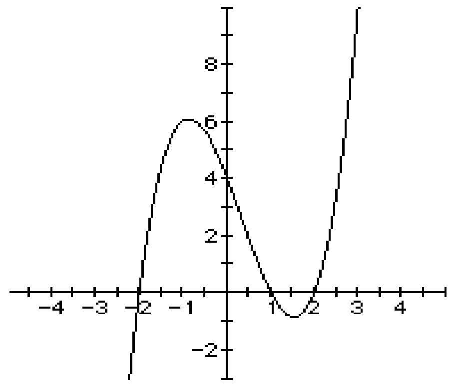
c.v.s: $x = \dfrac{2 \pm \sqrt{4 - 4(3)(-4)}}{2(3)} = 1.535 \text{ or } -0.869$

e.v.s: $y = 6.065 \text{ or } -0.879$

Rel. max: $(-0.869, 6.065)$ Rel. min: $(1.535, -0.879)$

6. Range: $y \in (-\infty, \infty)$ (the polynomial is an odd degree)

Sketch:



$y = x^3 - x^2 - 4x + 4$

EX 2 Find all the traits and sketch $y = x^4 - 8x^2$.

1. Domain: $x \in (-\infty, \infty)$

2. y-int: $x = 0$ gives $y = 0$, so $(0, 0)$

3. Zeros:
$$y = x^4 - 8x^2$$
$$y = x^2(x^2 - 8)$$
$$x = 2\sqrt{2}, -2\sqrt{2}, \text{ or } 0$$
$$(2\sqrt{2}, 0), (-2\sqrt{2}, 0), (0, 0)$$

4. End Behavior:
The right end goes up because x^4 is positive.
The left end also goes up. (Since the degree is even, the left does the same as the right.)

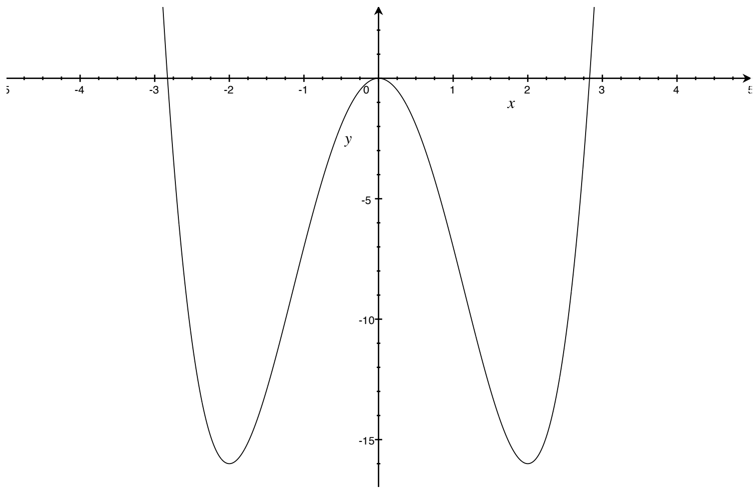
5. Extreme Points:
$$\frac{dy}{dx} = 4x^3 - 16x$$
$$= 4x(x^2 - 4) = 0$$

c.v.s: $x = 0, \pm 2$ e.v.s: $y = 0, -16$

Rel. max: $(0, 0)$ Abs. min: $(\pm 2, -16)$

6. Range: $y \in [-16, \infty)$

Sketch:



$$y = x^4 - 8x^2$$

EX 3 Find all the traits and sketch $y = -\frac{1}{4}x^4 + \frac{1}{3}x^3 + 3x^2$.

1. Domain: $x \in (-\infty, \infty)$

2. y-int: $x = 0$ gives $y = 0$, so $(0, 0)$

3. Zeros: $y = x^2\left(-\frac{1}{4}x^2 + \frac{1}{3}x + 3\right) = 0$

$$x = 0 \text{ or } \frac{-\frac{1}{3} \pm \sqrt{\frac{1}{9} - 4\left(-\frac{1}{4}\right)(3)}}{2\left(-\frac{1}{4}\right)} = -2.861 \text{ or } 4.194$$

$(0, 0)$, $(-2.861, 0)$, $(4.194, 0)$

4. End Behavior:

The right end goes up because $-\frac{1}{4}x^4$ is positive.

The left end also goes up. (Since the degree is even, the left does the same as the right.)

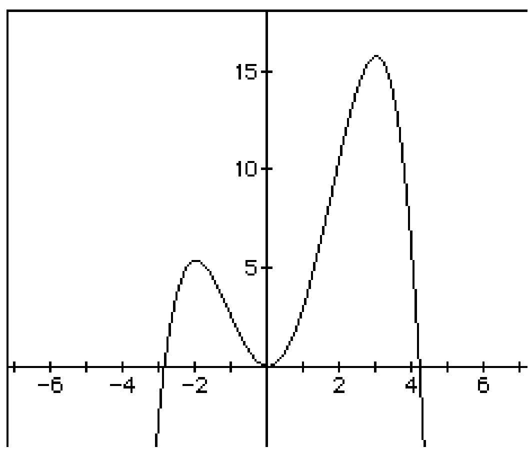
5. Extreme Points: $\frac{dy}{dx} = -x^3 + x^2 + 6x = 0$

$= -x(x-3)(x+2) = 0$

c.v.s: $x = 0$, 3 or -2 e.v.s: $x = 0$, 15.75 or 5.333

Rel. max: $(-2,\ 5.333)$ Abs. max: $(3,\ 15.75)$ Rel. min: $(0,\ 0)$

Sketch:



6. Range: $y \in (-\infty,\ 15.75]$

EX 4 Find all the traits and sketch $y = x^3 - 9x^2 - x + 9$ on $x \in [-2,\ 3]$.

1. Domain: $x \in [-2,\ 3]$

2. y-intercept: $x = 0$ gives $y = 9$, so $(0,\ 9)$

3. Zeros:
$$y = x^3 - 9x^2 - x + 9$$
$$y = x^2(x-9) - 1(x-9)$$
$$x = 1, -1, \text{or } 9$$
$$(\pm 1,\ 0)$$

4. End Behavior: None, because of the domain restriction.

5. Extreme Points: $\frac{dy}{dx} = 3x^2 - 18x - 1 = 0$

i) c.v.s: $x = \dfrac{18 \pm \sqrt{18^2 - 4(3)(-1)}}{2(3)} = 6.050 \text{ or } 0.050$

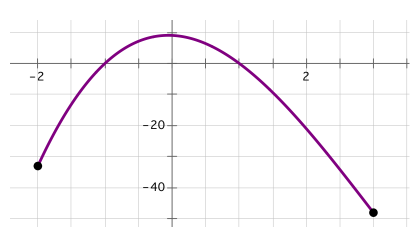
ii) $\dfrac{dy}{dx} = dne \to none$

iii) $x = -2, 3$

Rel. max: $(0.050,\ 7.426), (-2, -33), (3, -48)$

6. Range: $y \in [-48, 7.426]$

Sketch:



6-4 Free Response Homework

List the traits and sketch.

1. $f(x) = x^2 + 4x - 12$

2. $y = x^3 - x^2 - 9x + 9$

3. $h(x) = x^3 + x^2 - 7x - 15$

4. $y = x^4 - 5x^2 + 4$

5. $y = x^5 - 3x^3$

6. $g(x) = -x^3 + 4x^2 - x - 6$

7. $y = x^3 - 7x^2 + 11x + 3$

8. $y = -2x^3 + 2x^2 + 7x - 10$

9. $P(x) = -x^4 + 8x^2 - 12$

10. $y = 12x^2 - 6x^4$

11. $f(x) = x^4 + 4x^3 - 12x^2 - 32x$

12. $y = -2x^3 + 7x^2 + 50x - 175$

13. $F(x) = 4 + 3x^2 - x^4$

14. $y = 8x^3 + 20x^2 - 2x - 5$

15. $y = -3x^3 + 2x^2 + 147x - 98$

16. $y = 4x^4 - 37x^2 + 9$

17. $y = x^3 + x^2 - 5x - 3$ on $x \in [-3, 1]$

18. $y = 2x^3 + x^2 - 4x + 8$ on $x \in [0, 7]$

19. $y = -3x^4 + 24x^2 + 5$ on $x \in [-3, 5]$

20. $y = 2x^5 - 35x^3 + 135x - 29$ on $x \in [-2, 6]$

21. $y = x^4 - 6x^2 - 12$ on $x \in [-1, 2]$

22. $y = 3x^3 + 2x^2 - 27x - 18$ on $x \in [-3, 3)$

23. $y = x^4 - 4x^3$on $x \in [-1, 2]$

6-4 Multiple Choice Homework

1. Find an equation of the tangent line at $x = 2$ for $f(x) = 4 + x - 2x^2 - 3x^3$.

a) $y = -43x + 112$
b) $y = 43x - 112$
c) $y = -43x + 60$
d) $y = 43x - 60$
e) None of the above

---

2. It is given that $f(x)$ has a derivative with the following equation $f'(x) = -(x-3)^2(x+5)(x+1)$. Then

a) $f(x)$ has a relative minimum at $x = -5$ and $x = 3$
b) $f(x)$ has a relative maximum at $x = -5$ and a relative minimum at $x = -1$
c) $f(x)$ has a relative maximum at $x = -1$ and a relative minimum at $x = -5$ and $x = 3$
d) $f(x)$ has a relative maximum at $x = -1$ and a relative minimum at $x = -5$
e) $f(x)$ has a relative maximum at $x = -1$ and a relative minimum at $x = 3$

---

3. Given $y = -2x^3 + 2x^2 + 7x - 10$, the end behavior is

a) Both ends up
b) Both ends down
c) Right end up, left end down
d) Left end up, right end down
e) cannot be determined

---

4. If $y = 4x^4 - 37x^2 + 9$, the range is

a) $y \in [-2.131, \infty)$
b) All Reals
c) $y \in (-\infty, 2.151]$
d) $y \in [-76.562, \infty)$
e) cannot be determined

---

5. If $y = x^4 - 5x^2 + 4$, the zeros are

a) $(\pm 1, 0)$, $(\pm 2, 0)$
b) $(\pm 1, 0)$, $(\pm 2, 0)$, $(0, 4)$
c) $(0, 4)$, $(\pm 1.581, -2.25)$
d) $(\pm 1.581, -2.25)$
e) $(1, 0)$, $(2, 0)$

6-5: Multiple Sign Patterns

As we have seen, there are many purposes for sign patterns:

Remember:

A sign pattern is a tool, not a solution. There are different uses for sign patterns, depending on where the signs came from. The uses seen so far are:

I. Sign patterns of y shows where a function is above or below the x-axis and help solve inequalities.

II. Sign patterns of $\frac{dy}{dx}$ shows where a function is decreasing or increasing, and determines whether a critical value is at a maximum point, minimum point or neither.

III. Sign patterns of velocity determine which direction an object is moving.

IV. Sign patterns of acceleration indirectly determine whether an object in motion is speeding up or slowing down.

LEARNING OUTCOMES

Interpret pairs sign patterns in context of graphing.
Interpret pairs sign patterns in context of rectilinear motion.
Interpret pairs sign patterns in context of parametric motion.

Both the y and the $\frac{dy}{dx}$ equations give information about a function's graph. So will their sign patterns:

EX 1 The sign patterns associated with $f(x)$ are given.

$$\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & + & 0 & + \\ \hline x & & -3 & & -1 & & 1 & \end{array} \qquad \begin{array}{c|ccccccc} f'(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -2 & & 0 & & 1 & \end{array}$$

What can be said about the point where $x=-3$? Why?

Three things can be said about the point where $x=-3$:

1. The most obvious is that $x=-3$ is a zero of $f(x)$. The $f(x)$ sign pattern says so.
2. The $f(x)$ sign pattern also informs us that, at $x=-3$, $f(x)$ switches from above the x-axis to below the x-axis.
3. The thing most people forget is that, even though $x=-3$ is not one of the numbers on the $f'(x)$ sign pattern, it is still present to the left of $x=-2$. At $x=-3$, $f'(x)$ is negative, therefore, $f(x)$ is decreasing at $x=-3$.

EX 2 The sign patterns associated with $f(x)$ are given.

$$\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & + & 0 & + \\ \hline x & & -3 & & -1 & & 1 & \end{array} \qquad \begin{array}{c|ccccccc} f'(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -2 & & 0 & & 1 & \end{array}$$

What can be said about the point where $x=1$? Why?

Three things can be said about the point where $x=1$ as well:

1. The $f(x)$ sign pattern informs us that $x=1$ is a zero of $f(x)$.
2. The $f(x)$ sign pattern also informs us that $x=1$ is a bouncer—meaning that $x=1$ hits but does not cross the x-axis.
3. The $f'(x)$ sign pattern informs us that $x=1$ is at a minimum of $f(x)$ because the signs of $f'(x)$ switch from negative to positive.

EX 3 Given the $f(x)$ and its sign patterns in EX 2, for what values of x is $f(x)$ both below the x-axis and increasing?

From the $f(x)$ sign pattern, we can see that $f(x)$ is below the x-axis on $x \in (-3, -1)$. From the $f'(x)$ sign pattern, we know that $f(x)$ is decreasing before $x = -2$ and increasing after. Therefore,

$f(x)$ both below the x-axis and increasing on $x \in (-2, -3)$

Like the y and the $\frac{dy}{dx}$ equations and sign patterns, the velocity and acceleration equations and sign patterns both hold information about the motion of an object. In particular, questions about whether an object is speeding up or slowing down are answered by a combination of the velocity and acceleration.

> An object is speeding up when $v(t)$ and $a(t)$ have the same sign.
>
> An object is slowing when $v(t)$ and $a(t)$ have opposite signs.
>
> NB. Speeding up and slowing down is not determined by the sign of the acceleration.

EX 4 The position of a particle is described by $x(t) = t^3 - 3t^2 - 24t + 3$. Is the particle speeding up or slowing down at $t = 3$ seconds?

$$v(t) = x'(t) = 3t^2 - 6t - 24$$
$$v(3) = 3(3)^2 - 6(3) - 24 = -15$$

$$a(t) = v'(t) = 6t - 6$$
$$a(3) = v'(t) = 6(3) - 6 = 12$$

Since these have opposite signs, the particle is slowing down.

EX 5 The position of a particle is described by $x(t)=\frac{1}{2}t^4-3t^3+20t+3$. When is the particle speeding up?

First, we need the velocity sign pattern:

$$v(t)=2t^3-9t^2+20$$

Synthetic Division shows that $x=2$ is a zero and the velocity eqwuation factors into $(x-1)(2x^2-5x-10)$. The quadratic formula then shows the other zeros are $x=-1.312$ and 3.812. So

$$\begin{array}{c|ccccccc} v(t) & - & 0 & + & 0 & - & 0 & + \\ \hline t & & -1.312 & & 2 & & 3.812 & \end{array}$$

Next, we need the acceleration sign pattern:

$$a(t)=6t^2-18t$$

$$\begin{array}{c|ccccc} a(t) & + & 0 & - & 0 & + \\ \hline t & & 0 & & 3 & \end{array}$$

If we line up these sign patterns one above the other, we can see where the signs match and where they are opposite:

$$\begin{array}{c|ccccccc} v(t) & - & 0 & + & 0 & - & 0 & + \\ \hline t & & -1.312 & & 2 & & 3.812 & \end{array}$$

$$\begin{array}{c|ccccccccc} a(t) & + & & + & 0 & - & - & 0 & + & + \\ \hline t & & & & 0 & & & 3 & & \end{array}$$

Since the question was "When is the particle speeding up?" we need to name the intervals of time when $v(t)$ and $a(t)$ have the same sign; that is:

$$t\in(-1.312,0)\cup(2,3)\cup(3.812,\infty)$$

A third kind of problem that involves two sign patterns is the parametric motion problems.

EX 6 A particle's position $\langle x(t), y(t)\rangle$ at time t is described by

$$\left\langle \frac{1}{3}t^3 - t^2 - 8t + 1, -t^2 + 6t + 2 \right\rangle.$$

When is the particle moving both left and up?

Direction of motion is determined the velocity so:

$$\langle x'(t), y'(t)\rangle = \langle t^2 - 2t - 8, -2t + 6\rangle$$

$$\begin{array}{c|ccccc} x'(t) & + & 0 & - & 0 & + \\ t & & -2 & & 4 & \end{array}$$

$$\begin{array}{c|ccc} y'(t) & - & 0 & + \\ t & & 3 & \end{array}$$

According to these sign patterns, the particle is moving left on $t \in (-2, 4)$ and up on $t \in (3, \infty)$. The question was when is it doing both. So, the particle moving both right and down on

$$t \in (3, 4)$$

6-5 Free Response Homework

1. The sign patterns associated with $f(x)$ are given.

$$\begin{array}{c|ccccccc} f(x) & - & 0 & - & 0 & + & 0 & + \\ \hline x & & -3 & & -1 & & 1 & \end{array} \qquad \begin{array}{c|ccccccc} f'(x) & + & 0 & - & 0 & - & 0 & + \\ \hline x & & -3 & & 0 & & 1 & \end{array}$$

a) What can be said about the point where $x=-3$? Why?
b) What can be said about the point where $x=0$? Why?
c) What can be said about the point where $x=1$? Why?

2. The sign patterns associated with $g(x)$ are given.

$$\begin{array}{c|ccccccc} g(x) & + & 0 & - & 0 & + & 0 & - \\ \hline x & & -3 & & 0 & & 3 & \end{array} \qquad \begin{array}{c|ccccc} g'(x) & - & 0 & + & 0 & - \\ \hline x & & -2 & & 2 & \end{array}$$

a) What can be said about the point where $x=-3$? Why?
b) What can be said about the point where $x=2$? Why?
c) What can be said about the point where $x=0$? Why?

3. The sign patterns associated with $h(x)$ are given.

$$\begin{array}{c|ccccccccc} h(x) & + & 0 & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -2 & & -\sqrt{2} & & \sqrt{2} & & 2 & \end{array}$$

$$\begin{array}{c|ccccccc} h'(x) & - & 0 & - & 0 & + & 0 & + \\ \hline x & & -2 & & 0 & & 2 & \end{array}$$

a) What can be said about the point where $x=-\sqrt{2}$? Why?
b) What can be said about the point where $x=2$? Why?
c) What can be said about the point where $x=1$? Why?

4. The sign patterns associated with $g(x)$ are given.

$$\begin{array}{c|ccccccc} g(x) & + & 0 & - & 0 & + & 0 & - \\ \hline x & & -3 & & 0 & & 3 & \end{array} \qquad \begin{array}{c|ccccc} g'(x) & - & 0 & + & 0 & - \\ \hline x & & -2 & & 2 & \end{array}$$

a) What can be said about the point where $x=-2$? Why?
b) What can be said about the point where $x=1$? Why?
c) What can be said about the point where $x=3$? Why?

Find the sign patterns of y and $\frac{dy}{dx}$ for each of the following equations and answer the question asked.

5. Where is $y=x^2-8x$ both increasing and above the x-axis?

6. Where is $y=x^2-2x-8$ both decreasing and above the x-axis?

7. Where is $y=x^3-x^2-4x+4$ both increasing and below the x-axis?

8. Where is $y=x^3-7x^2-11x-3$ both decreasing and below the x-axis?

9. Where is $y=x^3-27x$ both decreasing and below the x-axis?

10. Where is $y=4x^3-x^2-36x+9$ both increasing and above the x-axis?

11. Where is $y=x^4+4x^3-12x^2-32x$ both decreasing and above the x-axis?

12. Where is $y=x^4-2x^2-8$ both increasing and below the x-axis?

13. Where is $y=x^4+7x^2+10$ both increasing and below the x-axis?

14. Where is $y=x^4-4x^3$ both increasing and below the x-axis?

Find the sign patterns of $v(t)$ and $a(t)$ for each of the following equations and answer the question asked.

15. If $x(t)=2t^3-21t^2+60t+4$, when is the object speeding up?

16. If $x(t)=t^3-6t^2+12t+5$, when is the object speeding up?

17. If $x(t)=9t^4-4t^3-240t^2+576t-48$, when is the object speeding up?

18. If $x(t)=12t^5-15t^4-220t^3+270t^2+1080t$, when is the object slowing down?

19. If $y(t)=5t^3-t^5$, when is the object slowing down?

20. If $y(t)=t^4-5t^2-36$, when is the object slowing down?

21. A particle's position $\langle x(t),\, y(t)\rangle$ is described by the parametric equations $x(t)=t^4-4t^2-5$ and $y(t)=6t-t^3$. When is the particle moving left and up?

22. A particle's position $\langle x(t),\, y(t)\rangle$ is described by the parametric equations $x(t)=t^2+4t+3$ and $y(t)=2t-t^2$. When is the particle moving right and down?

23. A particle's position $\langle x(t),\, y(t)\rangle$ is described by the parametric equations $x(t)=t^3-6t^2+12t+5$ and $y(t)=t^4-5t^2-36$. When is the particle moving left and down?

24. A particle's position $\langle x(t),\, y(t)\rangle$ is described by the parametric equations $x(t)=2t^3-21t^2+60t+4$ and $y(t)=5t^3-t^5$. When is the particle moving left and up?

6-5 Multiple Choice Homework

1. A particle moving in the *xy*-plane with its *x* coordinate given by $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+\frac{1}{2}t^2-1$ and its *y* coordinate given by $y(t)=\frac{1}{2}t^2-t+1$. When the particle is moving up it is also

a) moving right
b) moving left
c) at rest
d) cannot be determined
e) does not exist

2. A particle's position $\langle x(t),\ y(t)\rangle$ is described by the sign patterns below:

$$\begin{array}{c|ccccc} x(t) & - & 0 & + & 0 & - \\ t & & -1 & & 4 & \end{array} \qquad \begin{array}{c|ccccc} y(t) & + & 0 & - & 0 & + \\ t & & -3 & & 1 & \end{array}$$

On $t \in (-1,\ 1)$, the particle is in

a) Quadrant I
b) Quadrant II
c) Quadrant III
d) Quadrant IV

---

3. A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Which of the following statements is/are true?

I. The particle is moving right at $t = \frac{2}{3}$.

II. The particle is paused at $t = \frac{1}{3}$

III. The particle is speeding up at $t = 1$.

a) I only b) II only c) III only

d) I and II only e) I and III only

---

4. A particle moves on a plane so that its position vector is $p(t) = \langle t^2 + 2t - 8,\ t^2 - 5t + 6\rangle$. It is at rest

a) at $t = 2$ only b) at $t = -1$ & 2.5 c) at $t = -2$ only

d) $t = 2,\ 3,\ \&\ -4$ e) at no time

5. Consider the functions $f(x)=5+10x-x^2$, $g(x)=x^3-3x^2+3x$, and $h(x)=x^4+4x^3+6x^2+4x$. Which of the following statements is completely true?

a) $f(x)$ and $g(x)$ have absolute maximum points, $h(x)$ and $g(x)$ have absolute minimum points.

b) $f(x)$ and $h(x)$ have absolute maximum points, $g(x)$ has a local minimum point, and $h(x)$ has a local minimum point.

c) $g(x)$ has an absolute maximum point, $f(x)$ has an local minimum point, and $h(x)$ has an absolute minimum point.

d) $f(x)$ has an absolute maximum point, $g(x)$ has both a local minimum point and a local maximum point, $h(x)$ has an absolute minimum point.

e) None of the above is completely true.

Polynomial Functions Practice Test

Part 1: CALCULATOR ALLOWED

---

1. Consider the functions $f(x)=5+10x-x^2$, $g(x)=x^3-3x^2+3x$, and $h(x)=x^4+4x^3+6x^2+4x$. Which of the following statements is completely true?

a) $f(x)$ and $g(x)$ have absolute maximum points, $h(x)$ and $g(x)$ have absolute minimum points.
b) $f(x)$ and $h(x)$ have absolute maximum points, $g(x)$ has a local minimum point, and $h(x)$ has a local minimum point.
c) $g(x)$ has an absolute maximum point, $f(x)$ has a local minimum point, and $h(x)$ has an absolute minimum point.
d) $f(x)$ has an absolute maximum point, $g(x)$ has both a local minimum point and a local maximum point, $h(x)$ has an absolute minimum point.
e) None of the above is completely true.

---

2. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs $12 per square meter. Material for the sides costs $5 per square meter. Find the cost of materials for the cheapest such container.

a) $153.92
b) $158.10
c) $152.90
d) $151.60
e) $153.90

---

3. The function given by $f(x) = x^3 + 12x - 24$ is

(a) increasing for $x > -2$, decreasing for $-2 < x < 2$, increasing for $x > 2$

(b) decreasing for $x < 0$

(c) increasing for all x

(d) decreasing for all x

(e) decreasing for $x < -2$, increasing for $-2 < x < 2$, decreasing for $x > 2$

---

4. If g is a differentiable function such that $g(x) < 0$ for all real numbers x and if $f'(x) = (x^2 - 4) \bullet g(x)$, which of the following is true?

(a) f has a relative maximum at $x = -2$ and a relative minimum at $x = 2$.

(b) f has a relative minimum at $x = -2$ and a relative maximum at $x = 2$.

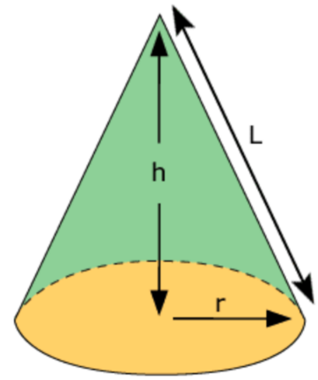
(c) f has relative minima at $x = -2$ and $x = 2$.

(d) f has relative maxima at $x = -2$ and $x = 2$.

(e) It cannot be determined if f has any relative extrema.

---

5. The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$ and the surface area is given by $SA = \pi r^2 + \pi r l$. A specific cone has surface area = 100π cm^2. [Note that Given that r, h, and l form a right triangle.]. Which of the following equations would be differentiated in order to find the radius that yields the maximum volume?



a) $V = \frac{1}{3}\pi r^2 h$ b) $SA = \pi r^2 + \pi r l$

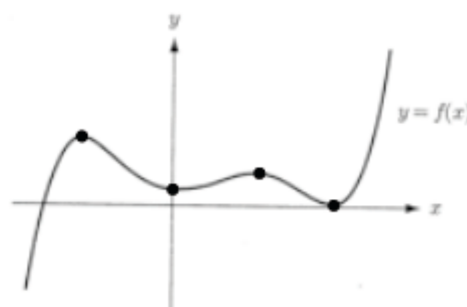
c) $r^2 + h^2 = l^2$ d) $V = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$

e) None of these as they are written.

---

6. If the graph of a fifth-degree polynomial, $f(x)$, is shown below, then the graph of $f'(x)$, the derivative of $f(x)$, will cross the x-axis in exactly how many points?

(a) None
(b) One
(c) Two
(d) Three
(e) Four



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7. Given this sign pattern $\begin{array}{c|ccccc} f'(x) & - & 0 & + & 0 & - \\ \hline x & & -3 & & 1 & \end{array}$, which of the following might be the sign pattern of $f(x)$?

a) $\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & + & 0 & - \\ \hline x & & -5 & & -1 & & 2 & \end{array}$

b) $\begin{array}{c|ccccccc} f(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -5 & & -1 & & 2 & \end{array}$

c) $\begin{array}{c|ccccccc} f(x) & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & -1 & & 1 & \end{array}$

d) $\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & + & 0 & - \\ \hline x & & -3 & & -1 & & 1 & \end{array}$

e) $\begin{array}{c|ccccccc} f(x) & + & 0 & - & 0 & - & 0 & + \\ \hline x & & -3 & & -1 & & 1 & \end{array}$

Polynomial Functions Practice Test
Part 2: CALCULATOR ALLOWED
Round answers to THREE DECIMAL PLACES.
Show all work.

---

1. A farmer with 2000 feet of fencing is wants to enclose a rectangular area and then divide it into three pens with fencing parallel to the **vertical** side of the rectangle. What is the largest possible total area of the three pens?

2. a) Find the extreme points of $y = t^3 + t^2 - 2t$.

   b) Find the extreme points and zeros of $y = x^4 - 29x^2 + 100$.

3. A cylindrical cola can has a volume of 16π in^3. What radius would minimize the surface area? If the top and bottom of the can cost 6 cents/square inch and the sides cost 3 cents/square inch, what radius would minimize the cost?

a) State the equation needed to minimize the cost per can.

b) State the secondary equation needed to eliminate the extra variable.

c) Eliminate the extra variable in the equation needed to minimize the cost per can.

d) Find the minimum cost per can.

Polynomial Functions Practice Test
Part 3: NO CALCULATOR ALLOWED
Show all work.

---

4. The sign pattern for the derivative of $H(x)$ is given.

$$\begin{array}{c|ccccccc} H'(x) & + & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & \tfrac{3}{4} & & 5 & \end{array}$$

a) Is $x = -3$ at a maximum, a minimum, or neither? Why?

b) Is $x = \tfrac{3}{4}$ at a maximum, a minimum, or neither? Why?

c) Is $x = 5$ at a maximum, a minimum, or neither? Why?

5. When is $y = 8x^3 - x^4$ both decreasing and above the x-axis? Explain your reasoning, showing the necessary sign patterns.

6. List the traits and **sketch** $y = t^3 + t^2 - 2t$.

Domain:

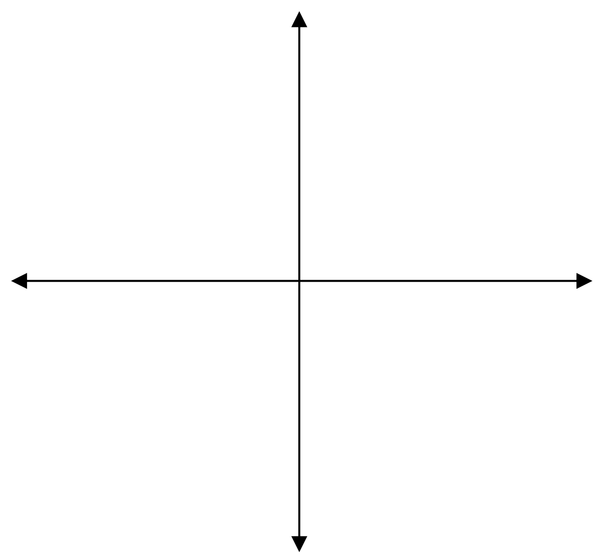
y-int:

Zeros:

Extreme Points:

End Behavior:

Range:



7. List the traits and **sketch** $y = x^4 - 29x^2 + 100$.

Domain:

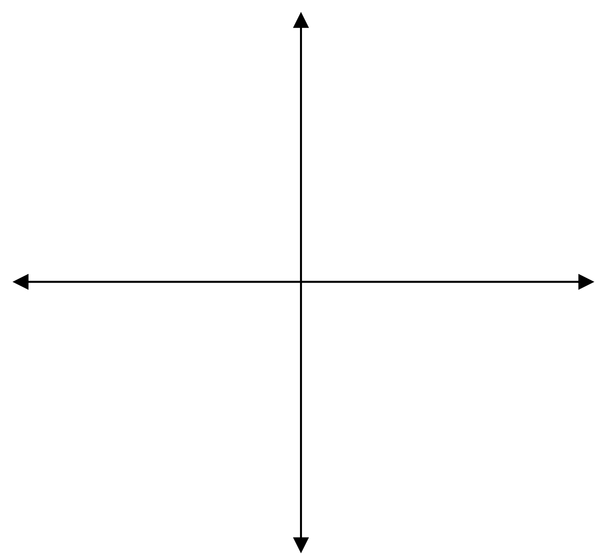
y-int:

Zeros:

Extreme Points:

End Behavior:

Range:



Polynomial Functions Homework Answer Key

6-1 Free Response Homework

1. $x=-2$ 2. $x=-\frac{5}{3},\ 1$ 3. $x=-1,\ \frac{2}{3}$

4. $x=-2$ 5. $x=-4,\ -1,\ 2$ 6. No critical values

7. $x=0,\ \pm 4$ 8. $x=\pm\sqrt{2},\ \pm\sqrt{6}$ 9. $x=-3,\ -\frac{5}{3},\ 1$

10. $x=0,\ \frac{2}{3},\ 7$ 11. $x=-3,\ 0,\ \pm 2,\ 5$ 12. $x=\pm\sqrt{\frac{3}{2}},\ -2,\ 3,\ 6$

13. $\left(\frac{2}{\sqrt[4]{5}},\ -5.120\right),\ \left(\frac{-2}{\sqrt[4]{5}},\ 29.120\right)$ 14. $(-7,\ 931),\ (4,\ -400)$

15. $(-1,-17),\ (0,-12),\ \left(\sqrt{3},-21\right),\ (2,\ -20)$

16. $(-3,0),\ (-1.968,\ 20.018),\ (1.524,\ -43.884)$

17. $y\in\left[-\frac{53}{4},\ \infty\right)$ 18. $y\in[-45.0625,\ \infty)$ 19. $y\in(-\infty,\ \infty)$

6-1 Multiple Choice Homework

1. C 2. E 3. A 4. C 5. D

6-2 Free Response Homework

1. $x\in(-3,\ 0)\cup(3,\ \infty)$ 2. $x\in\left(\frac{2}{3},\ 2\right)\cup(2,\ \infty)$

3a. Maximum point b. Neither

4. $x = -2$

5. $x \in (-\infty, -3) \cup \left(-3, \frac{2}{3}\right)$

6a. Neither

b. Minimum point

7. $x \in (-\infty, 0.131) \cup (2.535, \infty)$

8. $x \in (-\infty, -0.869) \cup (1.535, \infty)$

9. $x \in (-\infty, 1) \cup \left(\frac{11}{3}, \infty\right)$

10. $x \in (-\infty, -2) \cup \left(\frac{20}{9}, \infty\right)$

11. $x \in (-\infty, -0.797) \cup (1.464, \infty)$

12. $x \in (-\infty, -4) \cup (-1, 2)$

13. $x \in (-\infty, -0.236) \cup (3, 4.236)$

14. $x \in (-4, -1) \cup (-1, 2)$

15. Max pt. @ $x = 0.131$; Min pt. @ $x = 2.535$

16. Max pt. @ $x = -0.869$; Min pt. @ $x = 1.535$

17. Max pt. @ $x = 1$; Min pt. @ $x = \frac{11}{3}$

18. Max pt. @ $x = 2$; Min pt. @ $x = \frac{20}{9}$

19. Max pt. @ $x = 1.464$; Min pt. @ $x = -0.797$

20. Max pt. @ $x = -1$; Min pt. @ $x = -4, 2$

21. Max pt. @ $x = 3$; Min pt. @ $x = -0.236, 4.236$

22. Max pt. @ $x = -4$; Min pt. @ $x = 2$

23a. Min velocity = –13.5

b. No maximum accel.

24a. Relative max velocity = 1200

b. No max accel.

25. $x=-3,-\frac{5}{3},1$

26. $x=0,\frac{2}{3},7$

27. $x=-3,0,\pm 2,5$

28. $x=-2,\ \pm\sqrt{\frac{3}{2}},\ 3,\ 6$

29. $x=-1,\ 0,\sqrt{3},2$

30. $x=-3,-1.467,1.023$

31. $x=-1,\ 2$ on $x\in[-1,\ 2]$

6-2 Multiple Choice Homework

1.	B	2.	A	3.	C	4.	A	5.	A

6-3 Free Response Homework

1a.	\$110	b.	\$80	c.	500
2a.	6	b.	324	c.	$t\in[6,10]$

3. \$140,000

4. $t=8$

5. 2 in x 2 in

6. $r=\frac{36}{\pi}$ in, $l=36$ in, $V=\frac{36^3}{\pi}\text{in}^3$

7. Moon: $t=320$ sec, $h=66{,}560$ feet
   Earth: $t=52$ sec, $h=10{,}816$ feet

8. 16 cm

9. $x = \frac{1}{\sqrt{3}}$

10a. $r = 2$ in

10b. $r = \sqrt[3]{4}$ in=1.587 inches

11. 40 ft x 80 ft

12. $x = 18 \text{ in} \to V = 11{,}664 \text{ in}^3$

13. $A = 49.267$

14. 128

15. 30

16. $M = C$

17. $V = 971.104\ in^3$

18. 750 yd x 1500 yd; $A = 1{,}125{,}000 \text{ yd}^2$

19. 195.959

20. $262.91

21. 81.976

22. $40833\,{}^1\!/_3$

23a. Max at $t = 19.024$; min at $t = 0, 48$

23b. 4945

23c. $t = [0, 19.024]$

<u>6-3 Multiple Choice Homework</u>

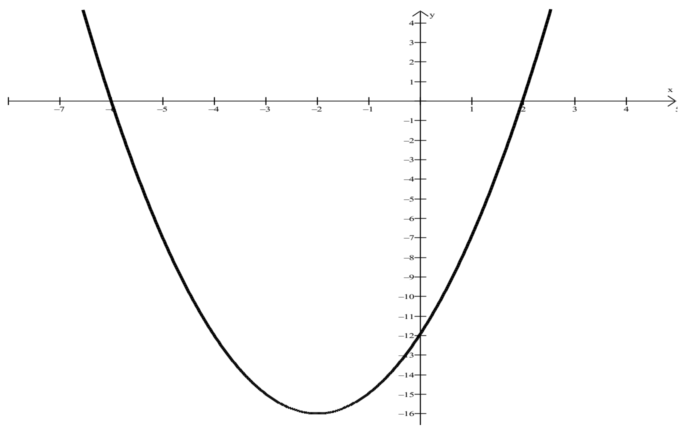
1. B 2. B 3. C 4. C 5. E 6. B

<u>6-4 Free Response Homework</u>

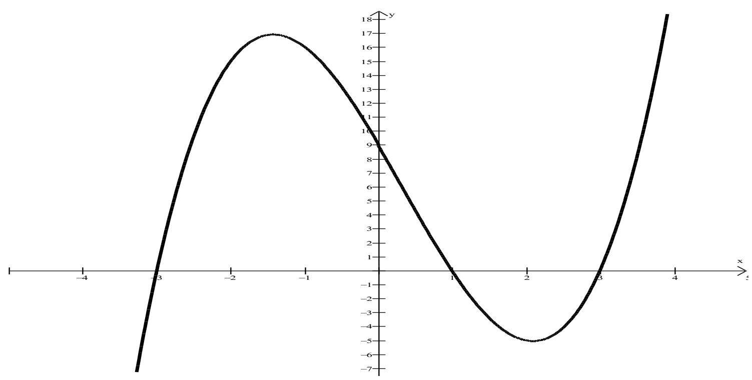
1. Domain: $x \in (-\infty, \infty)$ Range: $y \in [-16, \infty)$

Zeros: $(-6, 0)$, $(2, 0)$ y-int: $(0, -12)$

EB: Both ends up Extreme Point: $(-2, -16)$



2. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in (-\infty,\ \infty)$
Zeros: $(-3,\ 0)$, $(1,\ 0)$, $(3,\ 0)$ y-int: $(0,\ 9)$
EB: Left end down; right end up
Extreme Points: $(2.097,\ -5.049)$, $(-1.431,\ 16.901)$

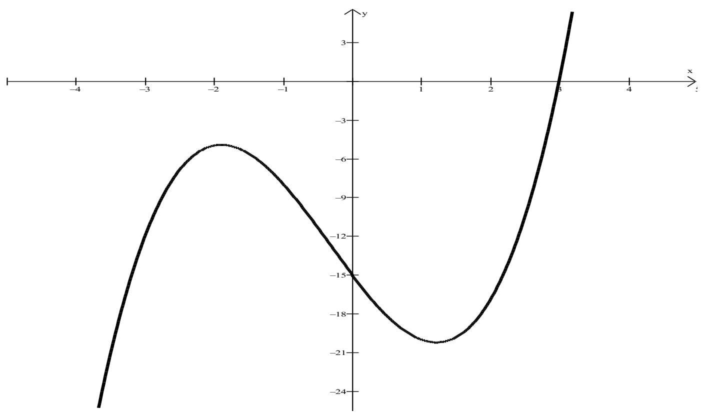


3. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in (-\infty,\ \infty)$

Zeros: $(3,\ 0)$ y-int: $(0,\ -15)$

EB: Left end down; right end up

Extreme Points: $(1.230,\ -6.738), (-1.897,\ 17.854)$

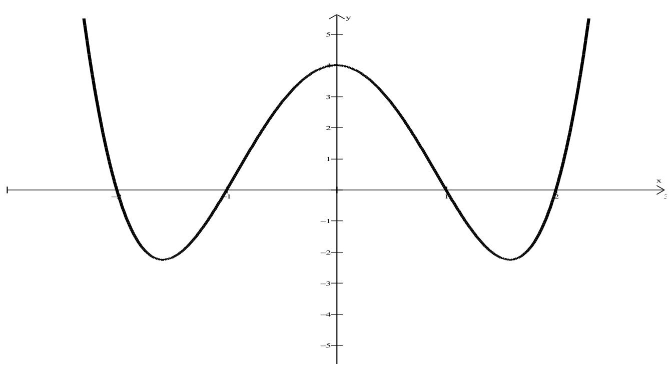


4. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in [-2.25,\ \infty)$

Zeros: $(\pm 1,\ 0),\ (\pm 2,\ 0)$ y-int: $(0,\ 4)$

EB: Both ends up

Extreme Points: $(0,\ 4), (\pm 1.581,\ -2.25)$

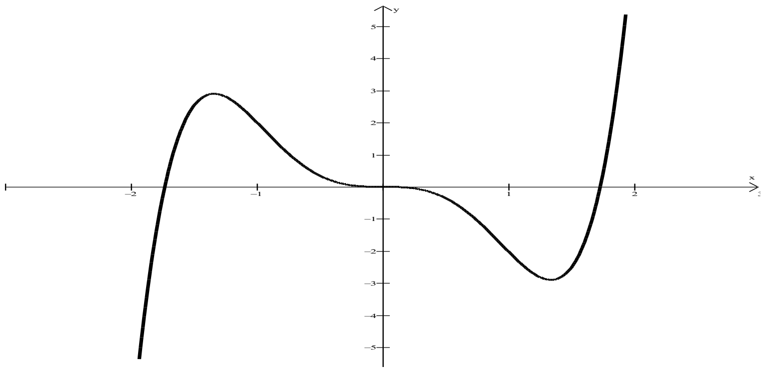


5. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-\infty, \infty)$

Zeros: $(\pm\sqrt{3}, 0)$, $(0, 0)$ y-int: $(0, 0)$

EB: Left end down; right end up

Extreme Points: $\left(\frac{3}{\sqrt{5}}, -2.898\right), \left(\frac{-3}{\sqrt{5}}, 2.898\right)$

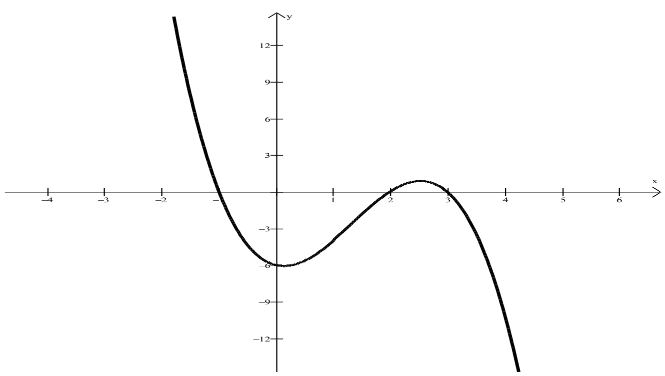


6. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-\infty, \infty)$

Zeros: $(3, 0)$, $(2, 0)$, $(-1, 0)$ y-int: $(0, -6)$

EB: Left end up; right end down

Extreme Points: $(0.131, -6.065)$, $(2.535, 0.879)$

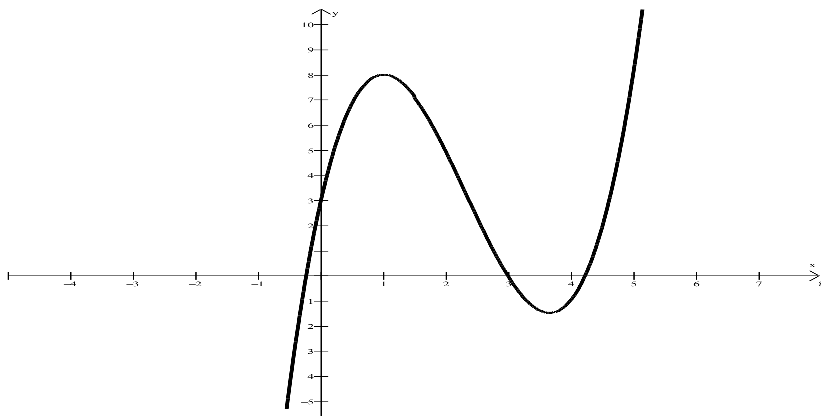


7. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-\infty, \infty)$

Zeros: $(-3, 0)$, $(-0.236, 0)$, $(4.236, 0)$ y-int: $(0, 3)$

EB: Left end down; right end up

Extreme Points: $(3.667, -1.481)$, $(1, 8)$

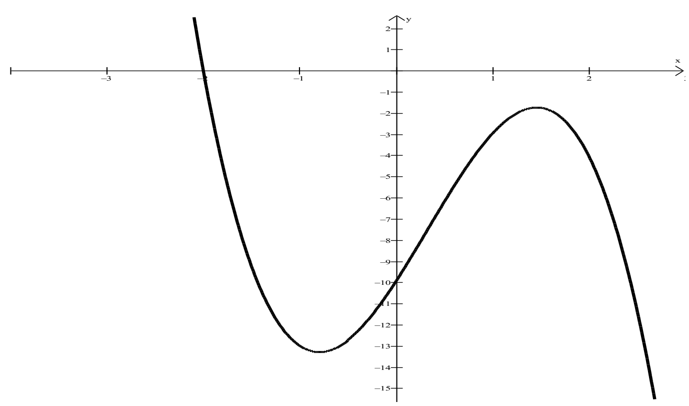


8. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-\infty, \infty)$

Zeros: $(-2, 0)$ y-int: $(0, -10)$

EB: Left end up; right end down

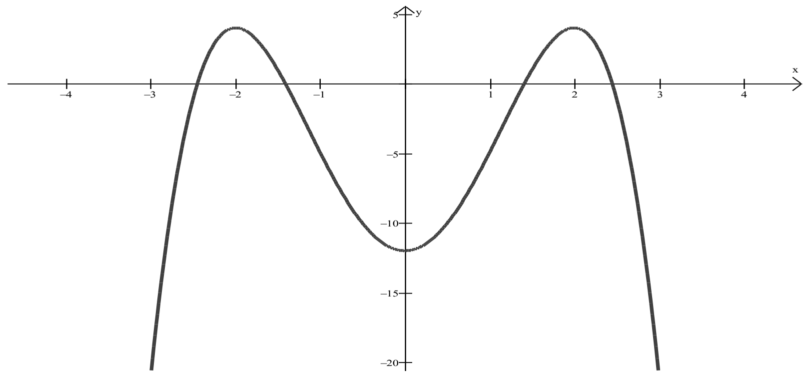
Extreme Points: $(1.464, -1.741)$, $(-0.797, -13.296)$



9. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in (-\infty,\ 4]$

Zeros: $(\pm\sqrt{2},\ 0)$, $(\pm\sqrt{6},\ 0)$ y-int: $(0,\ -12)$

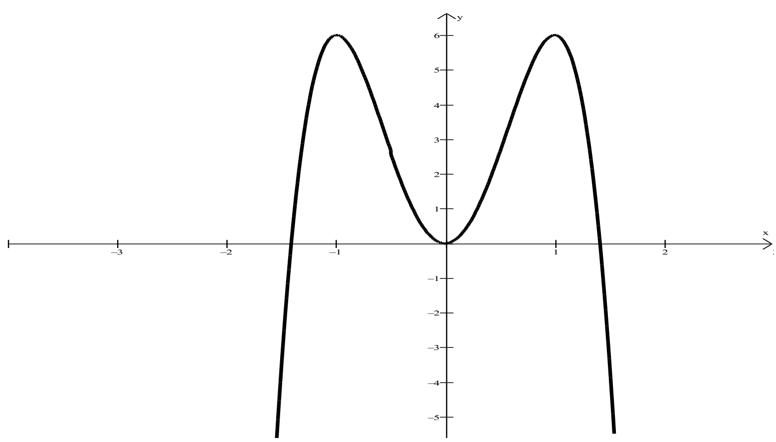
EB: Both ends down Extreme Points: $(0,\ -12)$, $(\pm 2,\ 4)$



10. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in (-\infty\ ,6]$

Zeros: $(\pm\sqrt{2},\ 0)$, $(0,\ 0)$ y-int: $(0,\ 0)$

EB: Both ends down Extreme Points: $(0,\ 0)$, $(\pm 1,\ 6)$

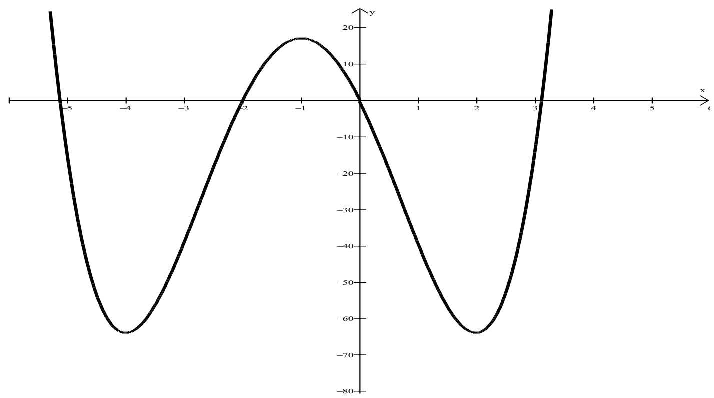


11. Domain: $x \in (-\infty, \infty)$ Range: $y \in [-64, \infty)$

Zeros: $(0, 0)$, $(-2, 0)$, $(3.123, 0)$, $(-5.123, 0)$

y-int: $(0, 0)$ EB: Both ends up

Extreme Points: $(-4, -64)$, $(-1, 17)$, $(2, -64)$

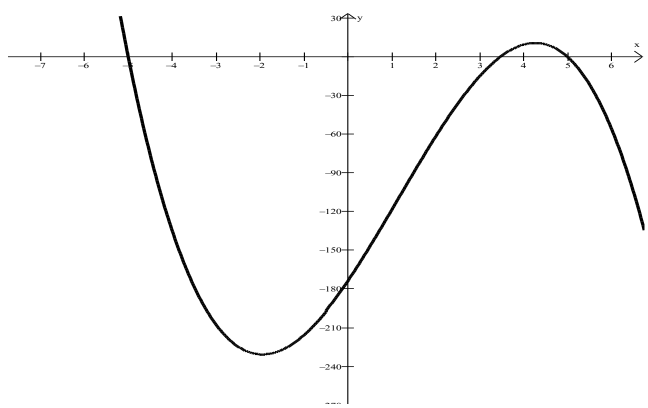


12. Domain: $x \in (-\infty, \infty)$ Range: $y \in (-\infty, \infty)$

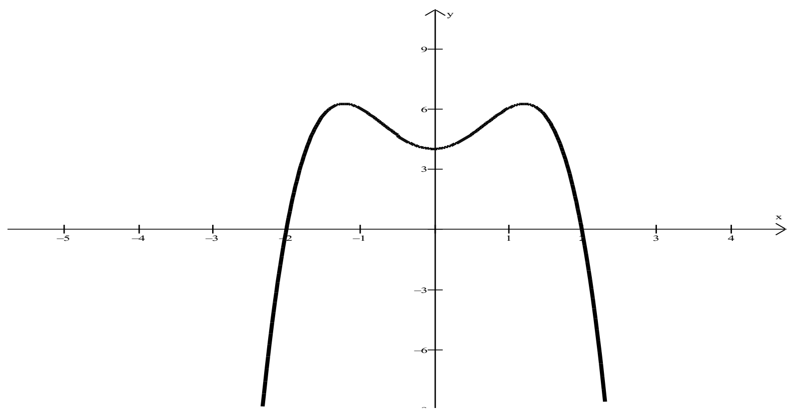
Zeros: $(\pm 5, 0)$, $\left(\frac{7}{2}, 0\right)$ y-int: $(0, -175)$

EB: Left end up; right end down

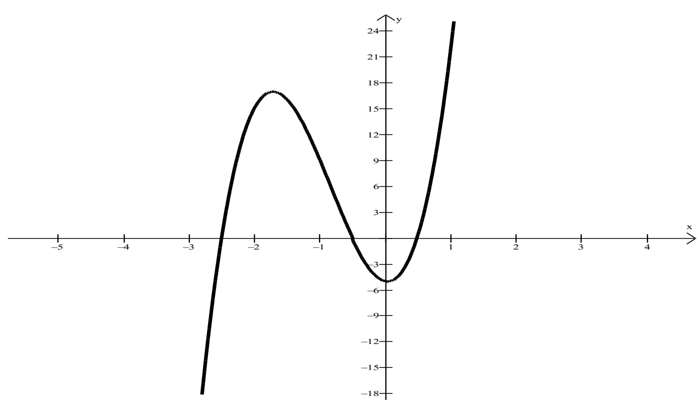
Extreme Points: $(4.280, 10.423)$, $(-1.947, -231.053)$



13. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in (-\infty,\ 6.25]$
Zeros: $(2,\ 0),\ (-2,\ 0)$ y-int: $(0,\ 4)$
EB: Both ends down Extreme Points: $(0,\ 4), (\pm 1.225,\ 6.25)$



14. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in (-\infty,\ \infty)$
Zeros: $\left(\pm\frac{1}{2},\ 0\right), \left(-\frac{5}{2},\ 0\right)$ y-int: $(0,\ -5)$
EB: Left end down; right end up
Extreme Points: $(0.049,\ -5.049),\ (-1.715,\ 16.901)$

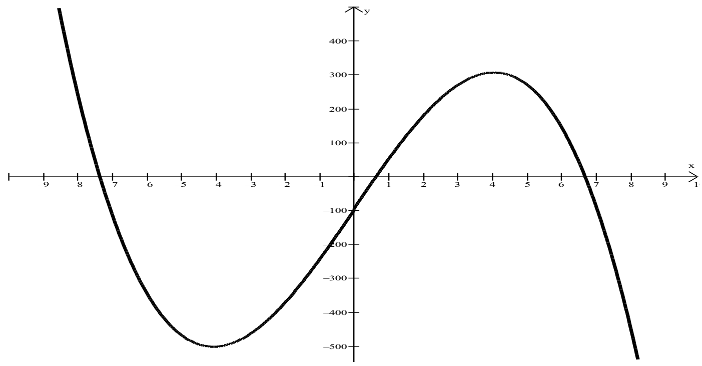


15. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in (-\infty,\ \infty)$

Zeros: $(\pm 7,\ 0),\ \left(\frac{2}{3},\ 0\right)$ y-int: $(0,\ -98)$

EB: Left end up; right end down

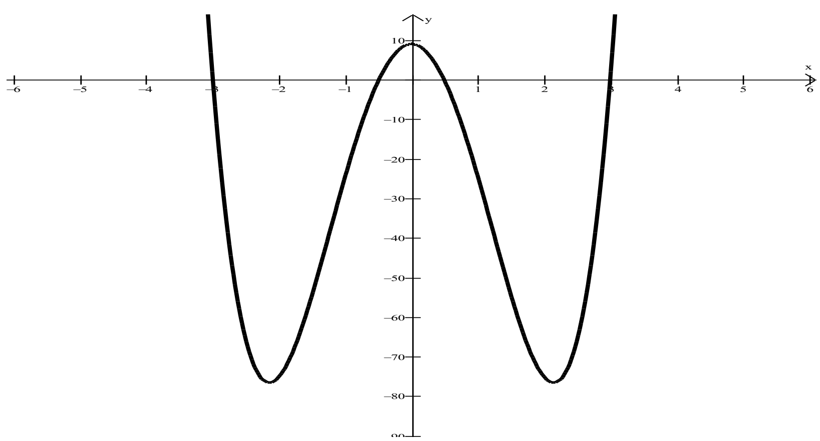
Extreme Points: $(4.270,\ 332.592),\ (-3.825,\ -463.127)$



16. Domain: $x \in (-\infty,\ \infty)$ Range: $y \in [-76.562,\ \infty)$

Zeros: $\left(\pm\frac{1}{2},\ 0\right),\ (\pm 3,\ 0)$ y-int: $(0,\ 9)$

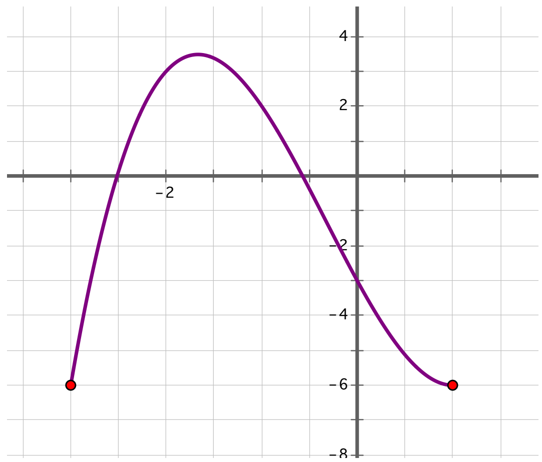
EB: Both ends up Extreme Points: $(0,\ 9),\ (\pm 2.151,\ -76.562)$



17. Domain: $x \in [-3, 1]$ Range: $y \in [-6, 3.481]$

Zeros: $(-2.514, 0), (-0.592, 0)$ y-int: $(0, 9)$

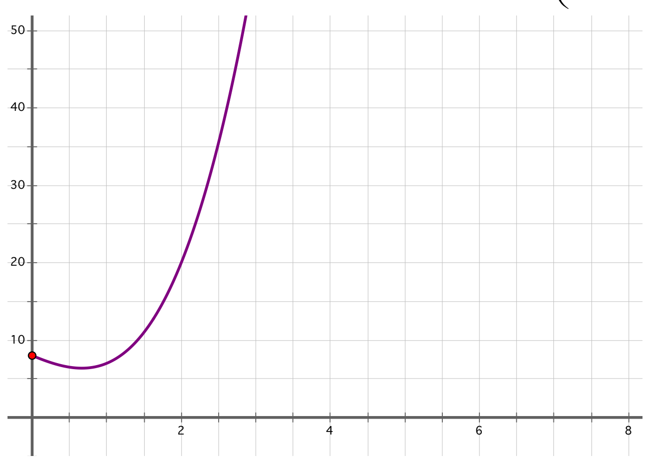
EB: None Extreme Points: $(-3, -6), \left(-\frac{5}{3}, 3.481\right), (1, -6)$



18. Domain: $x \in [0, 7]$ Range: $y \in [6.370, 715]$

Zeros: none y-int: $(0, 8)$

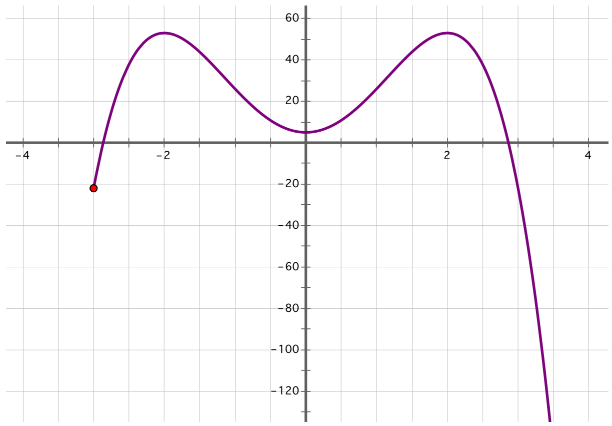
EB: none Extreme Points: $(0, 8), \left(\frac{2}{3}, 6.370\right), (7, 715)$



19. Domain: $x \in [-3,\ 5]$ Range: $y \in [-1270, 53]$

Zeros: $(\pm 2.864,\ 0)$ y-int: $(0,\ 5)$ EB: none

Extreme Points: $(0,\ 5)$, $(-3,\ -22)$, $(\pm 2, 53)$, $(5, -1270)$

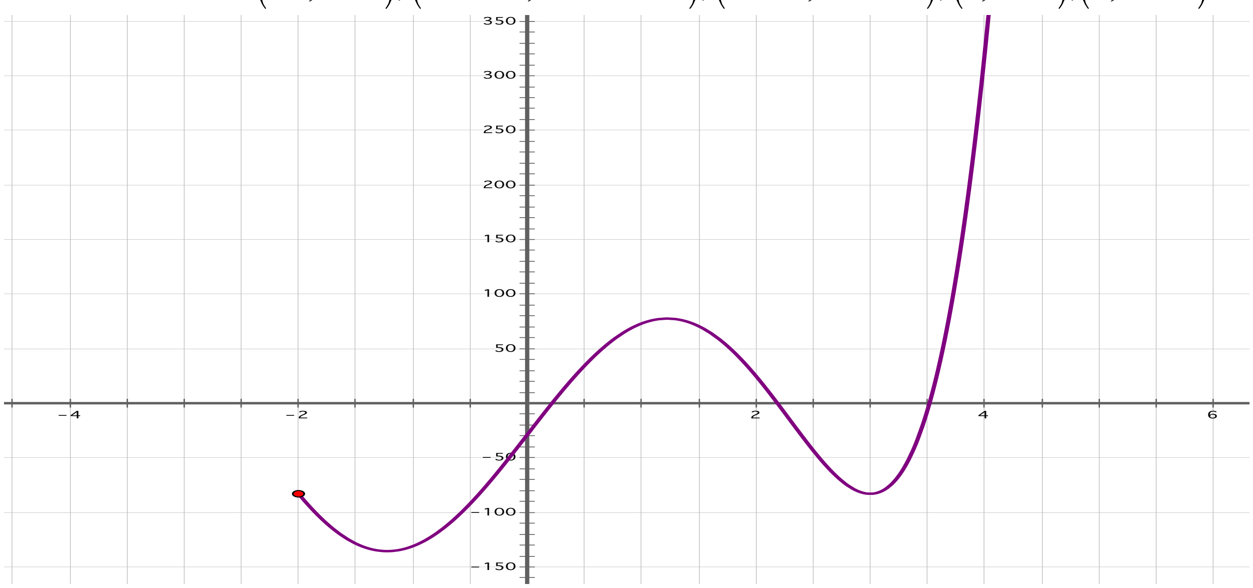


20. Domain: $x \in [-2,\ 6]$ Range: $y \in [-83, 8773]$

Zeros: $(-3.303,\ 0)$, $(-2.621,\ 0)$, $(0.217,\ 0)$, $(2.18,\ 0)$, $(3.519,\ 0)$

y-int: $(0,\ -29)$ EB: none

Extreme Points: $(-2, -83)$, $(-1.225, -135.552)$, $(1.225, 77.552)$, $(3, -83)$, $(6, 8773)$



21. Domain: $x \in [-1, 2]$ Range: $y \in [-21, -12]$

Zeros: none y-int: $(0, -12)$ EB: none

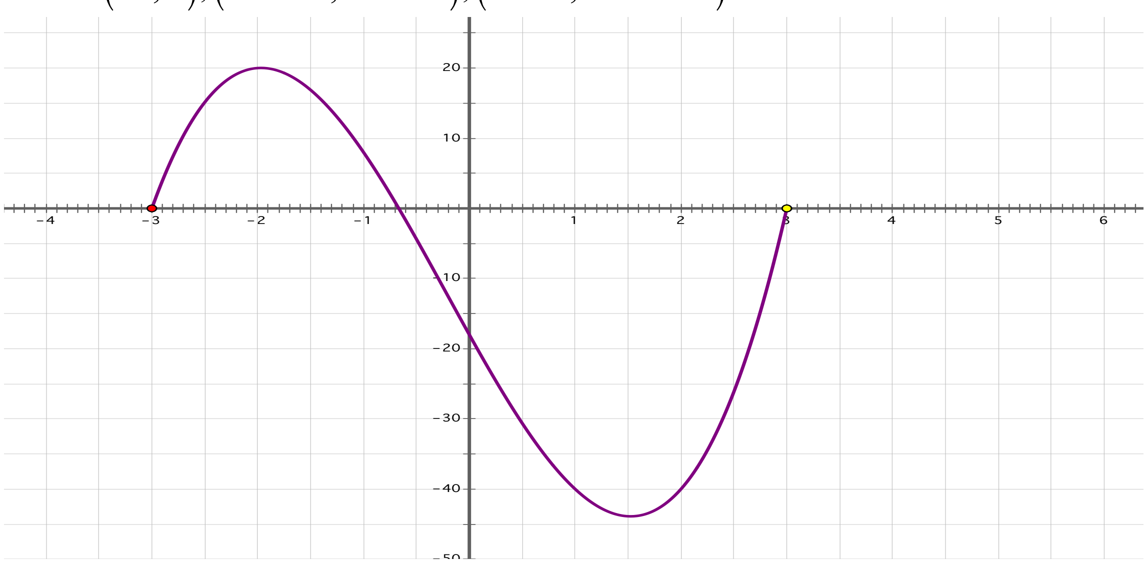
Extreme Points: $(-1, -17), (0, -12), (\sqrt{3}, -21), (2, -20)$

22. Domain: $x \in [-3, 3)$ Range: $y \in [-43.883, 20.016]$

Zeros: $\left(-\frac{2}{3}, 0\right), (-3, 0)$ y-int: $(0, -18)$

EB: none Extreme Points:

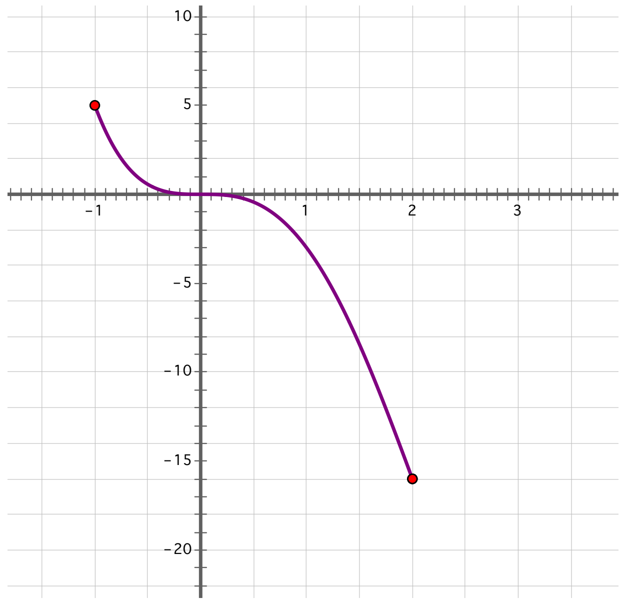
$(-3, 0), (-1.968, 20.016), (1.524, -43.884)$



23. Domain: $x \in [-1, 2]$ Range: $y \in [-16, 5]$

Zeros: $(0, 0)$ y-int: $(0, 0)$

EB: none Extreme Points: $(-1, 5), (2, -16)$



<u>6-4 Multiple Choice Homework</u>

1. C 2. D 3) D 4) D 5)

<u>6-5 Free Response Homework</u>

1a. At $x = -3$, there is a zero and a maximum
1b. At $x = 0$, there is above the x-axis and at a flat spot, but not an extreme.
1c. At $x = 1$, there is a zero and a minimum

2a. At $x = -3$, there is a zero and the curve is decreasing
2b. At $x = 2$, the curve is above the x-axis and at a maximum.
2c. At $x = 0$, there is a zero and the curve is increasing

3a. At $x = -\sqrt{2}$, there is a zero and the curve is increasing
3b. At $x = 2$, there is a zero and at a flat spot, but not an extreme
3c. At $x = 1$, the curve is increasing and above the x-axis.

4a. At $x = -2$, the curve is below the x-axis and is at a maximum
4b. At $x = 1$, the curve is below the x-axis and increasing
4c. At $x = 3$, there is a zero and the curve is decreasing

5. $x \in (8, \infty)$

6. $x \in (-\infty, 2)$

7. $x \in (-\infty, -2) \cup (1.535, 2)$

8. $x \in (0.359, 1.535)$

9. $x \in (0, 3)$

10. $x \in (-3, -1.144) \cup (3, \infty)$

11. $x \in (-\infty, -4) \cup (-1, 0)$

12. $x \in (-1, 0) \cup (1, 2)$

13. No interval

14. $x \in (3, 4)$

15. $t \in (2, 3.5) \cup (5, \infty)$

16. $t \in (2, \infty)$

17. $t \in (-4, -2) \cup \left(\frac{4}{3}, \frac{20}{9}\right) \cup (3, \infty)$

18. $t \in (-2.206, -1) \cup (0.399, 2) \cup (3, \infty)$

19. $t \in (-\infty, -\sqrt{3}) \cup \left(-\sqrt{\frac{3}{2}}, 0\right) \cup \left(\sqrt{\frac{3}{2}}, \sqrt{3}\right)$

20. $t \in (-\infty, 0) \cup (1, 2)$

21. $t \in \left(0, \sqrt{2}\right)$

22. $t \in (1, \infty)$

23. $t \in \left(0, \sqrt{2.5}\right) \cup (2, \infty)$

24. $t \in \left(-\sqrt{3}, 0\right) \cup \left(0, \sqrt{3}\right)$

<u>6-5 Multiple Choice Homework</u>

1. A 2. D 3. E 4. E 5. D

Polynomial Functions Practice Test Answer Key

Multiple Choice

1.	E	2.	E	3.	D	4.	E
5.	E	6.	E	7.	A		

Free Response

1. $x = -4$ is neither a maximum nor a minimum. $x = 4$ is at a maximum.

2a. $x = -2, \frac{1}{3}$ 2b. $x = -2, \frac{1}{3}$ 2c. 195 2d. 58 2e. Speeding up

3a. $C(r) = .12\pi r^2 + .06\pi rh$ 3b. $h = \frac{16}{r^2}$

3c. $C(r) = .12\pi r^2 + .96\pi r^{-1}$ 3d. $C(1.587) = \$2.86$

4a. $t \in (-1, 4)$ 4b. $t \in \left(\frac{5}{3}, \infty\right)$

5. $t \in (-\infty, -2)(4, \infty)$

6a. $x = -3$ is neither because the signs of $H'(x)$ do not change.

6b. $x = \frac{3}{4}$ is at a maximum because the signs of $H'(x)$ change from positive to negative.

6c. $x = 5$ is at a minimum because the signs of $H'(x)$ change from negative to positive.

7. $x \in (6, 8)$