Chapter 7:

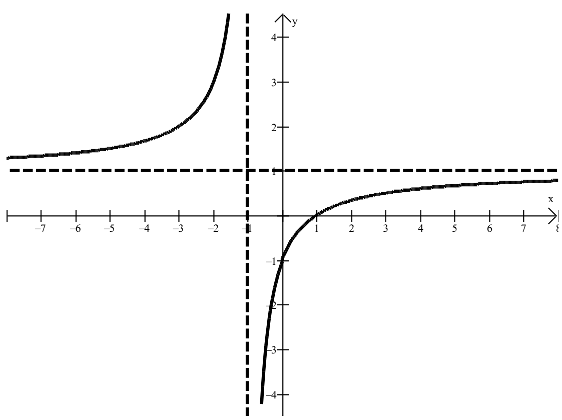
Rational Functions

Chapter 7 Overview: Types and Traits of Rational Functions

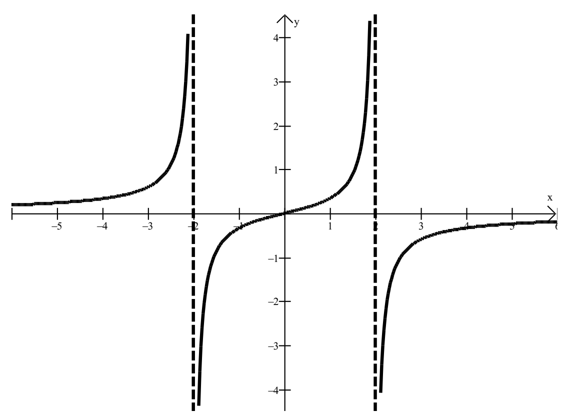
Vocabulary:
1. ***Rational Function*** – Defn: an expression that can be written as the ratio of one polynomial to another
Means: an equation with an *x* in the denominator

The General Rational Function is $y = \dfrac{A_n x^n + A_{n-1}x^{n-1}...}{B_m x^m + B_{m-1}x^{m-1}...}$.

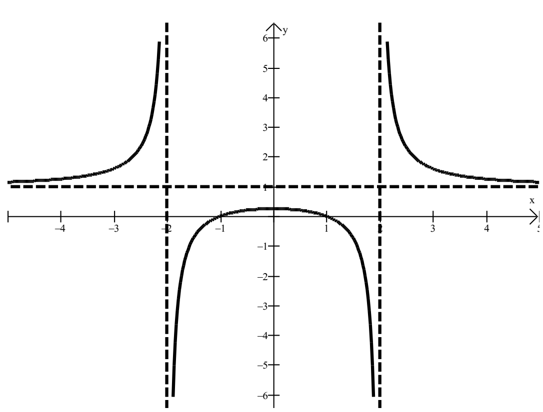
Unlike polynomials, rational functions are not grouped and classified at all, let alone by degree. There are a wide variety of generic rational graphs. The following are some:

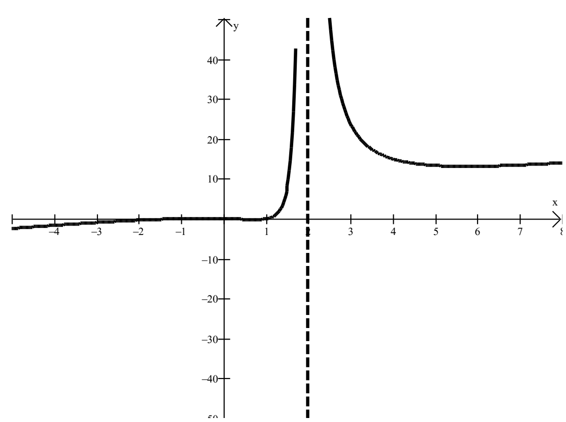


$$y = \frac{x-1}{x+1}$$



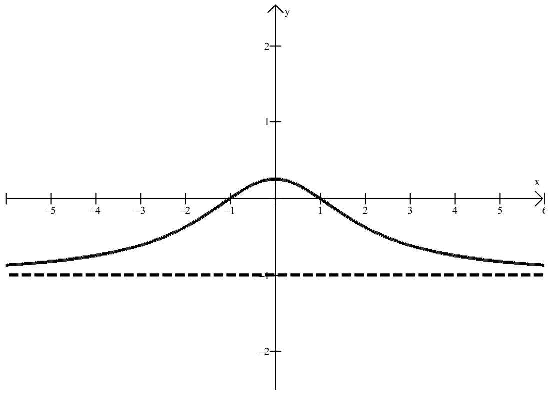
$$y = \frac{-x}{x^2-4}$$

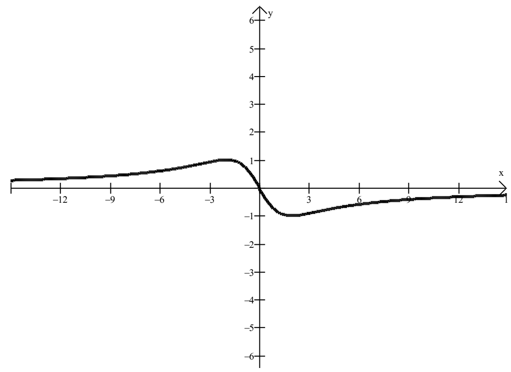




$$y=\frac{x^2-1}{x^2-4}$$

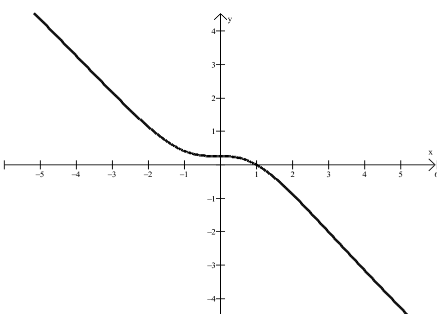
$$y=\frac{x^3-x}{x^2-4x+4}$$

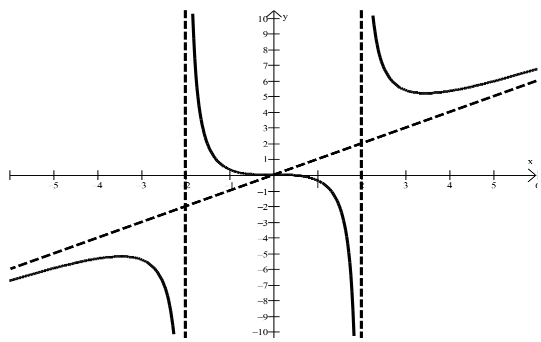




$$y=\frac{1-x^2}{x^2+4}$$

$$y=\frac{-4x}{x^2+4}$$





$$y=\frac{1-x^3}{x^2+4}$$

$$y=\frac{x^3}{x^2-4}$$

The traits are:

1. **Domain** – generally, the domain is all real numbers, except those where the denominator $= 0$

2. **Range** – varies greatly depending on the equation

3. ***y*-intercept** – the point where $x = 0$

4. **Zeros** – the points where $y = 0$; that is, where the numerator $= 0$, but the denominator $\neq 0$

5. **Vertical Asymptotes** – the *x*-value where the denominator $= 0$, but the numerator $\neq 0$

6. **Points of Exclusion** (a.k.a. removable discontinuities) – the points where both the numerator and denominator $= 0$

7. **End Behavior** – what the ends of the curve do; there are specific kinds of interest:
   a) **Horizontal Asymptote** – when the ends of the curve flatten out along a horizontal line
   b) **Slant Asymptote** – when the ends of the curve flatten out along a slanted line
   c) **End Behavior Asymptote** – when the ends of the curve approach a polynomial's end behavior

8. **Extreme Points** – the highest or lowest points in any section of the curve

Many rational traits are the same as the polynomial traits. The biggest distinction between polynomial and rational functions is that there are numbers for the *x* that cause the *y* to be a non-real number. This causes breaks in the curve, called discontinuities.

<u>An Important Note About Calculators</u>: The calculator is not infallible. It can only graph separate and distinct points. It cannot graph a point for every real number. The calculator sometimes will *<u>not</u>* show some of the necessary traits or will show things that are not part of the graph (like the vertical lines in the graphs above). Know what to expect and what not to expect of each graph and not just blindly copy what the machine shows. It may not be accurate!!!

7-1: Zeros, Vertical Asymptotes, and POEs

Zeros, vertical asymptotes, and points of exclusion (POEs) are easily confused because they all come from something equaling zero.

Vocabulary:

1. ***Indeterminate Form of a Number*** – Defn: a number for which further analysis is necessary to determine its value

   Means: the *y*-value equals $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, or some other strange thing

2. ***Transfinite Number*** – Defn: a number of the form $\frac{\text{non-zero}}{0}$

   Means: the denominator $=0$, but the numerator $\neq 0$

3. ***Vertical Asymptote**** – Defn: the line $x = a$ for which $\lim\limits_{x\to a^+} f(x) = \pm\infty$ or $\lim\limits_{x\to a^-} f(x) = \pm\infty$

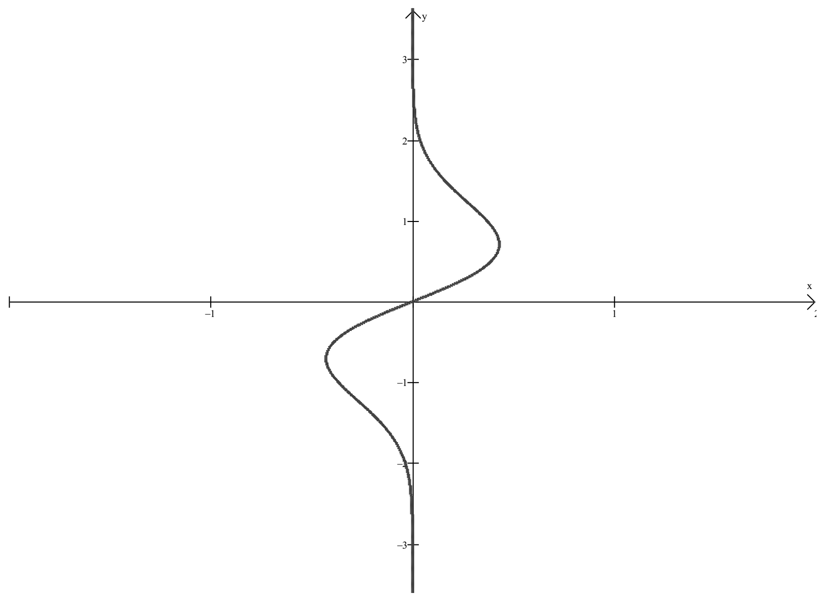
   Means: a vertical boundary line (sketched—not graphed—as a dotted line) on the graph at the *x*-value where the *y*-value equals $\pm\infty$

4. ***Point of Exclusion* (POE)*** – Defn: a number of the form $\frac{0}{0}$

   Means: the denominator and numerator $=0$

***The calculator may or may not show them, depending on the window used.**

In previous courses, it was stated that a curve could not cross an asymptote. This is not true, strictly speaking. A curve cannot cross a VERTICAL asymptote and still be a function. This is because of the "vertical" part, not the "asymptote" part. The curve can cross the vertical asymptote, but would have to bend back to it as the curve goes off the page.



Functions, by definition, cannot have more than one y-value for any x-value. This graph is not a function (it does not pass the Vertical Line Test). This does not mean the curve cannot cross the asymptote. It means that the graph is not that of a function if it does. A curve CAN cross a HORIZONTAL or SLANT asymptote and still be a function.

LEARNING OUTCOME
Find zeros, vertical asymptotes, and points of exclusion of a rational function, and distinguish them from one another.

EX 1 Find the zeros, vertical asymptotes, and points of exclusion (POEs) of $y=\frac{x+6}{x^2-4}$.

$$y=\frac{x+6}{(x-2)(x+2)}$$

Zero: $(-6,\ 0)$
VA: $x=2$ and $x=-2$
POE: None

As noted in the chapter overview, the domain of a rational function is generally all real numbers except where the denominator equals 0. Though not asked here, the domain of $y=\frac{x+6}{x^2-4}$ in EX 1 would be $x\neq\pm2$. In interval notation, that would be $x\in(-\infty,\ -2)\cup(-2,\ 2)\cup(2,\ \infty)$.

EX 2 Find the zeros, vertical asymptotes, and points of exclusion (POEs) of $y=\frac{x^3-x}{x^2-4x+4}$.

$$y=\frac{x(x-1)(x+1)}{(x-2)(x-2)}$$

Zeros: $(\pm1,\ 0)$ and $(0,\ 0)$
VA: $x=2$
POE: None

EX 3 Find the zeros, vertical asymptotes, and POEs of $y=\dfrac{x^2-1}{x^2-4x-5}$.

$$y=\frac{(x-1)(x+1)}{(x+1)(x-5)}$$

Zero: $(1,\ 0)$
VA: $x=5$
POE: $\left(-1,\ \dfrac{1}{3}\right)$

POE: $x=-1$, so $$y=\lim_{x\to-1}\frac{(x-1)(x+1)}{(x+1)(x-5)}$$
$$=\lim_{x\to-1}\frac{(x-1)}{(x-5)}$$
$$=\frac{-2}{-6}$$
$$=\frac{1}{3}$$

EX 4 Find the zeros, vertical asymptotes, and POEs of $y=\dfrac{x+1}{x^2-4x-5}$.

$$y=\frac{x+1}{(x-5)(x+1)}$$

Zeros: None
VA: $x=5$
POE: $\left(-1,\ -\dfrac{1}{6}\right)$

EX 5 Find the zeros, vertical asymptotes, and POEs of $y=\frac{2x^3-7x^2+2x+3}{x^3-3x^2+5x-15}$.

$$y=\frac{(2x+1)(x-1)(x-3)}{(x-3)(x^2+5)}$$

Zeros: $(1,\ 0)$ and $\left(-\frac{1}{2},\ 0\right)$
VA: None
POE: $(3,\ 1)$

EX 6 Find the zeros, vertical asymptotes, and POEs of $y=\frac{2x^3+x^2}{2x^2-5x-3}$.

$$y=\frac{x^2(2x+1)}{(2x+1)(x-3)}$$

Zero: $(0,\ 0)$
VA: $x=3$
POE: $\left(-\frac{1}{2},\ -\frac{1}{14}\right)$

Notice how the curve acts at $(0,\ 0)$. Instead of passing through the x-axis, the curve hits the axis and goes back down. Algebraically, x is a factor of the numerator twice.

EX 7 Find the zeros, vertical asymptotes, and POEs of $y=\frac{7x-1}{x^2-2x-3}-\frac{6x}{x^2-x-2}$

.

First, we need to make a single fraction, by creating a least common denominator:

$$\begin{aligned}
y&=\frac{7x-1}{(x+1)(x-3)}-\frac{6x}{(x+1)(x-2)}\\
&=\frac{(7x-1)(x-2)}{(x+1)(x-3)(x-2)}-\frac{6x(x-3)}{(x+1)(x-2)(x-3)}\\
&=\frac{(7x-1)(x-2)-6x(x-3)}{(x+1)(x-3)(x-2)}\\
&=\frac{7x^2-15x+2-6x^2+18x}{(x+1)(x-3)(x-2)}\\
&=\frac{x^2+3x+2}{(x+1)(x-3)(x-2)}\\
&=\frac{(x+1)(x+2)}{(x+1)(x-3)(x-2)}
\end{aligned}$$

Zero: $(-2,\ 0)$
VA: $x=2,\ 3$
POE: $\left(-1,\ \frac{1}{12}\right)$

7-1 Free Response Homework

Find the zeros, vertical asymptotes, and points of exclusion (POEs).

1. $y=\frac{(x+2)(x-3)}{(x-2)(x-3)(x-1)}$

2. $y=\frac{(x+3)(x-1)}{(x-2)(x+3)(x+1)}$

3. $y=\frac{5}{x^2-2x-8}$

4. $y=\frac{5}{x^4-x^2-20}$

5. $y=\frac{x^2+2x-15}{x+3}$

6. $y=\frac{x^2+2x-15}{x-3}$

7. $y=\frac{x^2+x-30}{x^3+x^2-x-1}$

8. $y=\frac{6x^2-x-1}{x^3+3x^2-6x-8}$

9. $y=\frac{x^3+2x^2-6x-4}{x^2+x-6}$

10. $y=\frac{x^2+x-6}{x^3+2x^2-6x-4}$

11. $y=\frac{x^3-2x}{x^4-5x^2+6}$

12. $y=\frac{x^3+2x^2-5x-6}{x^2-4x+3}$

13. $y=\frac{x^3+x^2-x-1}{x^2+x-30}$

14. $y=\frac{x^4+x^2-20}{2x^3-5x^2-19x+42}$

15. $y=\frac{x+3}{x^2+x-12}$

16. $y=\frac{x^3-5x^2-2x+24}{2x^2+7x-15}$

17. $y=\frac{2x}{x+2}-\frac{x-1}{x+4}$

18. $y=\frac{x}{x+2}-\frac{x+1}{x+3}$

19. $y=\frac{x}{x-3}-\frac{7}{x+5}-\frac{24}{x^2+2x-15}$

20. $y=\frac{4x}{x-1}-\frac{5x}{x-2}-\frac{2}{x^2-3x+2}$

21. $y=\frac{3x}{x+4}+\frac{4x}{x-3}-\frac{84}{x^2+x-12}$

7-1 Multiple Choice Homework

1. Which of the following functions has a zero at $(-3,\ 0)$, a POE at $\left(4,\ \frac{7}{20}\right)$, and no vertical asymptotes?

a) $y=\frac{x+3}{x^2+1}$ b) $y=\frac{x-3}{x^3-4x^2+4x-16}$ c) $y=\frac{x^2-x-12}{x^3-4x^2+4x-16}$

d) $y=\frac{x^2-x-12}{x^3-4x^2-4x+16}$ e) $y=\frac{x-4}{x^2+1}$

---

2. Which of the following functions has a vertical asymptote at $x=-1$ and a POE at $(2,\ 7)$?

a) $y=\frac{x+1}{x^2-x-2}$ b) $y=\frac{x-2}{x^2-x-2}$ c) $y=\frac{21x+42}{x^2-x-2}$

d) $y=\frac{21x-42}{x^2-x-2}$ e) $y=\frac{x^2-9x+14}{x^3-8x^2+5x+14}$

---

3. Which of the following has a vertical asymptote at $x=0$, a POE at $\left(-1,\ \frac{13}{7}\right)$, and a zero at $(12,\ 0)$?

a) $y=\frac{x^2-11x-12}{x^2+x}$ b) $y=\frac{x^2+11x-12}{x^2+x}$ c) $y=\frac{x^2-11x-12}{4x^2+4x}$

d) $y=\frac{x^2+11x-12}{7x^2+7x}$ e) $y=\frac{x^2-11x-12}{7x^2+7x}$

---

4. Which of the following is true about the graph of $f(x)=\frac{2x^2-7x+3}{x^2-5x+6}$?

a) $f(x)$ has a zero at $\left(\frac{1}{2},0\right)$, VA at $x=2$, and POE at $x=3$.

b) $f(x)$ has a zero at $(2,0)$, VA at $x=\frac{1}{2}$, and POE at $x=3$.

c) $f(x)$ has a zero at $\left(\frac{1}{2},0\right)$ and $(3,0)$, VAs at $x=2$ and $x=3$, and no POE.

d) $f(x)$ has a zero at $\left(\frac{1}{2},0\right)$ and $(3,0)$, VAs at $x=2$ and $x=3$, and POE at $x=3$.

e) $f(x)$ has a zero at $\left(\frac{1}{2},0\right)$ and $(3,0)$, VAs at $x=2$, and no POE.

---

5. Which of the following is true about the graph of $f(x)=\frac{x^2+7x+10}{x^3-2x^2+9x-18}$?

a) $f(x)$ has a zero at $(-5,0)$, VA at $x=\pm3$, and POE at $x=2$.

b) $f(x)$ has a zeros at $(-5,0)$ and $(-2,0)$, VA at $x=2$ and $x=\pm3$, and no POE.

c) $f(x)$ has a zeros at $(-5,0)$ and $(-2,0)$, VA at $x=2$, and no POE.

d) $f(x)$ has a zeros at $(-5,0)$, no VAs at, and POE at $x=-2$.

e) $f(x)$ has a zeros at $(-5,0)$ and $(-2,0)$, VA at $x=2$ and $x=\pm3$, and no POE.

---

6. Which of the following is true about the graph of $f(x)=\frac{x^2-7x+10}{x^3-2x^2-9x+18}$?

a) $f(x)$ has a zero at $(5,0)$, VA at $x=\pm3$, and POE at $x=2$.

b) $f(x)$ has a zeros at $(5,0)$ and $(-2,0)$, VA at $x=2$ and $x=\pm3$, and no POE.

c) $f(x)$ has a zeros at $(5,0)$ and $(-2,0)$, VA at $x=2$, and no POE.

d) $f(x)$ has a zeros at $(5,0)$, no VAs at, and POE at $x=-2$.

e) $f(x)$ has a zeros at $(5,0)$ and $(-2,0)$, VA at $x=2$ and $x=\pm3$, and no POE.

7-2: Sign Patterns and Rational Functions

Sign patterns of rational functions are very similar to polynomial sign patterns. The difference is that***, the sign pattern of a Rational Function must include both VAs and zeros***. Remember, though, that the VA and POE numbers are not in the domain, so it cannot be part of the solution set.

LEARNING OUTCOMES
Use sign patterns to solve rational inequalities. Apply sign patterns to velocity. Apply sign patterns to the first derivative.

EX 1 Find a rational inequality with this given sign pattern and solution:

$$\begin{array}{c} y \\ x \end{array} \begin{array}{c} + \quad VA \quad - \quad 0 \quad + \quad DNE \quad + \quad 0 \quad - \\ \longleftarrow\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\longrightarrow \\ -1 \qquad\quad 1 \qquad\quad 3 \qquad\quad 5 \end{array} ;\ x\in(-\infty,\ -1)\cup[1,\ 3)\cup(3,\ 5]$$

There are zeros at $x=1$ and $x=5$.

At $x=-1$, there is a VA because y does not exist and the signs change.

At $x=3$, there is either a POE or a repeated VA because y does not exist and the signs do not change.

The sign on the right is negative, so the rational function is negative.

Therefore, one possible rational function is

$$y=-\frac{(x-1)(x-5)(x-3)}{(x+1)(x-3)} \qquad \text{or} \qquad y=-\frac{(x-1)(x-5)}{(x+1)(x-3)^2}$$

The solution intervals indicate that the zeros and positives are the solution. Therefore,

$$-\frac{(x-1)(x-5)(x-3)}{(x+1)(x-3)} \geq 0 \qquad \text{or} \qquad -\frac{(x-1)(x-5)}{(x+1)(x-3)^2} \geq 0$$

EX 2 Solve $\frac{x-3}{x+1} \geq 0$

Sign Pattern:

y	+	VA	−	0	+
x		-1		3	

≥ 0 is + and 0, so $x \in (-\infty, -1) \cup [3, \infty)$

EX 3 Solve $\frac{2x^2+3x-2}{3x+1} < 0$

$$\frac{2x^2+3x-2}{3x+1} = \frac{(2x-1)(x+2)}{3x+1} < 0$$

Sign Pattern:

y	−	0	+	VA	−	0	+
x		-2		$-\frac{1}{3}$		$\frac{1}{2}$	

< 0 is −, so $x \in (-\infty, -2) \cup \left(-\frac{1}{3}, \frac{1}{2}\right)$

EX 4 Solve $\frac{x-3}{x+1} \le 3$

Remember that the inequality must be compared to 0 in order for the sign pattern to be interpreted as a solution to the inequality. In this case, some numbers less than 3 are positive and some are negative. Therefore, the 3 must move over to the left of the inequality.

$$\frac{x-3}{x+1} \le 3 \quad \rightarrow \quad \frac{x-3}{x+1} - 3 \le 0$$

We also know from the last section that there needs to be a single fraction in order to identify the zeros, vertical asymptotes and points of exclusion.

$$\frac{x-3}{x+1} - \frac{3(x+1)}{x+1} \le 0$$
$$\frac{-2x-6}{x+1} \le 0$$

Sign Pattern:

$$\begin{array}{c|ccccc} y & - & 0 & + & \text{DNE} & - \\ x & & -3 & & -1 & \end{array}$$

≤ 0 is – and 0, so $x \in (-\infty, -3] \cup (-1, \infty)$

Remember:

A sign pattern is a tool, not a solution. There are different uses for sign patterns, depending on where the signs came from. The uses seen so far are:

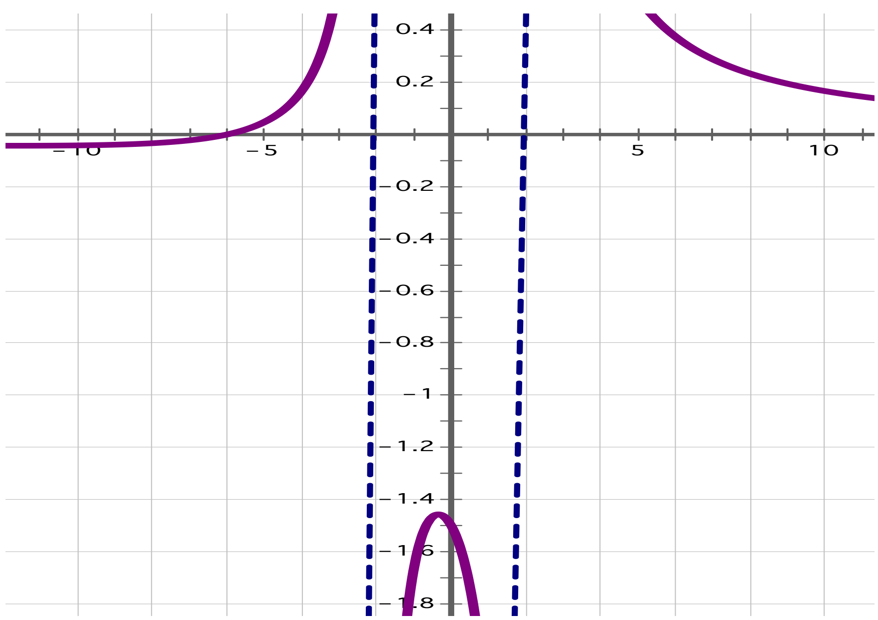
I. The sign patterns of y shows where a function is above or below the x-axis and help solve inequalities.

II. The sign patterns of velocity determine which direction an object is moving.

III. The sign patterns of $\frac{dy}{dx}$ shows where a function is decreasing or increasing, and determines whether a critical value is at a maximum point, minimum point or neither.

EX 5 Use the sign pattern of $y=\frac{x+6}{(x-2)(x+2)}$ to find a rough sketch of the graph.

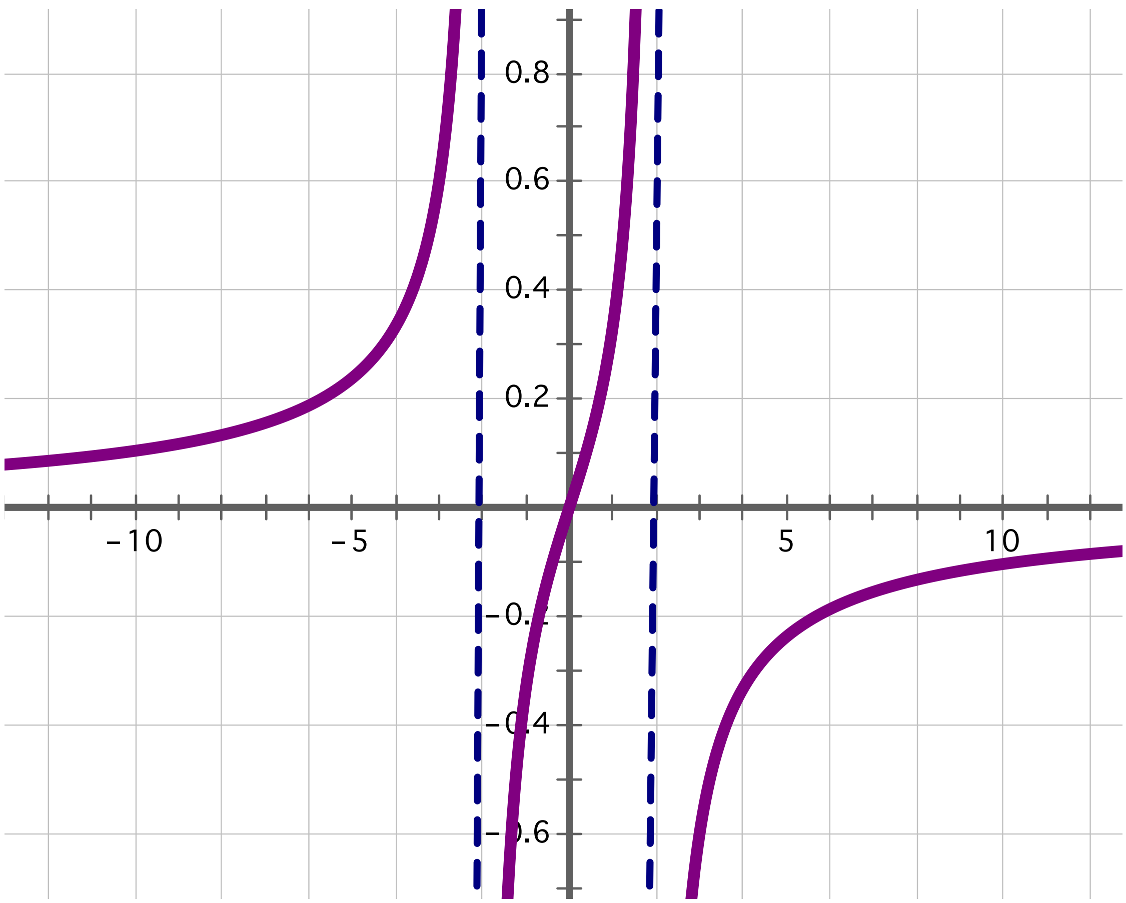
$f(x)$	$-$	0	$+$	VA	$-$	VA	$+$
x		-6		-2		2	



EX 6 Use the sign pattern of $y=\dfrac{-4x}{x^2-4}$ to find a rough sketch of the graph.

Zero: $(0,\ 0)$ VA: $x=2$ and $x=-2$ POE: None

$$\begin{array}{c|ccccccc} f(x) & + & \text{VA} & - & 0 & + & \text{VA} & - \\ x & & -2 & & 0 & & 2 & \end{array}$$

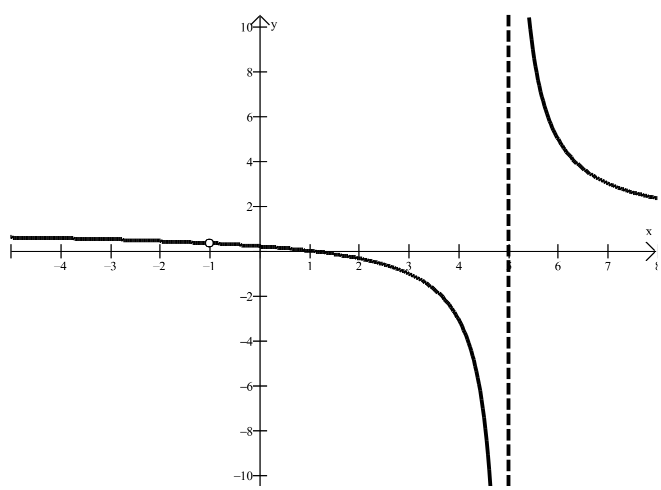


EX 7 Use the sign pattern of $y = \dfrac{x^2 - 1}{x^2 - 4x - 5}$ to find a rough sketch of the graph.

As was seen in Ex 3 in a previous section, $y = \dfrac{(x-1)(x+1)}{(x-5)(x+1)}$ and

Zero: $(1,\ 0)$ VA: $x = 5$ POE: $\left(-1,\ \dfrac{1}{3}\right)$

$$\begin{array}{c|ccccccc} f(x) & + & \text{POE} & + & 0 & - & VA & + \\ \hline x & & -1 & & 1 & & 5 & \end{array}$$

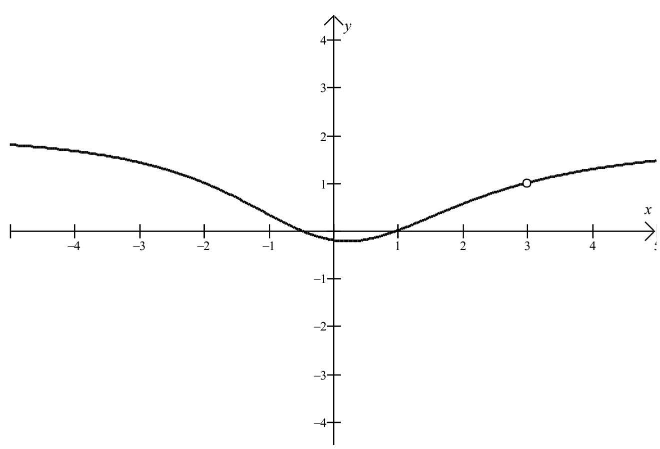


EX 8 Use the sign pattern of $y=\dfrac{2x^3-7x^2+2x+3}{x^3-3x^2+5x-15}$ to find a rough sketch of the graph.

As was seen in Ex 5 in a previous section, $y=\dfrac{(2x+1)(x-1)(x-3)}{(x-3)(x^2+5)}$ and

Zeros: $(1,\ 0)$ and $\left(-\dfrac{1}{2},\ 0\right)$ VA: None POE: $(3,\ 1)$

$$\begin{array}{lccccccccc} f(x) & + & 0 & - & 0 & + & poe & + \\ x & & -\tfrac{1}{2} & & 1 & & 3 & \end{array}$$



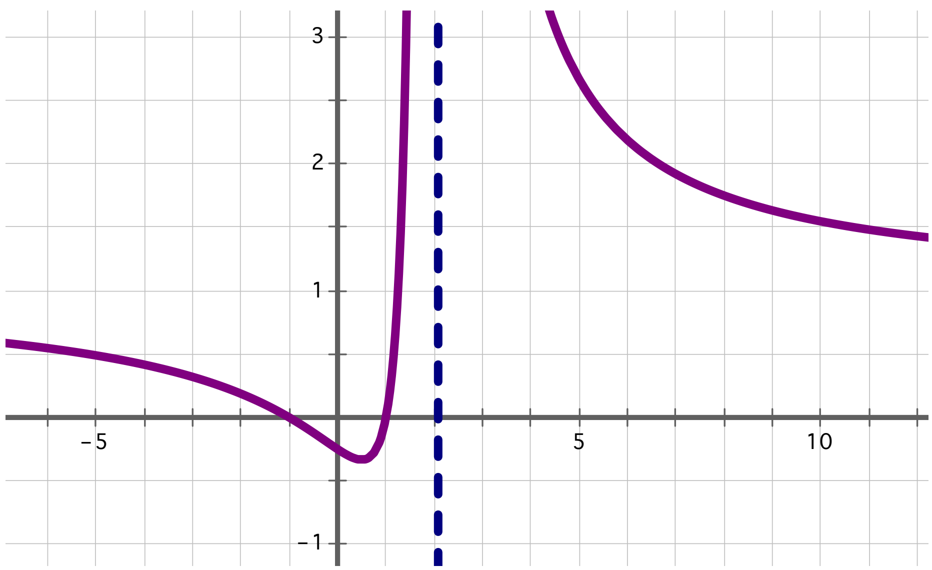
EX 9 Use the sign pattern of $y=\dfrac{x^2-1}{x^2-4x+4}$ to find a rough sketch of the graph.

$$y=\frac{x^2-1}{x^2-4x-4}=\frac{(x-1)(x-1)}{(x-2)(x-2)}=\frac{(x-1)(x-1)}{(x-2)^2}$$

Zeros: $(\pm 1,\ 0)$ VA: $x=2$ POE: None

\

$f(x)$	+	0	−	0	+	VA	+	
x		-1		1		2		



Notice what having two of the same factor in the denominator does to the curve: The curve goes up on both sides of the asymptote. Some calculators do not draw the vertical line in although **<u>it is there!!!</u>**

7-2 Free Response Homework

Find a rational inequality that would match the given sign pattern and solution.

1. $\begin{array}{c|c} y & - \quad VA \quad + \quad 0 \quad - \\ x & -1 \qquad 4 \end{array}$; $x \in (-1,\ 4)$

2. $\begin{array}{c|c} y & + \quad VA \quad - \quad 0 \quad + \quad VA \quad - \\ x & -\sqrt{2} \qquad 0 \qquad \sqrt{2} \end{array}$; $x \in \left(-\infty,\ -\sqrt{2}\right) \cup \left[0,\ \sqrt{2}\right)$

3. $\begin{array}{c|c} y & + \quad 0 \quad - \quad VA \quad - \quad 0 \quad + \quad POE \quad + \\ x & -\sqrt{6} \qquad -1 \qquad 2 \qquad \sqrt{6} \end{array}$;

$x \in \left(-\infty, -\sqrt{6}\right) \cup \left(2,\ \sqrt{6}\right) \cup \left(\sqrt{6},\ \infty\right)$

4. $\begin{array}{c|c} y & + \quad VA \quad - \quad 0 \quad - \quad 0 \quad + \\ x & -5 \qquad 0 \qquad 5 \end{array}$; $x \in (-\infty,\ -5) \cup [5,\ \infty)$ or $\{0\}$

5. $\begin{array}{c|c} y & + \quad VA \quad + \quad VA \quad - \quad 0 \quad + \\ x & -3 \qquad 4 \qquad 5 \end{array}$; $x \in (-\infty, -3) \cup (-3,\ 4) \cup (5,\ \infty)$

6. $\begin{array}{c|c} y & - \quad 0 \quad + \quad \text{VA} \quad + \quad 0 \quad - \\ x & -1/3 \qquad 2/3 \qquad 4/3 \end{array}$; $x \in \left[-1/3,\ 2/3\right) \cup \left(2/3,\ 4/3\right]$

Solve these inequalities.

7. $\dfrac{x^2+2x-3}{x+1} \le 0$

8. $\dfrac{x-3}{x^2+3x-4} > 0$

9. $\dfrac{x^2-4}{x^3-1} \ge 0$

10. $\dfrac{x-4}{x+2} \ge 5$

11. $\dfrac{x-7}{x^2+x-2} < \dfrac{8x}{x^2-6x+5}$

12. $\dfrac{2x-5}{x-3} \geq 3 + \dfrac{3}{2x^2-7x+3}$

13. $\dfrac{2x^2-1}{x^3+1} + \dfrac{x+2}{x^2-x+1} \geq \dfrac{7}{x+1}$

14. $\dfrac{2x^2-5}{x^2-4} - 1 > \dfrac{x^2+7}{x^2+1}$

Find and use the sign pattern of each equation below to find a rough sketch of the graph.

15. $y = \dfrac{x+3}{x^2+x-12}$

16. $y = \dfrac{x^2-1}{x^2-4x-4}$

17. $f(x) = \dfrac{3x^2-11x-4}{16-x^2}$

18. $y = \dfrac{x^2+x-6}{x^3+2x^2-6x-4}$

19. $y = \dfrac{x^2+x-30}{x^3+x^2-x-1}$

20. $y = \dfrac{6x^2-x-1}{x^3+3x^2-6x-8}$

21. $y = \dfrac{x^3-2x}{x^4-5x^2+6}$

22. $y = \dfrac{16x-12x^2}{3x^3-4x^2+6x-8}$

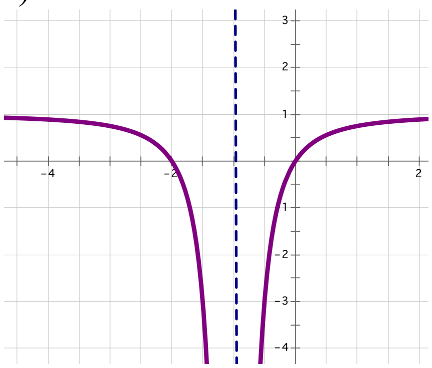
____

7-2 Multiple Choice Homework

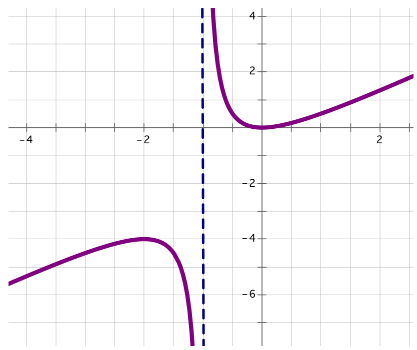
1. Determine which of the following graphs matches the sign pattern

$$\begin{array}{ll} y & \quad - \;\; 0 \;\; + \;\; \text{VA} \;\; + \;\; 0 \;\; - \\ & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\longrightarrow \\ x & \quad -2 \qquad -1 \qquad 0 \end{array}$$

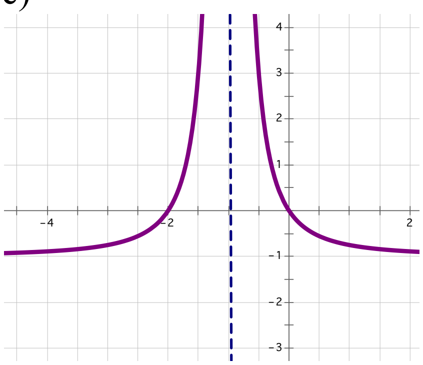
a)



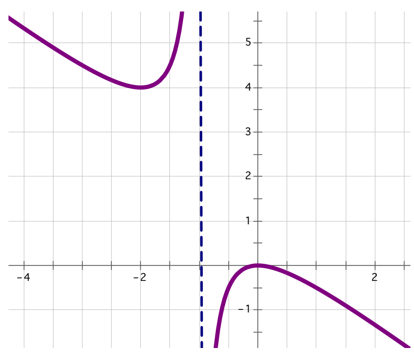
b)



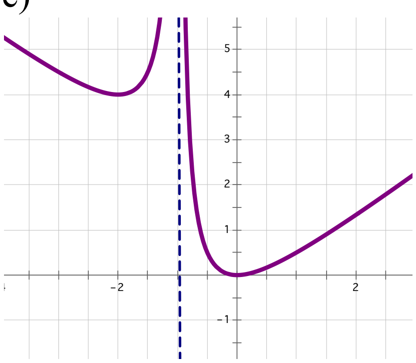
c)



d)



e)



2. Determine the sign pattern for $y = \dfrac{-4x^3 + 4x^2}{x-2}$.

a)

y	$-$	0	$+$	0	$-$	0	$+$
x		-1		0		2	

b)

y	$+$	0	$-$	0	$-$	VA	$+$
x		-1		0		2	

c)

y	$+$	VA	$-$	0	$+$	VA	$-$
x		-1		0		2	

d)

y	$-$	0	$+$	0	$+$	VA	$-$
x		-1		0		2	

e)

y	$+$	0	$+$	0	$+$	VA	$+$
x		-1		0		2	

---

3. Given this sign pattern

y	$-$	DNE	$+$	0	$-$	0	$-$
x		-3		$\frac{2}{3}$		2	

and the solution $x = (-\infty, -3)$ or $\left[\frac{2}{3}, \infty\right)$, which of the following **might** be the inequality?

a) $\dfrac{(3x+2)(x-2)^2}{(x+3)} \le 0$

b) $-\dfrac{(3x+2)(x-2)^2}{(x+3)} \ge 0$

c) $-\dfrac{(3x-2)(x-2)^2}{(x+3)} \le 0$

d) $-\dfrac{(3x-2)(x-2)^2}{(x+3)} < 0$

e) $-\dfrac{(3x-2)(x-2)^2}{(x+3)} \ge 0$

---

4. Given this sign pattern

$$\begin{array}{c|ccccccccc} y & + & \text{DNE} & + & 0 & - & 0 & + \\ \hline x & & -3 & & \frac{2}{3} & & 2 & \end{array}$$

and the solution $x = (-\infty, -3) \cup \left(-3, \frac{2}{3}\right]$ or $[2, \infty)$, which of the following **might** be the inequality?

a) $\frac{(3x+2)(x-2)}{(x+3)^2} \geq 0$

b) $-\frac{(3x+2)(x-2)}{(x+3)^2} \geq 0$

c) $\frac{(3x-2)^2(x-2)}{(x+3)^2} \leq 0$

d) $\frac{(3x-2)(x-2)}{(x+3)^2} < 0$

e) $\frac{(3x-2)^2(x-2)^2}{(x+3)^2} \geq 0$

---

5. Given this sign pattern

$$\begin{array}{c|cccccccc} f(x) & - & 0 & + & \text{DNE} & - & 0 & - \\ \hline x & & -4 & & -1 & & 2 & \end{array}$$

, which of the following **might** be the equation of $f(x)$?

a) $f(x) = \frac{(x+4)(x-2)}{(x+1)}$

b) $f(x) = -\frac{(x+4)(x-2)^2}{(x+1)}$

c) $f(x) = -\frac{(x+4)(x-2)}{(x+1)^2}$

d) $f(x) = -\frac{(x+4)(x-2)^2}{(x+1)^2}$

e) $f(x) = \frac{(x+4)^3(x-2)^4}{(x+1)}$

7-3: End Behavior: Horizontal Asymptotes

LEARNING OUTCOME

Determine the end behavior of a rational function from a model, polynomial long division, or infinite limits.

As previously stated, end behavior is what the curve does at the right and left ends of the *x*-axis, if the curve exists there. For polynomials, this was stated very broadly because the goal was to sketch, not find an exact graph. End behavior of rational functions requires more precision. First, consider end behavior algebraically:

Vocabulary:

1. ***End Behavior*** – Defn: a physical description of the $\lim_{x\to\pm\infty} f(x)$

 Means: what the *y*-value becomes as *x* gets numerically larger

End behavior is based on the mathematical idea that a fraction with an infinitely large denominator is an infinitely small number, ultimately approaching zero. In other words, $\frac{\text{non-zero}}{\infty}=0$. Previously, a transfinite number was defined as $\frac{\text{non-zero}}{0}$, so, algebraically, these statements are logically equivalent. A real mathematician would be appalled by these statements (since ∞ is not a real number, we cannot really talk about division), but they are essentially true.

EX 1 Find the end behavior of $y = \dfrac{x^2+1}{x^3+x^2}$.

The domain of this function is $x \in (-\infty,\ -1) \cup (-1,\ 0) \cup (0,\ \infty)$. Since both $\pm\infty$ are in the domain, consider the limit as *y* goes to $+\infty$ and $-\infty$.

$$\lim_{x \to \pm\infty} \frac{x^2+1}{x^3+x^2} = 0$$

Therefore, the end behavior is that the curve flattens along the line $y = 0$.

EX 2 Find the end behavior of $y = \dfrac{1-3x^2}{x^2+4}$.

The domain of this function is $x \in \mathbb{R} \Leftrightarrow x \in (-\infty,\ \infty)$. Since both $\pm\infty$ are in the domain, consider the limit as *y* goes to $+\infty$ and $-\infty$.

$$\lim_{x \to \pm\infty} \frac{1-3x^2}{x^2+4} = -3$$

Therefore, the end behavior is that the curve flattens along the line $y = -3$.

End Behavior Model (EBM):

For $y = \dfrac{A_n x^n + A_{n-1} x^{n-1} \ldots}{B_m x^m + B_{m-1} x^{m-1} \ldots}$, the EBM is $y = \dfrac{A_n x^n}{B_m x^m}$.

Implied here is that, as *x* gets larger and larger, the lower degree terms become negligible by comparison to the highest degree terms.

EX 3 Find the EBM of a) $y=\frac{x^5+3x^4-3x^2+1}{2x^3+4x^2-4x+7}$ and b) $y=\frac{-3x^6+2x^4-x^2+1}{4x^3+x^2-4x+12}$

a) The EBM is $y=\frac{x^5}{2x^3}=\frac{1}{2}x^2$. In other words, this curve does what $y=\frac{1}{2}x^2$ does—it goes up on both ends.

b) The EBM is $y=\frac{-3x^6}{4x^3}=\frac{-3}{4}x^3$. In other words, this curve goes up on the left and down on the right.

Vocabulary:
2. ***Horizontal Asymptote*** – the horizontal line that serves as the boundary for a curve as *x* becomes infinitely large

As can be seen from the End Behavior Models above, if the degrees of the numerator and denominator are equal, the *x*'s will cancel and there will be a horizontal asymptote at the fraction formed by the degree coefficients. Similarly, if the denominator degree is higher, the fraction will get smaller and smaller as the *x*-value increases, because *x*'s in the numerator will reduce to 1, while the denominator will still have an *x*. The numerator of the EBM will be a constant but the denominator will continue to grow. The fraction will tend toward 0. This leads to the following rule:

General Rule for Horizontal Asymptotes:

For $y=\frac{A_nx^n+A_{n-1}x^{n-1}\ldots}{B_mx^m+B_{m-1}x^{m-1}\ldots}$,

if $n<m$, there is a horizontal asymptote and it is $y=0$;

if $n=m$, there is a horizontal asymptote and it is $y=\frac{A_n}{B_m}$;

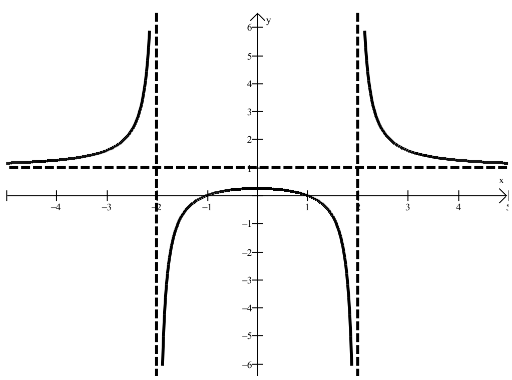
and if $n>m$, there is no horizontal asymptote.

EX 4 Find all the asymptotes of $y=\dfrac{x^2-1}{x^2-4}$.

The domain of this function is $x\in(-\infty,\ -2)\cup(-2,\ 2)\cup(2,\ \infty)$. Since both $\pm\infty$ are in the domain, consider the limit as y goes to $+\infty$ and $-\infty$.

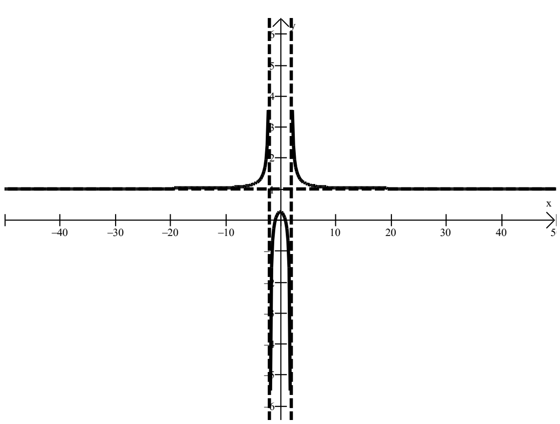
$$y=\frac{x^2-1}{x^2-4}=\frac{(x-1)(x+1)}{(x-2)(x+2)} \qquad \lim_{x\to\pm\infty}\frac{x^2-1}{x^2-4}=1$$

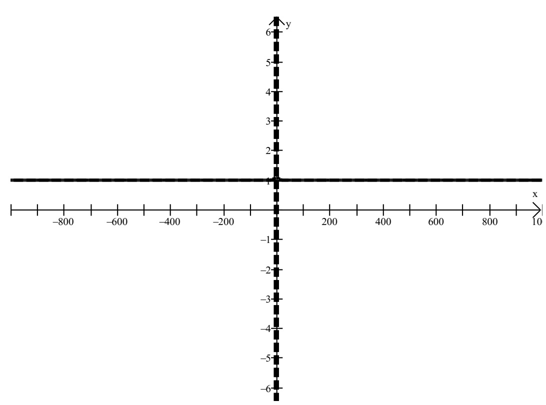
Vertical Asymptote (VA): $x=2$ and $x=-2$
Horizontal Asymptote (HA): $y=\dfrac{A_0}{B_0}=\dfrac{1}{1}=1$



$$y=\frac{x^2-1}{x^2-4}$$

Zoom out on the graph:





Zoom far enough out and what is seen is just the horizontal asymptote.

EX 5 Find all the asymptotes of $y=\dfrac{3x^2+5x-2}{6x^3-11x^2-3x+2}$.

$$y=\frac{(3x-1)(x+2)}{(3x-1)(x-2)(2x+1)}$$

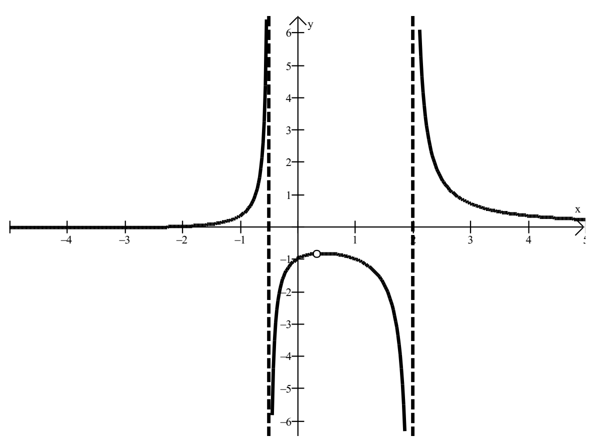
Domain: $x\in\left(-\infty,\ -\frac{1}{2}\right)\cup\left(-\frac{1}{2},\ \frac{1}{3}\right)\cup\left(\frac{1}{3},\ 2\right)\cup(2,\ \infty)\Leftrightarrow x\neq-\frac{1}{2},\ \frac{1}{3},\ 2$

$$\lim_{x\to\pm\infty}\frac{3x^2+5x-2}{6x^3-11x^2-3x+2}=0$$

HA: $y=0$

VA: $x=2$ and $x=-\dfrac{1}{2}$

Note that $x=\dfrac{1}{3}$ is not a VA; it is a POE. (Again, the calculator does not show the POE because of the window used.)



$$y=\frac{3x^2+5x-2}{6x^3-11x^2-3x+2}$$

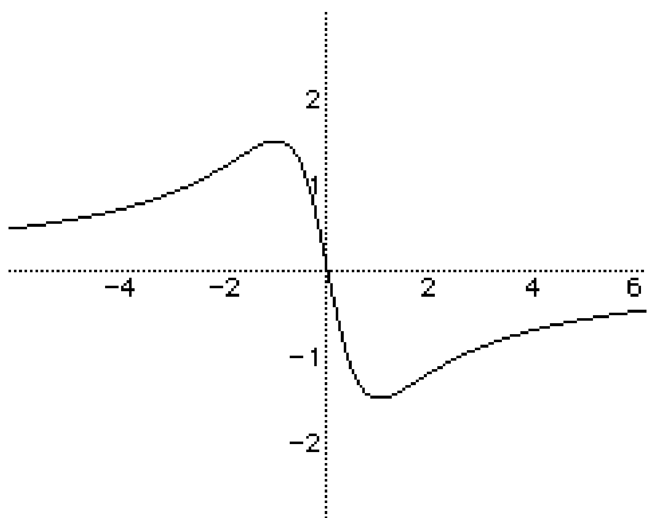
Another thing to notice about the graph in this example is that the curve crosses the horizontal asymptote. It is hard to see, but there is a zero at $x=-2$ despite the fact that $y=0$ is an asymptote. Contrary to what is known about vertical asymptotes, **the curve CAN cross a horizontal or slant asymptote.**

EX 6 Find all the asymptotes of $y=\frac{-3x}{x^2+1}$.

Domain: $x \in (-\infty,\ \infty)$

VA: None HA: $y=0$

Note the graph:



$$y=\frac{-3x}{x^2+1}$$

The curve crosses the HA and then bends back to follow it at the ends.

7-3 Free Response Homework

Evaluate each limit.

1. $\lim_{x\to\infty} \frac{x^2+7x-2}{3x^3-11x^2-5x+2}$

2. $\lim_{x\to\infty} \frac{3x^2-2x-21}{x^2-x+4}$

3. $\lim_{x\to\infty} \frac{(x-4)(x+1)}{(x+5)(x-4)(4x+1)}$

4. $\lim_{x\to\infty} \frac{6x^3-3x^2+5x-2}{2x^2-3x+2}$

5. $\lim_{x\to-\infty} \frac{x^2-6x+2}{x^2-4}$

6. $\lim_{x\to-\infty} \frac{x^3-8x^2+20x-14}{x^2-4x+4}$

Find the zeros, vertical asymptotes, POEs, and end behavior.

7. $y=\frac{2}{x^2-2}$

8. $y=\frac{x+1}{x^3+3x^2-6x-8}$

9. $y=\frac{24}{x^2+2x-15}$

10. $y=\frac{x^2+x-6}{x^3+2x^2-6x-4}$

11. $y=\frac{x+3}{x^2+x-12}$

12. $y=\frac{x^2-4x+3}{x^2-x-6}$

13. $y=\frac{(x+2)(x-3)}{(x-2)(x-3)(x-1)}$

14. $y=\frac{x^3-8}{x^4-13x^2+36}$

15. $y=\frac{x^2-1}{x^2+2x-3}$

16. $y=\frac{16-x^2}{x^2+9}$

17. $f(x)=\frac{3+2x-x^2}{x^2-9}$

18. $y=\frac{2x^3-7x^2-2x+7}{x^3+2x^2-9x-18}$

19. $y=\frac{3x^2-12x}{x^3-4x^2+9x-36}$

20. $y=\frac{(x-1)(x+2)(2x-3)}{(x+2)(x-2)(x+1)}$

7-3 Multiple Choice Homework

1. If the graph of $y = \frac{ax+b}{x+c}$ has a horizontal asymptote $y = 2$ and vertical asymptote $x = -3$, then $a + c =$

a) –5 b) –1 c) 0 d) 1 e) 5

---

2. $\lim_{x\to\infty} \frac{4x^5 + 3x^4 + 2x^3 + x^2 + 1}{3x^6 - 9x^4 + 4x^3 + 15} =$

a) 0 b) $\frac{3}{4}$ c) $\frac{4}{3}$ d) 3 e) DNE

---

3. $\lim_{n\to\infty} \frac{4n^2}{n^2 + 10{,}000n}$

a) 0 b) $\frac{1}{2500}$ c) 1 d) 4 e) DNE

---

4. For $x > 0$, the horizontal line $y = 2$ is an asymptote for the graph of the function *f*. Which of the following statements must be true?

a) $f(0) = 2$

b) $f(0) \neq 2$ for all $x \geq 0$

c) $f(2)$ is undefined

d) $\lim_{x\to 2} f(x) = \infty$

e) $\lim_{x\to\infty} f(x) = 2$

5. If $y = 7$ is a horizontal asymptote of a rational function *f*, then which of the following must be true?

a) $\lim_{x \to 7} f(x) = \infty$ b) $\lim_{x \to \infty} f(x) = 7$ c) $\lim_{x \to 0} f(x) = 7$

d) $\lim_{x \to 7} f(x) = 0$ e) $\lim_{x \to -\infty} f(x) = -7$

6. If *a* and *b* are positive constants, then $\lim_{x \to \infty} \frac{\ln(bx+1)}{\ln(ax^2+3)} =$

a) 0 b) $\frac{1}{2}$ c) $\frac{ab}{2}$ d) 2 e) ∞

7-4: End Behavior: Slant Asymptotes

LEARNING OUTCOME

Determine the end behavior of a rational function from a model, polynomial long division, or infinite limits.

Vocabulary:

3. ***Slant Asymptote*** – the slanted line that serves as the boundary for a curve as *x* becomes infinitely large

General Rule for Slant Asymptotes:

For $y = \frac{A_n x^n + A_{n-1}x^{n-1}\ldots}{B_m x^m + B_{m-1}x^{m-1}\ldots}$, if $n = m+1$, there is a slant asymptote.

The general rule above says that when $n = m+1$, there is a slant asymptote. That slant asymptote can be accurately defined by polynomial long division. The quotient is the asymptote.

EX 1 Find the end behavior asymptote of $y = \frac{x^2 + x + 1}{x - 2}$

The degree of the numerator is one higher than that of the denominator, therefore there is a slant asymptote. To find the slant asymptote, perform polynomial long division:

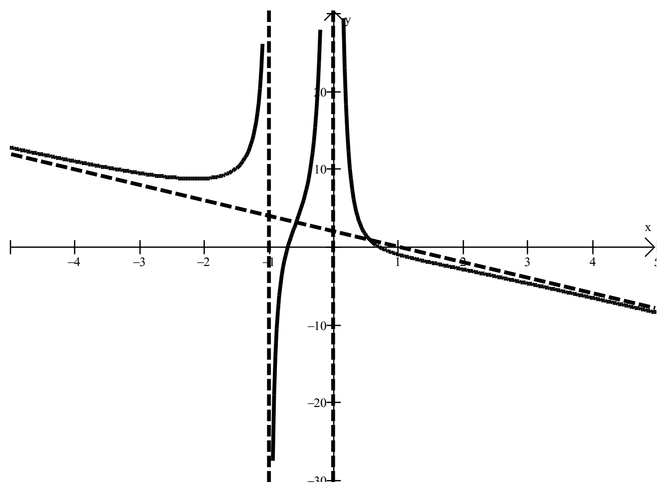
$$\begin{array}{r} x+3 \\ x-2 \overline{\big) x^2 + x + 1} \\ \underline{-\left(x^2 - 2x\right)} \\ 3x + 1 \\ \underline{-\left(3x - 6\right)} \\ 7 \end{array}$$

The Slant Asymptote is $y = x + 3$.

EX 2 Find the end behavior asymptote of $y = \frac{-2x^4 - x^2 + 1}{x^3 + x^2}$.

$$x^3 + x^2 \overline{\big) -2x^4 - x^2 + 1} \rightarrow \begin{array}{r} -2x+2 \\ x^3 + x^2 \overline{\big) -2x^4 + 0x^3 - x^2 + 0x + 1} \\ \underline{-2x^4 - 2x^3} \qquad\qquad\quad \\ 2x^3 - x^2 \qquad\quad \\ \underline{2x^3 + 2x^2} \qquad\quad \\ -3x^2 + 1 \end{array}$$

The slant asymptote is $y = -2x + 2$.



$$y = \frac{-2x^4 - x^2 + 1}{x^3 + x^2}$$

Note that the remainder does not affect the asymptote. As x gets infinitely large, the remainder becomes zero. Or, $\lim\limits_{x\to\infty} \frac{-3x^2+1}{x^3+x^2} = 0$, and the rational function becomes the line as x gets very large.

EX 3 Find all the asymptotes of $x^2 - xy + 2x + 3y - 3 = 0$.

At first, this may not appear to be a rational function, but if y is isolated:

$$y = \frac{x^2 + 2x - 3}{x - 3}$$

VA: $x = 3$
HA: None
SA: $y = x + 5$

$$\begin{array}{r} x+5 \\ x-3 \overline{) x^2 + 2x - 3} \\ \underline{x^2 - 3x} \\ 5x - 3 \\ \underline{5x - 15} \end{array}$$

Beyond the two end behavior asymptotes of interest, other end behaviors will be considered in the way polynomial end behavior was considered.

General Rule for End Behavior Asymptotes:

For $y = \dfrac{A_n x^n + A_{n-1} x^{n-1} \ldots}{B_m x^m + B_{m-1} x^{m-1} \ldots}$,

if $n < m$, there is a horizontal asymptote and it is $y = 0$;

if $n = m$, there is a horizontal asymptote and it is $y = \dfrac{A_n}{B_m}$;

if $n = m + 1$, there is a slant asymptote;

if $n > m + 1$, there is an end behavior model rather than an asymptote.

(Note that horizontal and slant asymptotes are mutually exclusive—a function cannot have both and still remain a function.)

7-4 Free Response Homework

Perform polynomial long division.

1. $\dfrac{x^2+6x+5}{x-4}$

2. $\dfrac{x^2-6x+2}{x^2-4}$

3. $\dfrac{x^2-6x+10}{2x-4}$

4. $\dfrac{x^3-8x^2+20x-14}{x^2-4x+4}$

Find the zeros, vertical asymptotes, POEs, and end behavior.

5. $y=\dfrac{x^3-3x^2-25x-21}{x^2-4x+4}$

6. $y=\dfrac{x^3+3x^2-6x-8}{x^2+6x+5}$

7. $x^2y-x^3+4y+1=0$

8. $x^2-xy+2y-4x=0$

9. $y=\dfrac{x^3+x^2-x-1}{x^2+x-30}$

10. $y=\dfrac{x^3-5x^2-2x+24}{2x^2+7x-15}$

11. $y=\dfrac{2x^3-9x^2+7x+6}{6-x-x^2}$

7-4 Multiple Choice Homework

1. Find the asymptotic end behavior of $f(x)=\dfrac{x^2+7}{x}$.

a) $y=x^3$ b) $y=x^2$ c) $y=1$

d) $y=x$ e) $y=x^7$

2. Which of the following functions has a slant asymptote with a positive slope?

a) $y=\dfrac{x^3-5x^2-2x+24}{2x^2+7x-15}$

b) $y=\dfrac{2x^3-9x^2+7x+6}{6-x-x^2}$

c) $y=\dfrac{x^3-8}{x^4-13x^2+36}$

d) $y=\dfrac{x^2-1}{x^2+2x-3}$

e) $y=\dfrac{1-x^2}{x^2+2x-3}$

---

3. Which of the following functions has a slant asymptote with a negative slope?

a) $y=\dfrac{x^3-5x^2-2x+24}{2x^2+7x-15}$

b) $y=\dfrac{2x^3-9x^2+7x+6}{6-x-x^2}$

c) $y=\dfrac{x^3-8}{x^4-13x^2+36}$

d) $y=\dfrac{x^2-1}{x^2+2x-3}$

e) $y=\dfrac{1-x^2}{x^2+2x-3}$

---

4. Which of the following functions has a slant asymptote with a positive slope?

a) $x^2y-x^3+4y+1=0$

b) $x^2+xy+2y-4x=0$

c) $x^2y-x^2+4y+1=0$

d) $x^2-x^3y+2y-4x=0$

e) $x^2y-x^3+4y+1=0$

---

5. Which of the following functions does not have a slant asymptote?

I. $x^2 + xy + 2y - 4x = 0$ II. $x^2 y - x^2 + 4y + 1 = 0$

III. $x^2 - x^3 y + 2y - 4x = 0$

a) I only b) II only c) III only

d) I and II only e) II and III only

7-5: Derivatives: The Quotient Rule

In a previous chapter extreme values could be found by setting the derivative equal to zero, and that the derivative of a polynomial was easy to find by The Power Rule. But most rational functions cannot be written as x^n, so The Power Rule would not apply. There is a rule for rational functions.

The Quotient Rule:

If $f(x)=\frac{U}{V}$, where U and V are functions of x, then $f'(x)=\frac{V\bullet\frac{du}{dx}-U\bullet\frac{dv}{dx}}{V^2}$.

In the case of a rational function, $\frac{du}{dx}$ and $\frac{dv}{dx}$ can be found by The Power Rule.

LEARNING OUTCOMES

- Find the derivative of a rational function.
- Find the extreme points of a rational function.

EX 1 $\frac{d}{dx}\left(\frac{x^2+2x-3}{x-4}\right)$

$$U=x^2+2x-3,\text{ so }\frac{du}{dx}=2x+2 \qquad V=x-4,\text{ so }\frac{dv}{dx}=1$$

$$f'(x)=\frac{V\cdot\frac{du}{dx}-U\cdot\frac{dv}{dx}}{V^2}=\frac{(x-4)\cdot(2x+2)-(x^2+2x-3)\cdot 1}{(x-4)^2}$$

$$= \frac{2x^2 - 6x - 8 - x^2 - 2x + 3}{(x-4)^2}$$

$$= \frac{x^2 - 8x - 5}{(x-4)^2}$$

EX 2 $\frac{d}{dx}\left(\frac{x^4 - 4x^2}{x^2 - x - 3}\right)$

$$\frac{d}{dx}\left(\frac{x^4 - 4x^2}{x^2 - x - 3}\right) = \frac{(x^2 - x - 3)(4x^3 - 8x) - (x^4 - 4x^2)(2x - 1)}{(x^2 - x - 3)^2}$$

$$= \frac{(4x^5 - 4x^4 - 20x^3 + 8x^2 + 24x) - (2x^5 - x^4 - 8x^3 + 4x^2)}{(x^2 - x - 3)^2}$$

$$= \frac{2x^5 - 3x^4 - 12x^3 + 4x^2 + 24x}{(x^2 - x - 3)^2}$$

EX 3 $\frac{d}{dx}\left(\frac{x^2-4x+3}{2x^2-5x-3}\right)$

Notice that this problem becomes much easier if the fraction is simplified before applying the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2-4x+3}{2x^2-5x-3}\right) &= \frac{d}{dx}\left(\frac{(x-1)(x-3)}{(2x+1)(x-3)}\right)\\ &= \frac{d}{dx}\left(\frac{x-1}{2x+1}\right)\\ &= \frac{(2x+1)(1)-(x-1)(2)}{(2x+1)^2}\\ &= \frac{3}{(2x+1)^2}\end{aligned}$$

As before, find the critical values and extreme values by setting the derivative equal to zero, finding where the derivative is undefined, or endpoints of an arbitrarily stated domain.

> REMEMBER: A critical value is where
>
> i) $\frac{dy}{dx}=0$ at that value;
>
> ii) $\frac{dy}{dx}$ does not exist at that value;
>
> or iii) a value at the end of a domain restriction.

Steps to Finding the Extreme Points of a Rational Function

1. Check for POEs before differentiating.

   a. If there is a POE, reduce the function.

2. Find $\frac{dy}{dx}$ using the Quotient Rule.

3. Find the Critical Values:

   i. $\frac{dy}{dx} = 0 \rightarrow$ numerator $= 0 \rightarrow$ solve for *x*.

   ii. $\frac{dy}{dx}$ *dne* $\rightarrow$ denominator $= 0 \rightarrow$ solve for *x*.
   [*Note that the numbers that solve this equation will the vertical asymptotes of the y-equation, therefore they will not lead to extreme points.*]

   iii. Identify the *x*-coordinates of any given domain.

4. Find the *y*-coordinates for each of the critical values (from the *y*-equation).

   a. Note that this can be done as each critical value is found.

5. List answers as points.

EX 4 Find the extreme points of EX 1.

i) $f'(x)=0$

$$f'(x)=\frac{x^2-8x-5}{(x-4)^2}=0$$

$$x^2-8x-5=0$$

$$x=\frac{8\pm\sqrt{8^2-4(1)(-5)}}{2(1)}=8.583 \text{ or } -0.583$$

ii) $f'(x)=\text{DNE}$

$\Rightarrow x=4$, but $x=4$ is not in the domain

iii) Endpoints of a given domain: none given

The critical values are $x=8.583,\ -0.583$.

The extreme points are $(8.583,\ 19.165)$ and $(-0.583,\ 0.835)$.

EX 5 Find the extreme points of $y=\dfrac{-3x}{x^2+1}$.

i) $\dfrac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{\left(x^2+1\right)(-3)-(-3x)(2x)}{\left(x^2+1\right)^2}=0$$

$$\frac{-3x^2-3+6x^2}{\left(x^2+1\right)^2}=0$$

$$\frac{3x^2-3}{\left(x^2+1\right)^2}=0$$

$$3x^2-3=0$$

c.v.s: $x=\pm 1$

ii) $f'(x)=\text{DNE}$

$\dfrac{dy}{dx}$ is never a DNE

iii) Endpoints of a given domain: none given

The critical values are $x=1,\ -1$.

The extreme points are $\left(1,\ -\dfrac{3}{2}\right)$ and $\left(-1,\ \dfrac{3}{2}\right)$.

EX 6 Find the critical values of $y = \frac{-3x}{x^2 - 2x - 3}$.

i) $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{(x^2 - 2x - 3)(-3) - (-3x)(2x - 2)}{(x^2 - 2x - 3)^2} = 0$$

$$\frac{3x^2 + 9}{(x^2 - 2x - 3)^2} = 0$$

$$3x^2 + 9 = 0$$

$$x^2 = -3$$

ii) $\frac{dy}{dx} = \text{DNE} \Rightarrow x = -1,\ 3$

iii) Endpoints of a given domain: none given

$$x = -1,\ 3$$

Note, though, that these two critical values do not lead to extreme points.

EX 7 Find the equations of the lines tangent and normal to $y=\dfrac{x^2+2x-3}{x-4}$ at $x=-2$.

The point of tangency is $\left(-2,\ f(-2)\right)=\left(-2,\ \dfrac{1}{2}\right)$.

The slope of the tangent line will be $f'(-2)$.

$$f'(x)=\frac{x^2-8x-5}{(x-4)^2}$$
$$f'(-2)=\frac{15}{36}$$

$$m_{\text{tangent}}=f'(-2)=\frac{15}{36} \text{ and } m_{\text{normal}}=\frac{-1}{f'(-2)}=-\frac{36}{15}$$

$$y-\frac{1}{2}=\frac{15}{36}(x+2) \text{ and } y-\frac{1}{2}=-\frac{36}{15}(x+2)$$

7-5 Free Response Homework

Find the derivatives.

1. $y=\left(\dfrac{x^2-3}{x^2-4}\right)$; find $\dfrac{dy}{dx}$

2. $D_x\left(\dfrac{3x^2+4x-3}{x^2-9}\right)$

3. $f(x)=\dfrac{x^2+2x-8}{x^2-x-3}$; find $f'(x)$

4. $\dfrac{d}{dx}\left(\dfrac{x^3-2x^2-5x+6}{x+2}\right)$

5. $\dfrac{d}{dx}\left(\dfrac{3x+3}{x^3+1}\right)$

6. $y=\dfrac{x^2+2x-3}{x-4}$; find y'

7. $\dfrac{d}{dx}\left[\dfrac{x^5-12x^3-19x}{3x^3}\right]$

8. $D_x\left[\dfrac{x-4}{x^2-9x+20}\right]$

9. Find the first derivative for the following function: $y=\dfrac{x^2+2x-15}{x-3}$

Find the equations of the lines tangent and normal to the given functions at the given x-value.

10. $y=\dfrac{-2x}{x^2+16}$ at $x=-1$.

11. $y=\dfrac{x^2-3}{x^2-4}$ at $x=1$

12. $y=\dfrac{-3x}{x^2+1}$ at $x=1$

13. $y=\dfrac{x^2-4x+3}{2x^2-5x-3}$ at $x=2$

Find the critical values algebraically.

14. $y=\dfrac{x-3}{x+1}$

15. $y=\dfrac{3-x^2}{1+x^2}$

16. $y=\dfrac{x^2-3}{x^2-10}$

17. $y=\dfrac{-3x^3}{x^2+2x-3}$

Find the extreme points algebraically.

18. $y=\dfrac{x-3}{x^2+5}$

19. $y=\dfrac{3x^2+5x-2}{6x^3-11x^2-3x+2}$

20. $y=\dfrac{x^2-1}{x^2+2x-3}$

21. $y=\dfrac{4x}{x^2+4x-5}$

22. $y=\dfrac{4-x^2}{x^2-1}$

23. $y=\dfrac{x^2-4x+3}{x^2-x-6}$

24. $y=\dfrac{4x^3-7x^2+12x-21}{4x^2-3x-7}$ on $x\in[-6,2]$

25. $y=\dfrac{x}{x^2-4}$ on $x\in[-1,5]$

26. $y=\dfrac{x^2-4}{x^2+4}$ on $x\in[-3,5]$

27. $y=\dfrac{-4x}{x^2+4}$ on $x\in[-3,\infty)$

28. $y=\dfrac{x^2-9}{x^2+16}$ on $x\in(-\infty,3]$

29. $y=\dfrac{x^2-1}{x^3+1}$ on $x\in(-\infty,3]$

30. $y=\dfrac{x^2-6x+9}{x^2-4x-5}$ on $x\in[-3,\infty)$

7-5 Multiple Choice Homework

1. If $y = \dfrac{3}{4+x^2}$, then $\dfrac{dy}{dx} =$

a) $y = \dfrac{-6x}{\left(4+x^2\right)^2}$ b) $y = \dfrac{3x}{\left(4+x^2\right)^2}$ c) $y = \dfrac{6x}{\left(4+x^2\right)^2}$

d) $y = \dfrac{-3x}{\left(4+x^2\right)^2}$ e) $\dfrac{3}{2x}$

---

2. If $f(x) = \dfrac{x^3}{(x+2)^2}$, find $f'(-1)$.

a) 8 b) 5 c) 10 d) 4 e) 20

---

3. Find the slope of the tangent line to the curve $y = \dfrac{1}{1+x}$ at the point $\left(1, \dfrac{1}{2}\right)$.

a) $-\dfrac{1}{4}$ b) 2 c) -1 d) $\dfrac{1}{4}$ e) $-\dfrac{1}{2}$.

---

4. Given the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
–3	7	–7	–6	7
–2	1	–5	0	5
–1	–3	–3	4	3
0	–5	–1	6	1
1	–5	1	6	–1
2	–3	3	4	–3
3	1	5	0	5

Find $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$, when $x = 0$.

a) $\frac{7}{36}$ b) $\frac{5}{36}$ c) $-\frac{5}{36}$ d) $\frac{1}{36}$ e) $-\frac{1}{36}$.

---

5. An equation of the line tangent to the graph of $y = \frac{2x+3}{3x-2}$ at $(1, 5)$ is

a) $13x - y = 8$

b) $13x + y = 18$

c) $x - 13y = 64$

d) $x + 13y = 66$

e) $-2x + 3y = 13$

---

6. The equation of the line normal to the graph of $y = \dfrac{x}{2x-3}$ at the point $(1, f(1))$ is

(a) $3x + y = 4$ (b) $3x + y = 2$ (c) $x - 3y = -2$

(d) $x - 3y = 4$ (e) $x + 3y = 2$

---

7. The limit $\lim\limits_{x \to 2} \dfrac{x^2 - 4x + 4}{x - 2}$ represents the derivative of a function *f* at number *a*. What is $f(x)$?

a) x^2 b) $-4x$ c) $x^2 - 4x$

d) $x^2 - 4x - 4$ e) $2x - 4$

---

8. Given this sign pattern

$f'(x)$	$-$	0	$+$	0	$-$	DNE	$-$
x		-4		-1		2	

, at what value of *x* does *f* have a local maximum?

a) -4 b) -1 c) 2 d) 1 e) no value

7-6: General Rational Curve Sketching

The next task will be to put together all of the traits and do a general sketch algebraically.

REMEMBER: Rational Traits

1. Domain
2. y-intercept
3. Zero(s)
4. Point(s) of Exclusion
5. Vertical Asymptote(s)
6. End Behavior
7. Extreme Point(s)
8. Range

LEARNING OUTCOME

Find all the traits and sketch a fairly accurate rational curve algebraically.

EX 1 Given the traits below, sketch a graph of the function.

Domain: $x \neq -4, -3, 2$

Zeros: $(-2, 0)$, $(2.2, 0)$

y-intercept: $(0, 1)$

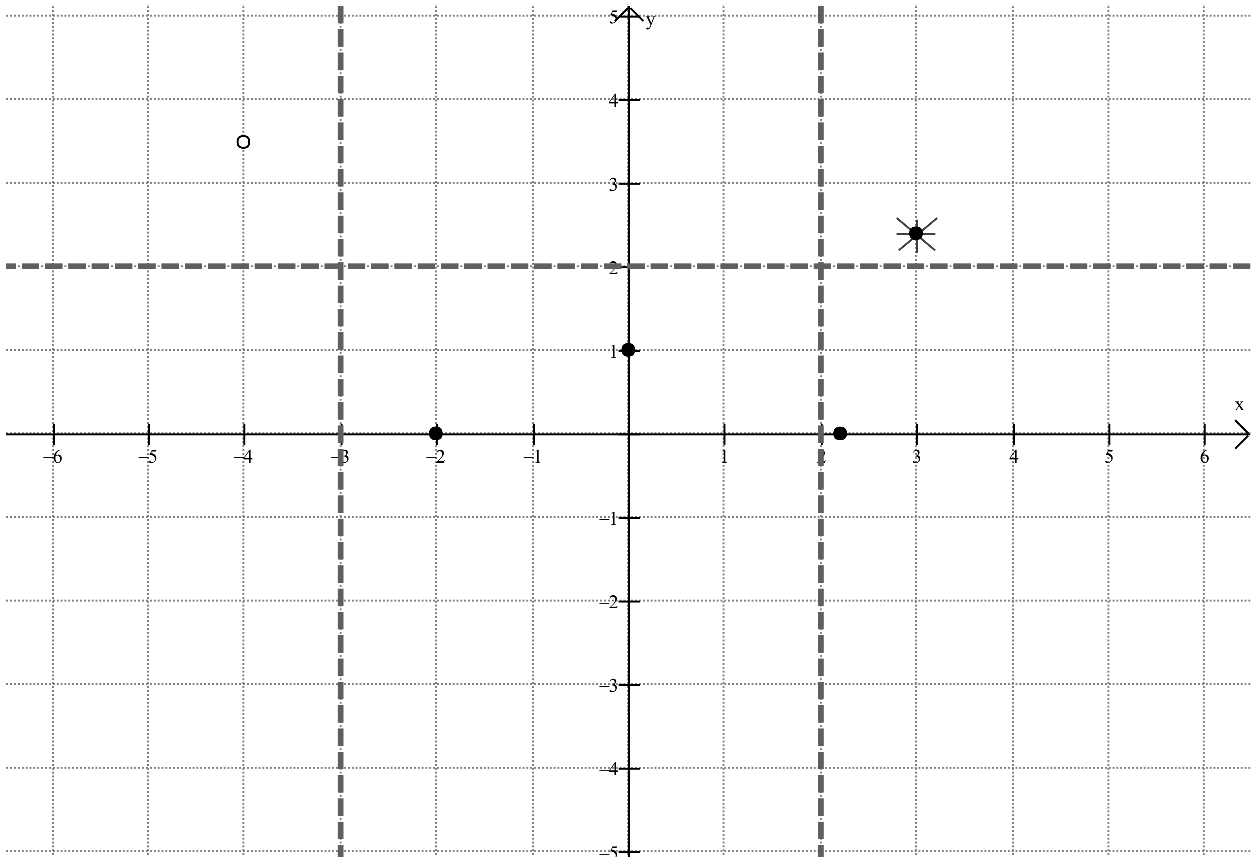
Extreme Point: $(3, 2.4)$

VA: $x = -3, x = 2$

POE: $(-4, 3.5)$

End Behavior: Left side $y = 2$; right side $y = 2$

Range: $y \in (-\infty, \infty)$



For $x \in (-\infty, -3)$**:**

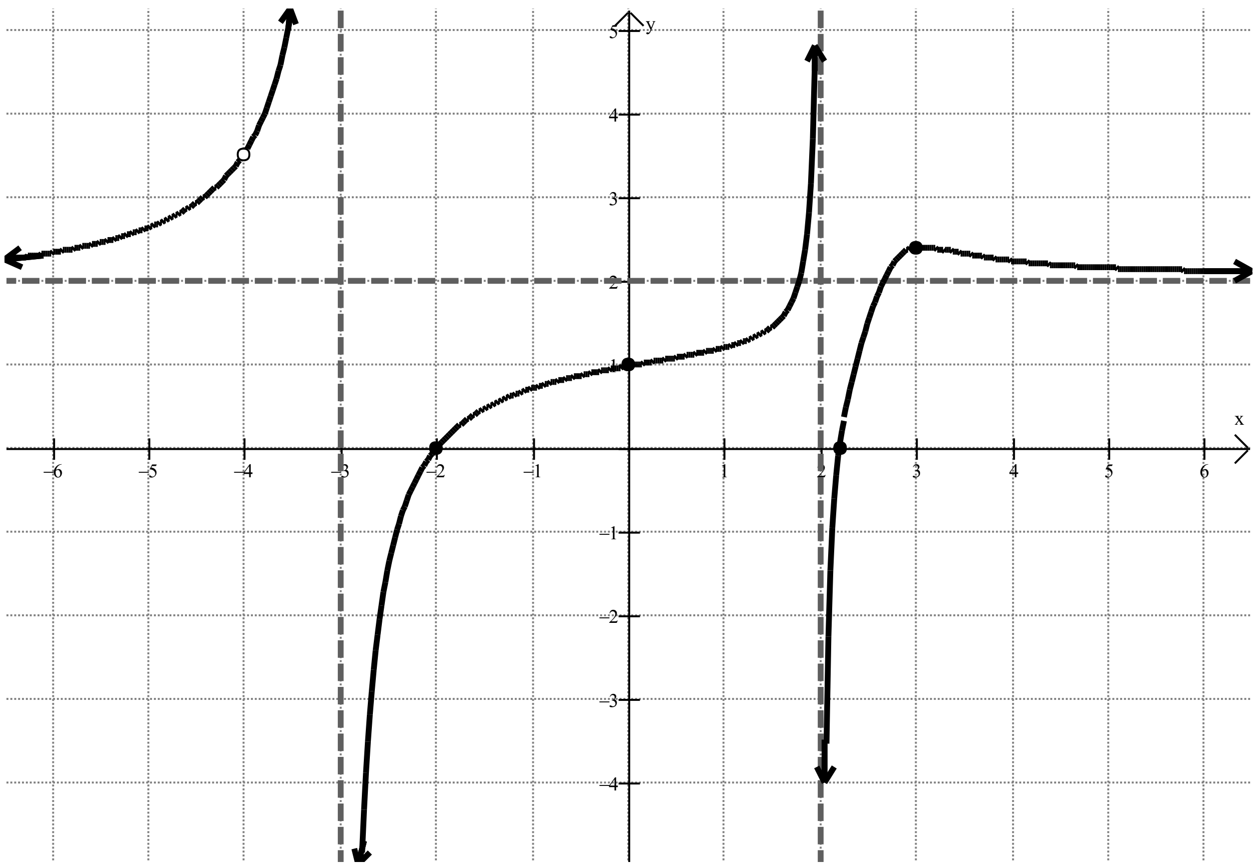
The curve crosses the POE and heads towards both asymptotes; there are no extreme points on that side, so the curve does not cross the horizontal asymptote over there.

For $x \in (-3, 2)$**:**

Notice that $(-2, 0)$ and $(0, 1)$ are in the same region, and there are no extreme points or other zeros in that region, so when they are connected, the curve heads down along the asymptote $x = -3$ and up along the asymptote $x = 2$. (Note that a curve can and does cross the horizontal asymptote.)

For $x \in (2, \infty)$**:**

There is an extreme point at $(3, 2.4)$ and a zero at $(2.2, 0)$. Because of their relative positions, the extreme point is a maximum point. There are no other zeros in this region of the graph, so the curve heads toward the end behavior asymptote $(y = 2)$.



y
x
5
4
3
2
1
−1
−2
−3
−4
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6

EX 2 Find the traits and sketch $y = \dfrac{x^2 - 1}{x^2 - 4}$.

1. Domain: $x \neq 2, -2$

2. y-intercept: $x = 0$ gives $y = \frac{1}{4}$, so $\left(0, \frac{1}{4}\right)$

3. Zeros: $y = 0$ gives $x = \pm 1$, so $(\pm 1, 0)$

4. POE: None, since y cannot equal $\frac{0}{0}$

5. Vertical Asymptotes: $x = 2,\ x = -2$

6. End Behavior: The denominator and numerator have the same power. There is a horizontal asymptote at $y = \frac{A_0}{B_0} = \frac{1}{1} = 1$, i.e. $y = 1$.

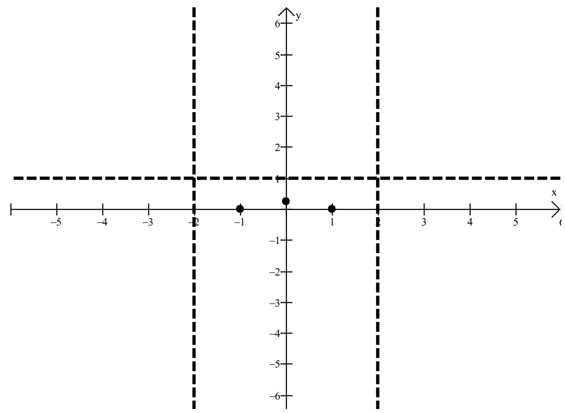
   $y = 1$ is the end behavior on both sides.

7. Extreme Points:
$$\frac{dy}{dx} = \frac{(x^2 - 4)(2x) - (x^2 - 1)(2x)}{(x^2 - 4)^2} = 0$$
$$\frac{-6x}{(x^2 - 4)^2} = 0 \Rightarrow x = 0$$
$$\frac{dy}{dx} = \text{DNE} \Rightarrow x = \pm 2, \text{ neither of which are in the domain}$$
c.v.: $x = 0$

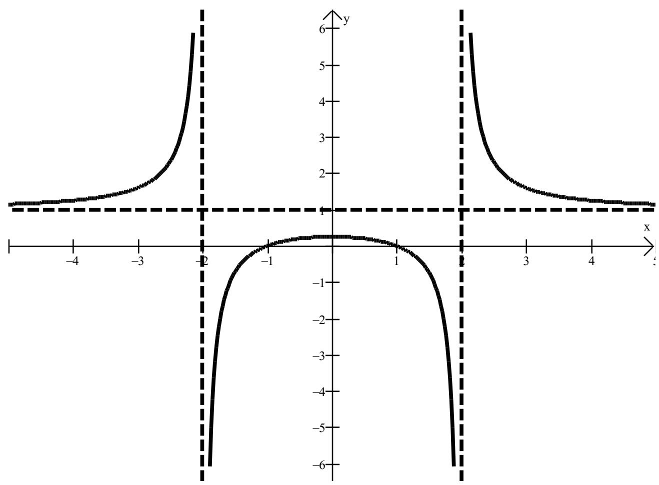
e.v.: $y(0) = \frac{1}{4}$

The extreme point is $\left(0, \frac{1}{4}\right)$.

Plotting the traits:



Connecting the dots:



$$y = \frac{x^2 - 1}{x^2 - 4}$$

8. Range: $y \in (-\infty,\ 0.25] \cup (1,\ \infty)$

EX 3 Find the traits and sketch $y = \frac{-4x}{x^2+4}$ $x \in [-3, \infty)$.

1. Domain: since the denominator cannot equal 0, $x \in \text{All Reals}$ is the domain of the equation. With the restriction, the domain of this particular problem becomes $x \in [-3, \infty)$.

2. y-intercept: $x = 0$ gives $y = 0$, so $(0,\ 0)$

3. Zero: $y = 0$ gives $x = 0$, so $(0,\ 0)$

4. POE: None, since the denominator cannot equal 0

5. VA: None, since the denominator cannot equal 0

6. End Behavior:
   Left: None because of the domain restriction
   Right: $y = 0$ because the denominator degree is higher than the numerator.

7. Extreme Points:

$$\frac{dy}{dx} = \frac{(x^2+4)(-4) - (-4x)(2x)}{(x^2+4)^2} = 0$$

$$\frac{-4x^2 - 16 + 8x^2}{(x^2+4)^2} = 0$$

$$\frac{4x^2 - 16}{(x^2+4)^2} = 0$$

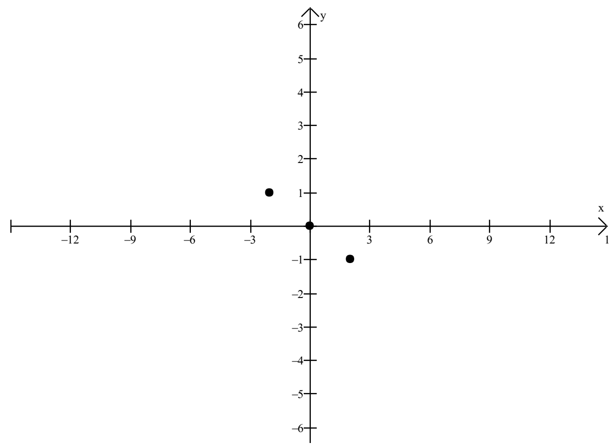
$$4x^2 - 16 = 0$$

c.v.s: $x = \pm 2$

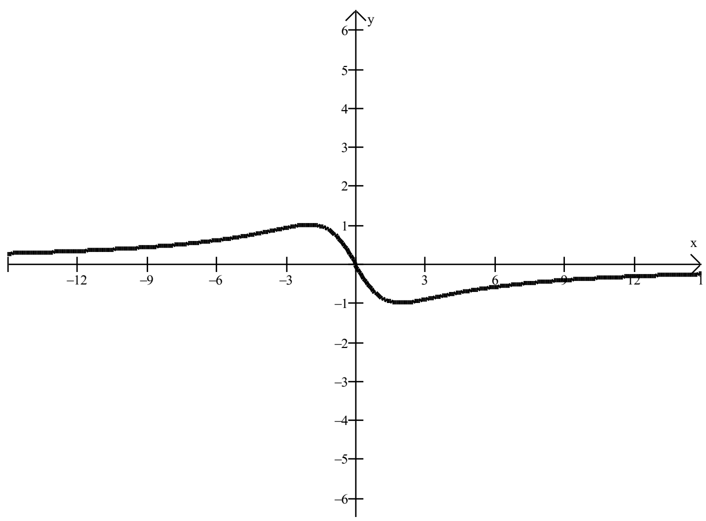
e.v.s: $y(2) = -1$ and $y(-2) = 1$

The extreme points are $(2,\ -1)$ and $(-2,\ 1)$.

Plotting the traits:



Connecting the dots:



$$y = \frac{-4x}{x^2 + 4}$$

8. Range: because of the extreme values, $y \in [-1, 1]$

EX 4 Find the traits and sketch $y = \dfrac{x^2 - 2x}{x^3 - 2x^2 - x + 2}$.

1. Domain: $x \neq \pm 1,\ 2$

2. y-intercept: $x = 0$ gives $y = 0$, so $(0,\ 0)$

3. Zero: $y = 0$ gives $x = 0$, so $(0,\ 0)$

4. POE: $\left(2,\ \dfrac{2}{3}\right)$

5. Vertical Asymptotes: $x = 1,\ x = -1$

6. End Behavior: The denominator degree is higher than the numerator, so there is a horizontal asyptote at $y = 0$.

7. Extreme Points: $\dfrac{dy}{dx} = \dfrac{(x^2-1)(1)-(x)(2x)}{(x^2-1)^2} = 0$

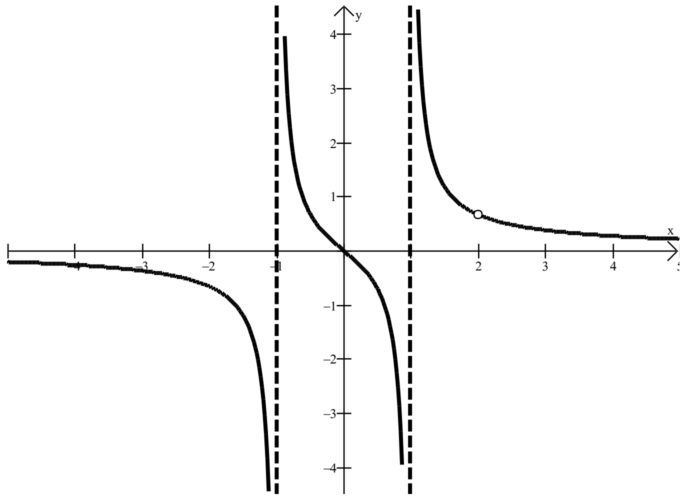
$$\frac{-x^2-1}{(x^2-1)^2} = 0$$

$x^2 = -1 \Rightarrow$ no solution

$\dfrac{dy}{dx} = \text{DNE} \Rightarrow x = \pm 1,$ neither of which are in the domain

There are no critical values, hence no extreme points.

8. Range: $y \in (-\infty,\ \infty)$



$$y=\frac{x^2-2x}{x^3-2x^2-x+2}$$

EX 5 Find the traits and sketch $y=\dfrac{(x+1)(x-2)(2x-1)}{(x-3)(x-2)}$.

1. Domain: $x \neq 2,\ 3$

2. y-intercept: $x=0$ gives $y=\frac{1}{3}$, so $\left(0,\ \frac{1}{3}\right)$

3. Zeros: $y=0$ gives $x=\frac{1}{2},\ -1$, so $\left(\frac{1}{2},\ 0\right)$ and $(-1,\ 0)$

4. POE: $x=2$ gives $y=\frac{0}{0}$.

$$\lim_{x\to 2}\frac{(x+1)(x-2)(2x-1)}{(x-3)(x-2)}=\lim_{x\to 2}\frac{(x+1)(2x-1)}{(x-3)}$$
$$=\frac{(3)(3)}{-1}$$
$$=-9$$

The POE is $(2, \ -9)$.

5. Vertical Asymptote: $x = 3$

6. End Behavior: The degree of the numerator is one more than the degree of the denominator, so there is a slant asymptote

$$y = \frac{(x+1)(x-2)(2x-1)}{(x-3)(x-2)} \approx \frac{2x^2+x-1}{x-3}$$

$$\begin{array}{r} 2x+7 \\ x-3\overline{)2x^2+x-1} \\ \underline{2x^2-6x} \\ 7x-1 \end{array}$$

The end behavior is $y = 2x+7$.

7. Extreme Points: $\frac{dy}{dx} = \frac{(x-3)(4x+1)-(2x^2+x-1)(1)}{(x-3)^2} = 0$

$$\frac{4x^2-11x-3-(2x^2+x-1)}{(x-3)^2} = 0$$

$2x^2-12x-2=0 \Rightarrow x = -0.162, 6.162$

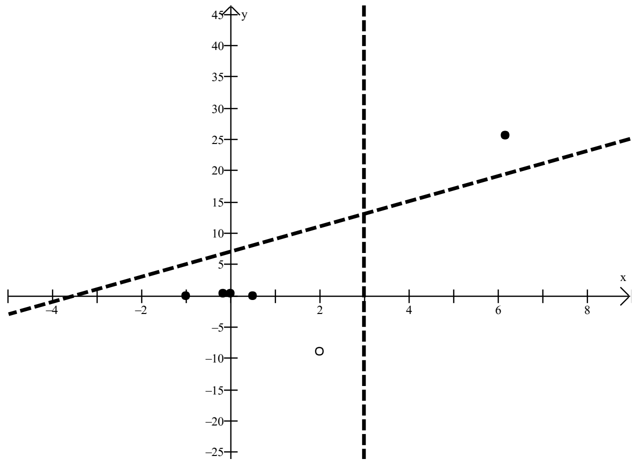
$\frac{dy}{dx} = \text{DNE} \Rightarrow x = 3$, which is not in the domain

c.v.s: $x = -0.162, 6.162$

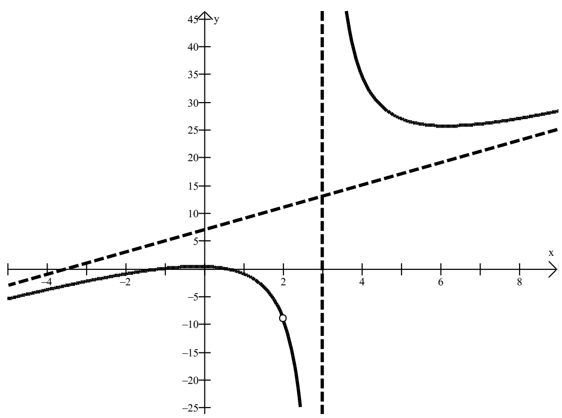
e.v.s: $y(-0.162) = 0.351$ $y(6.162) = 25.649$

The extreme points are $(-0.162, \ 0.351)$ and $(6.162, \ 25.649)$.

Plotting the traits:



Connecting the dots:



$$y=\frac{(x+1)(x-2)(2x-1)}{(x-3)(x-2)}$$

8. Range: $y\in(-\infty,\ 0.351]\cup[25.649,\ \infty)$

EX 6 Create an equation of a rational function that has zero at $(3,\ 0)$, VA $x=2$, POE at $(1,\ 1)$, and HA $y=\frac{1}{2}$.

We know that the solution should be of the format $y=\frac{(x\quad)(x\quad)}{(x\quad)(x\quad)}$ where the traits fill in the blanks. A zero at $(3,\ 0)$ and a VA at $x=2$ means:

$$y=\frac{(x-3)(x\quad)}{(x-2)(x\quad)}$$

A POE at $x=1$ means:

$$y=\frac{(x-3)(x-1)}{(x-2)(x-1)}$$

A quick check on the limiting y value shows -1, not 1. Therefore,

$$y=\frac{(3-x)(x-1)}{(x-2)(x-1)}$$

Finally, an HA at $y=\frac{1}{2}$ means:

$$y=-\frac{1(3-x)(x-1)}{2(x-2)(x-1)}$$

7-6 Free Response Homework

1. Given the traits below, sketch a graph of the function.

Domain: $x \neq -3, -1, 3, 6$
Zeros: $(\pm 3.3,\ 0)$
y-intercept: $(0,\ 4)$
Extreme Points: $(4.5,\ -4),\ (1,\ 2),\ (-2,\ -4),\ (-5,\ -3.5)$
VA: $x=-3,\ x=-1,\ x=3$
POE: $(6,\ -3.2)$
End Behavior: $y=0$
Range: $y \in (-\infty,\ \infty)$

2. Given the traits below, sketch a graph of the function.

Domain: $x \neq 2, 6$
Zeros: $(-1.9,\ 0),\ (3,\ 0),\ (6.3,\ 0)$
y-intercept: $(0,\ 1)$
Extreme Points: None
VA: $x=2,\ x=6$
POE: None
End Behavior: $y=\frac{1}{2}x+1$
Range: $y \in (-\infty,\ \infty)$

3. Given the traits below, sketch a graph of the function.

Domain: $x \neq \pm 2, 4, 5$
Zeros: $(3,\ 0),\ (6,\ 0)$
y-intercept: $(0,\ 3)$
Extreme points: $(0,\ 3),\ (4.5,\ 0.5),\ (7.5,\ -0.5)$
VA: $x=\pm 2$
POE: $(4,\ 0.2),\ (5,\ 0.2)$
End Behavior: $y=0$
Range: $y \in (-\infty,\ 0.2) \cup (0.2,\ 0.5] \cup [3,\ \infty)$

Find the traits and sketch.

4. $y=\dfrac{(x-4)(x+1)}{(x+5)(x-4)(4x+1)}$

5. $y=\dfrac{4x}{x^2+4x-5}$

6. $y=\dfrac{x^2-1}{x^2+2x-3}$

7. $y=\dfrac{x^2-4x+3}{x^2-x-6}$

8. $y=\dfrac{4-x^2}{x^2-1}$

9. $y=\dfrac{x^3+3x^2-6x-8}{x^2+6x+5}$

10. $x^2y-x^3-4y+1=0$

11. $x^2-xy+2y-4x=0$

12. $y=\dfrac{5x-x^2}{x^2-4x+4}$

13. $y=\dfrac{6x^3+5x^2-3x-2}{2x^2-3x-2}$

14. $y=\dfrac{-4x}{x^2+4}$ on $x\in[-3,3]$

15. $y=\dfrac{x^2-9}{x^2+16}$ on $x\in[-5,5]$

16. $y=\dfrac{x^2-1}{x^3+1}$ on $x\in[-2,3]$

17. $y=\dfrac{x^2-6x+9}{x^2-4x-5}$ on $x\in[-3,5]$

18. $y=\dfrac{x^2-9}{x^2-2x-8}$ on $x\in[-7,\infty)$

19. $f(x)=\dfrac{3+2x-x^2}{x^2-9}$ on $x\in(-\infty,5]$

20. $y=\dfrac{6x^2-24x}{x^3-4x^2+9x-36}$ on $x\in(-\infty,4]$

21. $y=\dfrac{x^2-2x-3}{x^3-3x^2+x-3}$ on $x\in[-4,\infty)$

Find the equation of the rational function that has the following traits.

22. $(2,\ 0)$ and $(-1,\ 0)$, HA $y=1$, and VA $x=\dfrac{1}{2}$ and $x=3$

23. No zeros, POE $(0,\ 1)$, HA $y=0$, and VA $x=\dfrac{1}{3}$

24. $(3,\ 0)$, no VA, and HA $y=0$

25. $(-1,\ 0)$ and $(-5,\ 0)$, VA $x=1$, and SA $y=2x+14$

7-6 Multiple Choice Homework

1. The zero(s) of $y=\dfrac{x^2+2x-3}{x^2+5x+6}$ is/are at

a) $x=1$ b) $x=-3$ c) $x=1\ \&\ -3$

d) $x=-2$ e) $x=-2\ \&\ -3$

---

2. Which of the following would be a rational function that has *x*–intercepts at (-5, 0), VA at *x* = 6, a POE at *x* = 2, and a HA at $y=\dfrac{3}{7}$?

a) $y=\dfrac{(x+5)(x-2)}{(x-2)(x-6)}$

b) $y=\dfrac{3(x-5)(x+2)}{7(x+2)(x+6)}$

c) $y=\dfrac{3(x-6)(x-2)}{7(x-2)(x+5)}$

d) $y=\dfrac{3(x-2)^2}{7(x+5)(x-6)}$

e) $y=\dfrac{3(x+5)(x-2)}{7(x-2)(x-6)}$

---

3. If $y=\frac{x}{x+1}$, find $f''(0)$.

a) –2 b) 4 c) 8 d) –6 e) –8

---

4. The VA(s) of $y=\frac{x^2+2x-3}{x^2+5x+6}$ is/are at

a) $x=1$ b) $x=-3$ c) $x=1$ & -3

d) $x=-2$ e) $x=-2$ & -3

---

5. The end behavior of $y=\frac{3-x-x^2}{4x^2+x-1}$ is

a) 0 b) $\frac{3}{4}$ c) $\frac{1}{4}$ d) $-\frac{1}{4}$ e) DNE

---

6. The POE(s) of $y=\frac{x^2+2x-3}{x^2+5x+6}$ is/are at

a) $x=1$ b) $x=-3$ c) $x=1$ & -3

d) $x=-2$ e) $x=-2$ & -3

7-7: Sign Patterns of the Derivative of a Rational Function

Remember from Chapter 6:

The First Derivative Test

As the sign pattern of the 1st Derivative is viewed left to right, the critical value represents the *x*-value of a

i) relative maximum point if the sign changes from + to –
ii) relative minimum point if the sign changes from – to +
or iii) non-extreme if the sign does not change

Corollaries:

An endpoint is at a maximum if
i) it is the left end and followed by a –, or
ii) it is the right end and preceded by a +.

An endpoint is at a minimum if
i) it is the left end and followed by a +, or
ii) it is the right end and preceded by a –.

Process for the First Derivative Test:

1. Differentiate the equation.
2. Find the critical values.
3. Sketch a sign pattern **without** any endpoints of a given domain.
4. Add the endpoints to the sign pattern
5. Determine which critical values are at maximum points vs. minimum points from the sign change.

LEARNING OUTCOMES

Find the sign pattern of the derivative of a rational function.
Apply the First Derivative test to Rational Functions.

EX 1: Determine at which x-value of $y=\frac{-10x}{x^2+25}$ there is a maximum point.

$$\frac{dy}{dx}=\frac{(x^2+25)(-10)-(-10x)(2x)}{(x^2+25)^2}$$
$$=\frac{-10x^2-250+20x^2}{(x^2+25)^2}$$
$$=\frac{10x^2-250}{(x^2+25)^2}$$

i) $\frac{dy}{dx}=0\rightarrow 10x^2-250=0\Rightarrow x=\pm 5$

ii) $\frac{dy}{dx}=\text{DNE}\Rightarrow x^2+25=0\Rightarrow$ no solution

iii) No domain restrictions given.

$\frac{dy}{dx}$	+	0	−	0	+
x		-5		5	

$x=-5$ is at a maximum point

EX 2: Determine at which x-values $y=\frac{9x^2+1}{1-3x}$ there is a maximum point and at which x-values there is a minimum point.

$$\frac{dy}{dx}=\frac{(1-3x)(18x)-(9x^2+1)(-3)}{(1-3x)^2}$$
$$=\frac{18x-54x^2+27x^2+3}{(1-3x)^2}$$
$$=\frac{-27x^2+18x+3}{(1-3x)^2}$$

i) $\frac{dy}{dx}=0\rightarrow -27x^2+18x+3=0$

$$x=\frac{-18\pm\sqrt{18^2-4(-27)(3)}}{2(-27)}=\begin{cases}-0.138\\ 0.805\end{cases}$$

ii) $\frac{dy}{dx}=\text{DNE}\Rightarrow 1-3x=0$

$$x=\frac{1}{3}$$

iii) No restricted domain given.

$\frac{dy}{dx}$	$-$	0	$+$	DNE	$+$	0	$-$
x		-0.138		$\frac{1}{3}$		0.805	

$x=-0.138$ is at a minimum point
$x=0.805$ is at a maximum point

EX 3: Find the interval(s) on which $y=\frac{x^2+6x-5}{x-4}$ is increasing.

$$\frac{dy}{dx}=\frac{(x-4)(2x+6)-(x^2+6x-5)(1)}{(x-4)^2}=0$$

$$\frac{2x^2-2x-24-x^2-6x+5}{(x-4)^2}=0$$

$$x^2-8x-19=0$$

$$x=-1.916,\ 9.916$$

Sign Pattern:

$\frac{dy}{dx}$	+	0	−	DNE	−	0	+
x		-1.916		4		9.916	

The intervals of increasing are $x\in(-\infty,\ -1.916)\cup(9.916,\ \infty)$.

EX 4: The position of a particle moving along the x-axis at time t is described by $x(t)=\frac{t}{t^2+1}$. When is the particle moving left?

$$v(t)=\frac{(t^2+1)(1)-(t)(2t)}{(t^2+1)^2}$$

$$=\frac{1-t^2}{(t^2+1)^2}=0$$

$$1-t^2=0$$

$$t^2=1$$

$$t=\pm1$$

Sign Pattern:

v	−	0	+	0	−
t		-1		1	

The particle is moving left at $t\in(-\infty,\ -1)\cup(1,\ \infty)$.

7-7: Free Response Homework

Determine if the critical values are at a maximum point, a minimum point, or neither.

1. $y=\dfrac{x-5}{x^2-9}$

2. $y=\dfrac{x-5}{x^2+9}$

3. $y=\dfrac{x^2+1}{1-3x}$

4. $y=\dfrac{2x^2+3x-2}{x+1}$

5. $y=\dfrac{-4x}{x^2+4}$

6. $y=\dfrac{x^2-9}{x^2+16}$

7. $y=\dfrac{x^2-1}{x^3+1}$

8. $y=\dfrac{x^2-6x+9}{x^2-4x-5}$

9. The motion of a particle has distance $x(t)$ at time $t\geq 0$ is described by $x(t)=\dfrac{t^2+1}{t+1}$. Where is it when it stops?

10. The motion of a particle has distance $x(t)$ at time $t\geq 0$ is described by $x(t)=\dfrac{3t}{t^2+9}$. Where is it when it stops?

11. A particle's position $\langle x(t), y(t)\rangle$ at time $t\geq 0$ is described by the parametric equations $x(t)=\dfrac{-5t}{t^2+4}$ and $y(t)=t^3-27t$. When is the particle moving right and down?

12. A particle's position $\langle x(t), y(t)\rangle$ at time $t \geq 0$ is described by the parametric equations $x(t) = \dfrac{9-t^2}{t^2+4}$ and $y(t) = t^2 - 8t$. When is the particle moving left and down?

7-7: Multiple Choice Homework

1. Which of the following sign patterns apply to the equation $f(x) = \dfrac{x^4 - 2x^2 - 8}{x-2}$?

I.

$f(x)$	$-$	0	$+$	DNE	$+$
x		-2		2	

II.

$f'(x)$	$+$	DNE	$+$
x		2	

III.

$f(x)$	$+$	0	$-$	0	$+$	0	$-$	DNE	$+$
x		-2		$-\sqrt{2}$		$\sqrt{2}$		2	

a) I only b) II only c) III only

d) I and II only e) I, II and III

---

2. If the line tangent to the graph of f at the point $(1, 7)$ passes through the point $(2, 2)$, then $f'(1)$ is

a) –5 b) 1 c) 3 d) 7 e) undefined

---

3. Suppose $f'(x)=\dfrac{(x+4)^3(x-2)^2}{(x^4+1)}$. Which of the following statements must be **true**?

I. The slope of the line tangent to $y=f(x)$ at $x=1$ is 125.

II. $f(x)$ is decreasing on $x\in(-4,\ 2)$

III. $f(x)$ has a minimum at $x=-4$

a) I only b) II only c) III only

d) II and III only e) I, II and III

---

4. Given this sign pattern

$f'(x)$	−	0	+	DNE	+	0	−
x		-2		0		2	

which of the following might be the sign pattern of $f(x)$?

a)

$f(x)$	+	0	−	DNE	+	0	−
x		-3		0		3	

b)

$f(x)$	−	0	+	DNE	−	0	+
x		-3		0		3	

c)

$f(x)$	−	0	+	DNE	−	0	+
x		-2		0		2	

d)

$f(x)$	+	0	−	DNE	+	0	−
x		-2		0		2	

e)

$f(x)$	+	0	−	DNE	+	0	−
x		-2		0		2	

---

5. A function is defined as $g(x)=\frac{(x-3)^2}{x-7}$. Which of the following is **false**?

a) $g(x)$ is increasing for $x > 11$.

b) $g(x)$ is decreasing on [3, 11].

c) $g(x)$ has a local maximum at $x = 3$.

d) $g(x)$ has a horizontal asymptote at $y = 0$.

e) $g(x)$ has a vertical asymptote at $x = 7$.

Rational Functions Practice Test
Part 1: CALCULATOR REQUIRED

Round to 3 decimal places. Show all work.

1. An equation of the line normal to the graph of $y=\dfrac{2x+3}{3x-2}$ at $(1,\,5)$ is

a) $13x-y=8$

b) $13x+y=18$

c) $x-13y=-64$

d) $x+13y=66$

e) $-2x+3y=13$

---

2. A function is defined as $g(x)=\dfrac{kx}{x^2+1}$, where *k* is a constant. For what values of *k*, if any, is f strictly increasing on the interval $(-1,\,1)$?

(a) $k < 0$
(b) $k > 0$
(c) $k > 1$ only
(d) $-1<x<1$
(e) No such Values of *k*

---

3. An equation of the line tangent to the curve $y=\frac{kx+8}{k+x}$ at $x=-2$ is $y=x+4$. What is the value of k?

a) -3
b) -1
c) 1
d) 3
e) 4

---

4. If $y=\frac{1-x}{x-1}$, then $\frac{dy}{dx}=$

a) -1
b) 0
c) $\frac{-1}{x-1}$
d) $\frac{-2}{x-1}$
e) $\frac{-2x}{(x-1)^2}$

---

5. Let $f(x)$ be a differentiable function. The table below gives the value of $f(x)$ and $f'(x)$, at several values of x. If $g(x)=\frac{1}{f(x)}$, what is the value of $g'(2)$?

x	1	2	3	4
$f(x)$	-3	-8	-9	0
$f'(x)$	-5	-4	3	16

a) $-\frac{1}{8}$ b) 0 c) $\frac{1}{16}$ d) $\frac{1}{64}$ e) 16

---

6. $\lim_{x\to\infty} \frac{10^8 x^5 + 10^6 x^4 + 10^4 x^2}{10^9 x^6 + 10^7 x^5 + 10^5 x^3} =$

a) 0 b) 1 c) -1 d) $\frac{1}{10}$ e) $-\frac{1}{10}$

---

7. Suppose $f'(x) = \frac{(x+1)^3 (x-4)^2}{(x^4+1)}$. Which of the following statements must be true?

I. The slope of the line tangent to $y = f(x)$ at $x = 1$ is 36.

II. $f(x)$ is increasing on $x \in (1, 4)$

III. $f(x)$ has a minimum at $x = 4$

a) I only b) II only c) III only d) I and II only e) I, II and III

---

8. For what value of c will $x^2 + \frac{c}{x}$ have a relative minimum at $x = -1$?

a) -4
b) -2
c) 2
d) 4
e) None of these

---

9. A particle moves along the x-axis at that its position at any time $t \geq 0$ is given by $x(t) = \dfrac{t}{4+t^2}$. The particle is at rest at $t =$

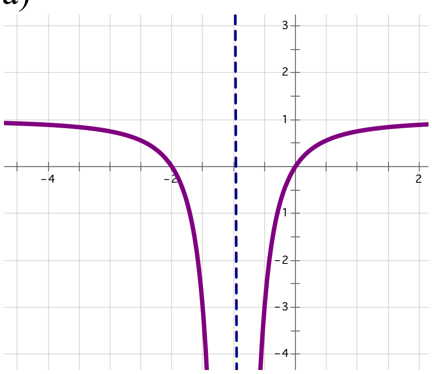
a) 0 b) $\dfrac{1}{4}$ c) 1 d) 2 e) 4

---

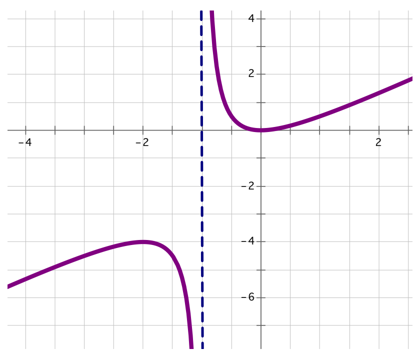
10. Determine which of the following graphs matches the sign pattern

$f'(x)$	$-$	0	$+$	VA	$+$	0	$-$
x		-2		-1		0	

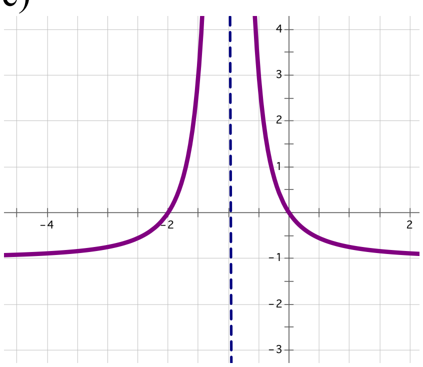
a)



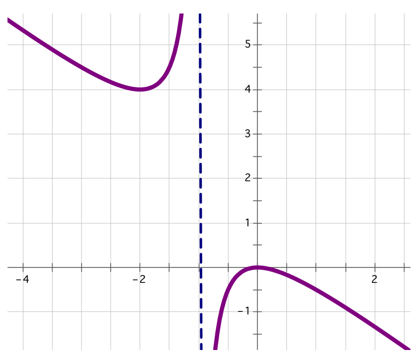
b)



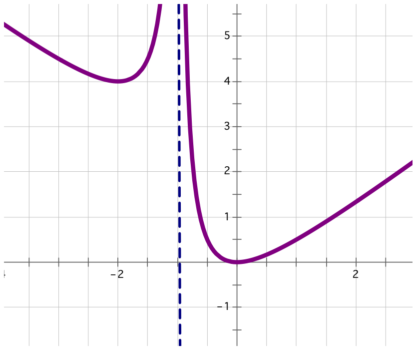
c)



d)



e)



Rational Functions Practice Test
Part 2: CALCULATOR ALLOWED

1. Find the zeros and VAs of $y=\dfrac{4x^2-16x}{x^3-4x^2+x-4}$. Show the supporting algebraic work.

Zeros:

VAs:

POE:

2. Find the extreme points of $y=\dfrac{4x^2-16x}{x^3-4x^2+x-4}$. Show the derivative and algebra to support the critical values.

3. Find the equations of the lines tangent to and normal to $y = \frac{2x^2 - x - 3}{3 + 2x - x^2}$ at $x = 0$?

Tangent:

Normal:

4. Find the zeros, VAs, POEs and HA of $y = \frac{16 - x^2}{x^2 - 25}$. Show the supporting algebraic work.

Zeros:

VAs:

EB:

POE:

5. Find the extreme points of $y=\frac{16-x^2}{x^2-25}$ on $x\in\left[-4,\ 4\right]$. Show the derivative and algebra to support the critical values.

6. Find the traits and **sketch** $y=\frac{2x^3-5x^2+x+2}{2x^2+5x+2}$.

Domain:

Y – Intercept:

Zeros:

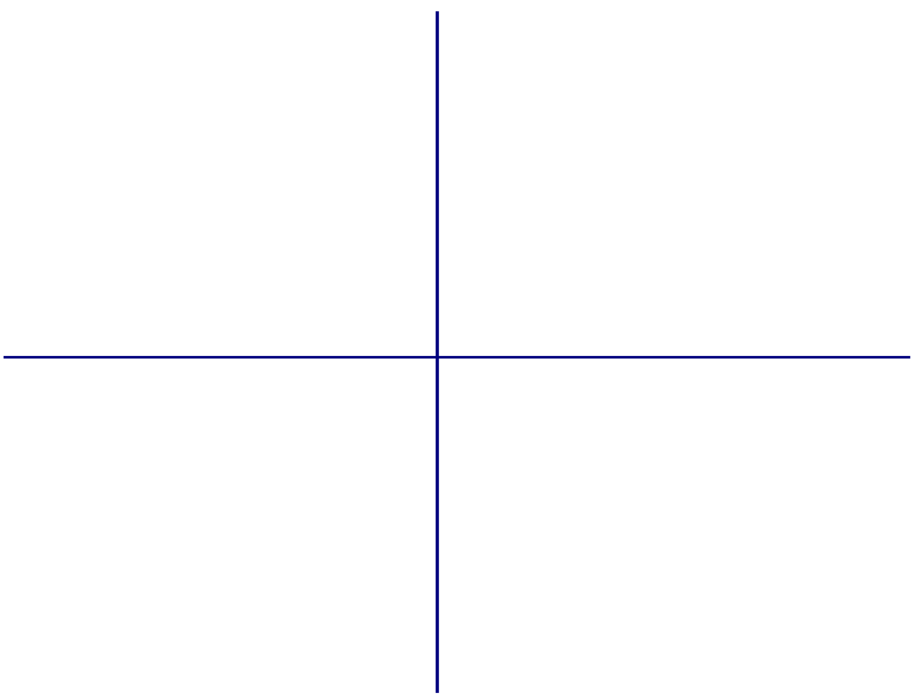
Range:

VAs:

End Behavior:

POEs:

Extreme Points:



Rational Functions Practice Test
Part 2: NO CALCULATOR ALLOWED

Round to 3 decimal places. Show all work.

7. Write an equation of a rational function that has x–intercepts at (-2, 0), VA at $x = 4$, a POE at $x = -3$, and a HA at $y = \frac{3}{7}$.

8. Show the sign pattern and solve $\frac{2x^2 - x - 3}{3 + 2x - x^2} \le 0$.

9. Find the traits and **sketch** $y = \dfrac{16 - x^2}{x^2 - 25}$ on $x \in [-4, 4]$.

Domain:

Range:

Y – Intercept:

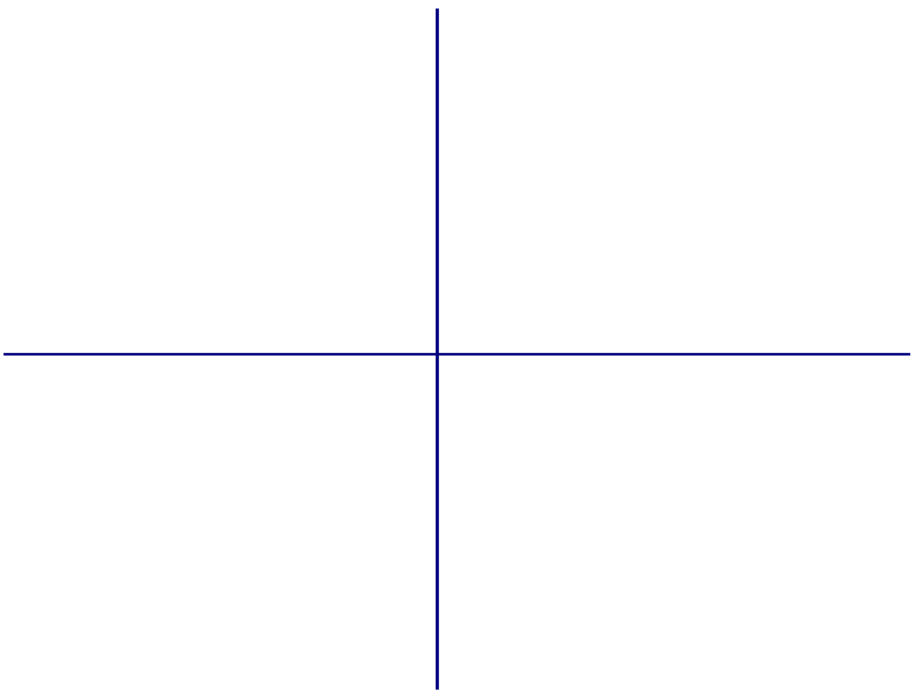
End Behavior:

Zeros:

Extreme Points:

VAs:

POE:



10. Find the traits and **sketch** of $y = \dfrac{4x^2 - 16x}{x^3 - 4x^2 + x - 4}$.

Domain:

Y – Intercept:

Zeros:

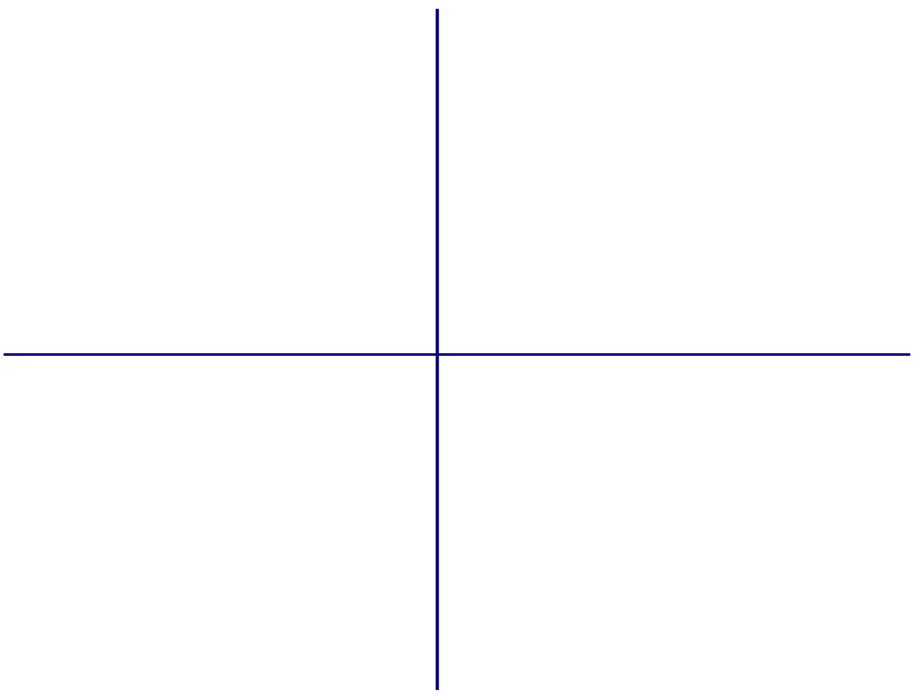
Range:

VAs:

End Behavior:

POEs:

Extreme Points:



Rational Functions Homework Answer Key

7-1 Free Response Homework

1. Zero: $(-2,\ 0)$; VA: $x=1,\ x=2$; POE: $\left(3,\ \frac{5}{2}\right)$

2. Zero: $(1,\ 0)$; VA: $x=2,\ x=-1$; POE: $\left(-3,\ -\frac{2}{5}\right)$

3. Zeros: None; VA: $x=4$ and $x=-2$; POE: None

4. Zeros: None; VA: $x=\pm\sqrt{5}$; POE: None

5. Zeros: $(-5,\ 0)$, $(3,\ 0)$; VA: $x=-3$; POE: None

6. Zero: $(-5,\ 0)$; VA: None; POE: $(3,\ 8)$

7. Zeros: $(-6,\ 0)$, $(5,\ 0)$; VA: $x=1,\ x=-1$; POE: None

8. Zeros: $\left(-\frac{1}{3},\ 0\right)$, $\left(\frac{1}{2},\ 0\right)$; VA: $x=-4,\ x=-1,\ x=2$; POE: None

9. Zeros: $\left(-2\pm\sqrt{2},\ 0\right)$; VA: $x=-3$; POE: $\left(2,\ \frac{14}{5}\right)$

10. Zero: $(-3,\ 0)$; VA: $x=-2\pm\sqrt{2}$; POE: $\left(2,\ \frac{5}{14}\right)$

11. Zero: $(0,\ 0)$; VA: $x=\pm\sqrt{3}$; POE: $\left(\sqrt{2},\ -\sqrt{2}\right)$, $\left(-\sqrt{2},\ \sqrt{2}\right)$

12. Zeros: $(-3,\ 0)$, $(-1,\ 0)$, $(2,\ 0)$; VA: $x=1,\ x=3$; POE: None

13. Zeros: $(\pm 1,\ 0)$; VA: $x=-6,\ x=5$; POE: None

14. Zeros: $(-2,\ 0)$; VA: $x=-3,\ x=\dfrac{7}{2}$; POE: $\left(2,\ -\dfrac{36}{15}\right)$

15. Zero: $(-3,\ 0)$; VA: $x=-4,\ x=3$; POE: None

16. Zeros: $(-2,\ 0),\ (3,\ 0),\ (4,\ 0)$; VA: $x=-5,\ x=\dfrac{3}{2}$; POE: None

17. Zeros: $(-0.298,\ 0),\ (-6.702,\ 0)$; VA: $x=-2,\ x=-4$; POE: None

18. Zeros: None; VA: $x=-3,\ x=-2$; POE: None

19. Zero: $(-1,\ 0)$; VA: $x=-5$; POE: $\left(3,\ \dfrac{1}{2}\right)$

20. Zeros: $(-2,\ 0),(-1,\ 0)$;VA: $x=1,\ x=2$; POE: None

21. Zero: None; VA: None; POE: $(3,\ 7),\ (-4,\ 7)$

<u>7-1 Multiple Choice Homework</u>

1. C 2. D 3. E 4. A 5. C 6. A

<u>7-2 Free Response Homework</u>

1. $-\dfrac{x-4}{x+1}>0$

2. $-\dfrac{x^2}{x^2-2}\geq 0$

3. $\dfrac{\left(x-\sqrt{6}\right)(x-2)\left(x+\sqrt{6}\right)}{(x+1)^2\left(x+\sqrt{6}\right)}>0$

4. $\dfrac{x(x-5)}{(x+5)}\geq 0$

5. $\dfrac{x-5}{(x+3)^2(x-4)}>0$

6. $-\dfrac{(3x+1)(3x-4)}{(3x-2)^2}\ge 0$

7. $x\in(-\infty,\ -3]\cup(-1,\ 1]$

8. $x\in(-4,\ 1)\cup(3,\ \infty)$

9. $x\in[-2,\ 1)\cup[2,\ \infty)$

10. $x\in\left[-\dfrac{7}{2},\ -2\right)$

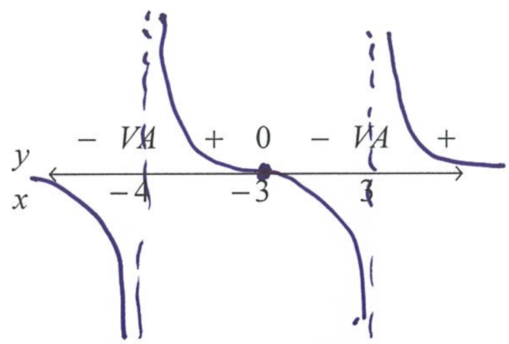
11. $x\in(-5,\ -2)\cup(5,\ \infty)$

12. $x\in\left(-\infty,\ \dfrac{1}{2}\right)\cup[1,\ 3)\cup\left[\dfrac{7}{2},\ \infty\right)$

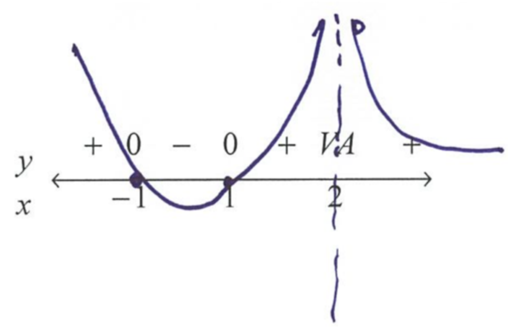
13. $x\in(-1,\ 1]\cup\left[\dfrac{3}{2},\ \infty\right)$

14. $x\in(-\infty,-3]\cup(-2,2)\cup[3,\ \infty)$

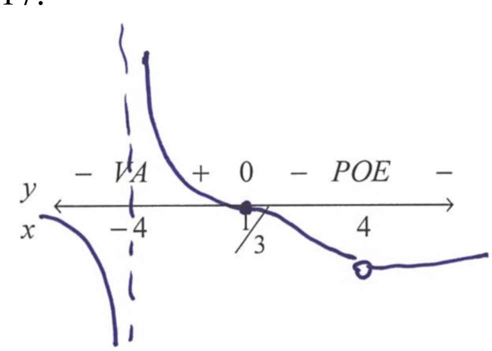
15.



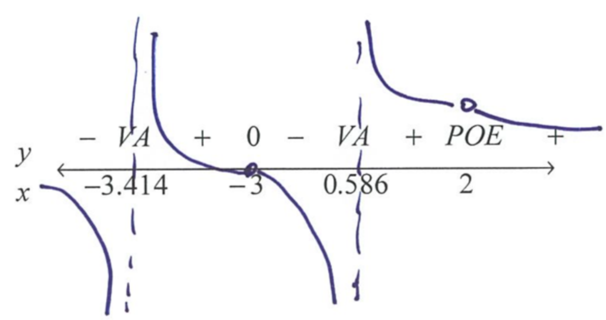
16.



17.

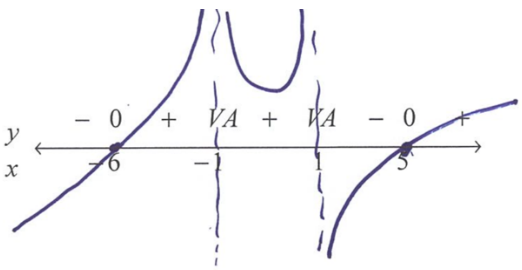


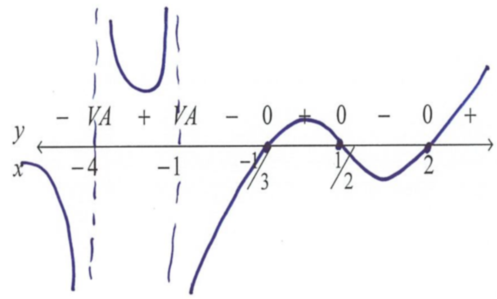
18.



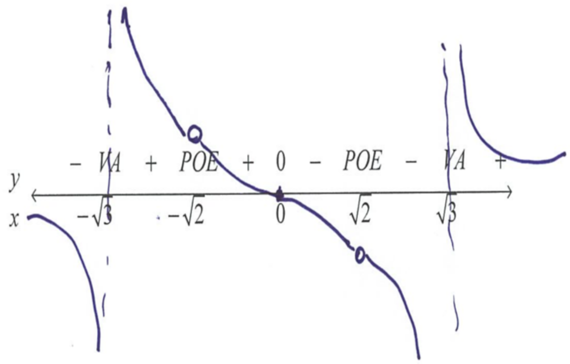
19.

20.

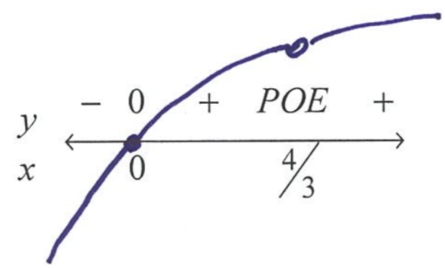




21.



22.



<u>7-2 Multiple Choice Homework</u>

1. C 2. D 3. C 4. A 5. B

<u>7-3 Free Response Homework</u>

1. 0 2. 3 3. 0 4. ∞

5. 1 6. $-\infty$

7. Zeros: None; POE: None; VA: $x=\pm\sqrt{2}$; HA: $y=0$

8. Zeros: None; POE: $\left(-1,-\frac{1}{9}\right)$; VA: $x=2,\ x=-4$; HA: $y=0$

9. Zeros: None; POE: None; VA: $x=-5,\ x=3$; HA: $y=0$

10. Zero: $(-3,\ 0)$; POE: $\left(2,\ \frac{5}{14}\right)$; VA: $x=2\pm\sqrt{2}$; HA: $y=0$

11. Zero: $(-3,\ 0)$; POE: None; VA: $x=-4,\ x=3$; HA: $y=0$

12. Zeros: $(3,\ 0),\ (1,\ 0)$ POE: None; VA: $x=6,\ x=-1$; HA: $y=1$

13. Zero: $(-2,\ 0)$; POE: $\left(3,\ \frac{5}{2}\right)$; VA: $x=2,\ x=1$; HA: $y=0$

14. Zeros: None; POE: $\left(2,\ -\frac{3}{5}\right)$; VA: $x=\pm 3,\ x=-2$; HA: $y=0$

15. Zeros: $(-1,\ 0)$; POE: $\left(1,\ \frac{1}{2}\right)$; VA: $x=-3$; HA: $y=1$

16. Zeros: $(\pm 4,\ 0)$; POE: None; VA: None; HA: $y=-1$

17. Zero: $(-1,\ 0)$; POE: $\left(3,\ -\frac{2}{3}\right)$; VA: $x=-3$; HA: $y=-1$

18. Zero: $(\pm 1,\ 0)$ and $\left(\frac{7}{2},\ 0\right)$; POE: none; VA: $x=-2,\pm 3$; HA: $y=2$

19. Zero: $(0,\ 0)$; POE: $\left(4,\ \frac{12}{25}\right)$; VA: *none*; HA: $y=0$

20. Zero: $(1,\ 0)$ and $\left(\frac{2}{3},\ 0\right)$; POE: $\left(-2,\ \frac{21}{4}\right)$; VA: $x=-1, 2$; HA: $y=2$

<u>7-3 Multiple Choice Homework</u>

1. E 2. A 3. D 4. 5. B 6. B

7-4 Free Response Homework

1. $x+10+\dfrac{45}{x-4}$ 2. $1+\dfrac{-6x+6}{x^2-4}$ 3. $\dfrac{1}{2}x-2+\dfrac{1}{x-2}$

4. $x-4+\dfrac{2}{x^2-4x+4}$

5. Zeros: $(1,\ 0),(5,\ 0)$; POE: None; VA: x=4; SA: $y=x+1$

6. Zeros: $(0.354,\ 0),(5.646,\ 0)$; POE: None; VA: $x=\pm 2$; SA: $y=x-3$

7. Zero: $(1,\ 0)$; POE: None; VA: None; SA: $y=x$

8. Zeros: $(0,\ 0),(4,\ 0)$; POE: None; VA: $x=2$; SA: $y=x-2$

9. Zeros: $(\pm 1,\ 0)$; POE: None; VA: $x=-6,\ x=5$; SA: $y=x$

10. Zeros: $(-2,\ 0),\ (3,\ 0),\ (4,\ 0)$; POE: None; VA: $x=\dfrac{3}{2},\ x=-5$;
SA: $y=\dfrac{1}{2}x-\dfrac{17}{4}$

11. VA: $x=-3$; SA: $y=-2x+11$;Zeros: $\left(-\dfrac{1}{2},\ 0\right),(3,\ 0)$; POE: $(2,\ -1)$

7-4 Multiple Choice Homework

1. D 2. A 3. B 4. D 5. E

7-5 Free Response Homework

1. $\dfrac{-2x}{\left(x^2-4\right)^2}$ 2. $\dfrac{-4x^2-48x-36}{\left(x^2-9\right)^2}$ 3. $\dfrac{-3x^2+10x-14}{\left(x^2-x-3\right)^2}$

4. $2x-4$

5. $\dfrac{-3(2x-1)}{(x^2-x+1)^2}$

6. $\dfrac{x^2-8x-5}{(x-4)^2}$

7. $\dfrac{2}{3}x+\dfrac{38}{3x^3}$

8. $\dfrac{-1}{x^2-10x+25}$

9. 1

10. Tangent: $y-\dfrac{2}{17}=-\dfrac{38}{289}(x+1)$ Normal: $y-\dfrac{2}{17}=\dfrac{289}{38}(x+1)$

11. Tangent: $y-\dfrac{2}{3}=-\dfrac{2}{9}(x-1)$ Normal: $y-\dfrac{2}{3}=\dfrac{9}{2}(x-1)$

12. Tangent: $y=-\dfrac{3}{2}$ Normal: $x=1$

13. Tangent: $y-\dfrac{1}{5}=\dfrac{3}{25}(x-2)$ Normal: $y-\dfrac{1}{5}=-\dfrac{25}{3}(x-2)$

14. No solution

15. $x=0$

16. $x=0$

17. $x=0,\ -2\pm\sqrt{13}$

18. $(6.742,\ 0.074),\ (-0.742,\ -0.674)$

19. $(-2+\sqrt{6},\ -0.832),(-2-\sqrt{6},\ -0.048)$

20. None

21. None

22. $(0,\ -4)$

23. None

24. $(-3,-6),(1,2)\left(-6,\ \dfrac{39}{5}\right),\left(2,\ \dfrac{7}{3}\right)$

25. $\left(-1,\ \dfrac{1}{3}\right),\left(5,\ \dfrac{5}{21}\right)$

26. $(0,-1),\left(-3,\ \dfrac{5}{13}\right),\left(5,\ \dfrac{21}{29}\right)$

27. $(2,\ -1),(-2,\ 1),\left(-3,\ \dfrac{12}{13}\right)$

28. $\left(0, -\frac{9}{16}\right), (3,\ 0)$

29. $(0,\ -1), \left(2,\ \frac{1}{3}\right), \left(3, \frac{2}{7}\right)$

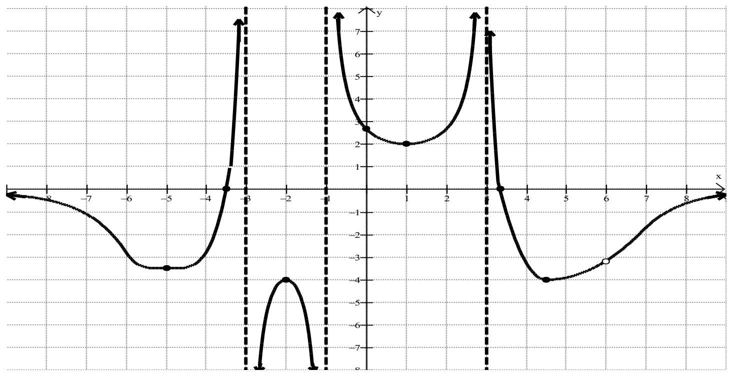
30. $(3, 0), (11,\ .889), (12,\ .890), (-3,\ 2.25)$

7-5 Multiple Choice Homework

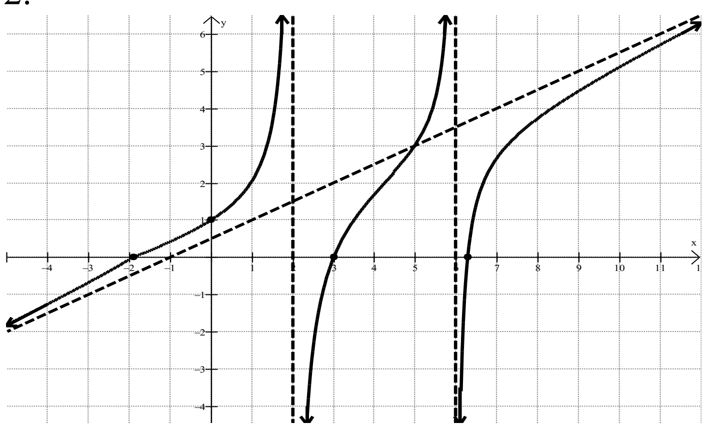
1.	A	2.	B	3.	A	4.	E
5.	D	6.	B	7.	C	8.	B

7-6 Free Response Homework

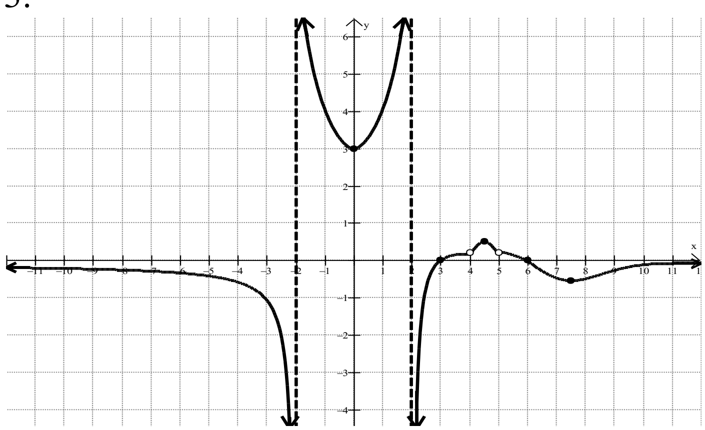
1.



2.



3.



4. Domain: $x \neq -5,\ -\dfrac{1}{4},\ 4$ Range: $y \in (-\infty,\ \infty)$

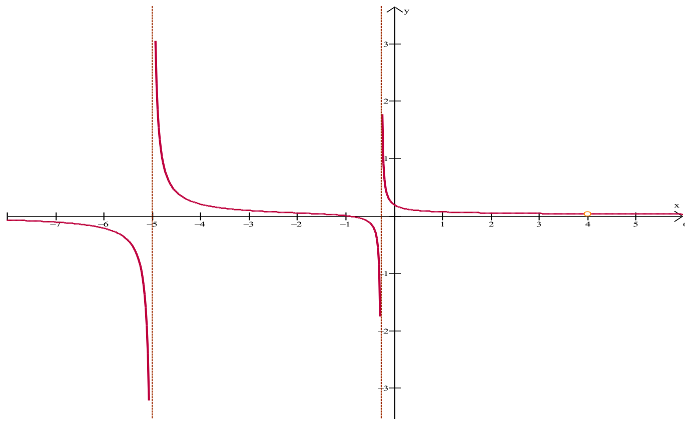
Zero: $(-1,\ 0)$ y-int: $\left(0,\ \dfrac{1}{5}\right)$

VA: $x = -5,\ x = -\frac{1}{4}$

HA: $y = 0$

POE: $\left(4,\ \frac{5}{153}\right)$

Extreme Points: None



5. Domain: $x \neq -5,\ 1$

Zeros: $(0, 0)$

VA: $x = -5,\ x = 1$

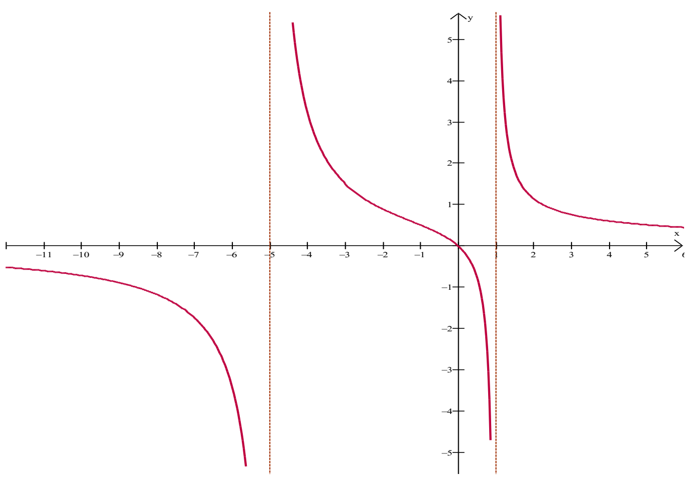
HA: $y = 0$

Range: $y \in (-\infty,\ \infty)$

y-int: $(0, 0)$

POE: None

Extreme Points: None



6. Domain: $x \neq -3,\ 1$

Zero: $(-1,\ 0)$

VA: $x = -3$

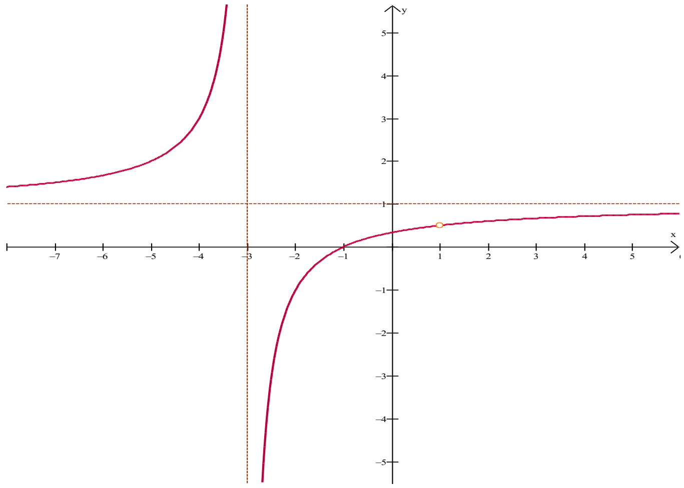
HA: $y = 1$

Range: $y \neq 1,\ \frac{1}{2}$

y-int: $\left(0,\ \frac{1}{3}\right)$

POE: $\left(1,\ \frac{1}{2}\right)$

Extreme Points: None

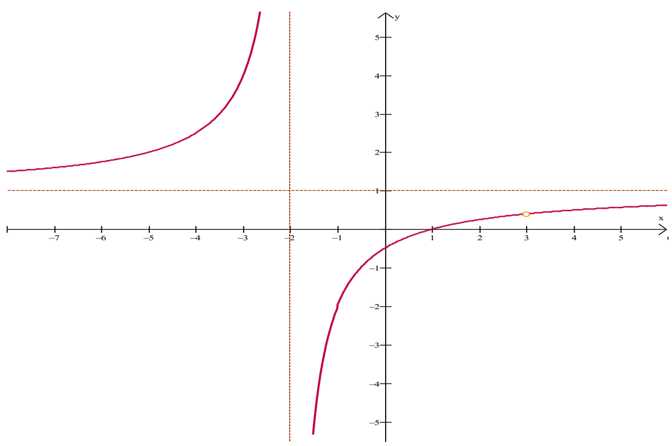


7. Domain: $x \neq -2,\ 3$ — Range: $y \neq 1,\ \frac{2}{5}$

Zero: $(1,\ 0)$ — y-int: $\left(0,\ -\frac{1}{2}\right)$

VA: $x = -2$ — POE: $\left(3,\ \frac{2}{5}\right)$

HA: $y = 1$ — Extreme Points: None

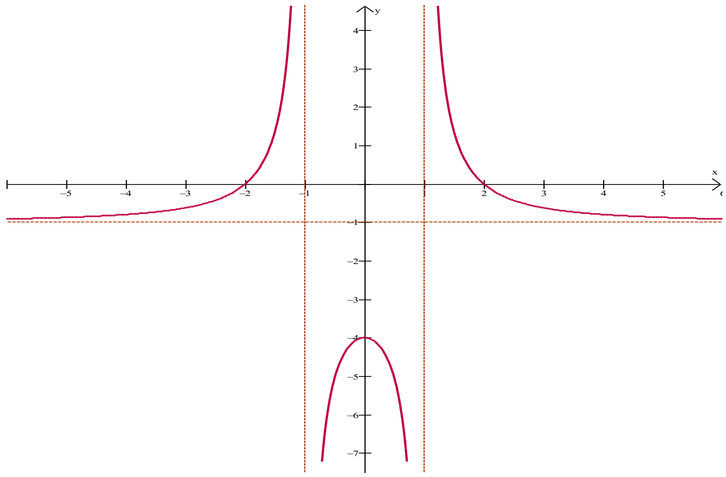


8. Domain: $x \neq -1,\ 1$ — Range: $y \in (-\infty,\ -4] \cup (1,\ \infty)$

Zeros: $(2,\ 0)$, $(-2,\ 0)$ — y-int: $(0,\ -4)$

VA: $x = -1,\ x = 1$ — POE: None

HA: $y = -1$ — Extreme Point: $(0,\ -4)$

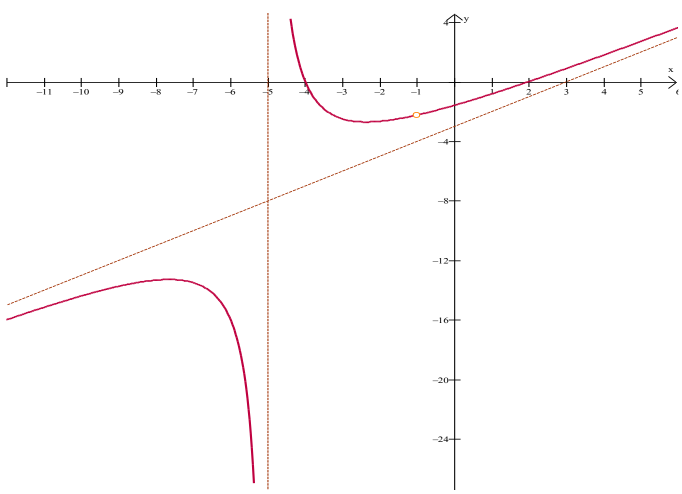


9. Domain: $x \neq -5, -1$ Range: $y \in (-\infty, -13.292] \cup [-2.708, \infty)$

Zeros: $(2, 0)$, $(-4, 0)$ y-int: $\left(0, -\frac{8}{5}\right)$

VA: $x = -5$ POE: $\left(-1, -\frac{9}{4}\right)$

SA: $y = x - 3$ Extreme Points: $(-2.354, -2.708)$, $(-7.645, -13.292)$



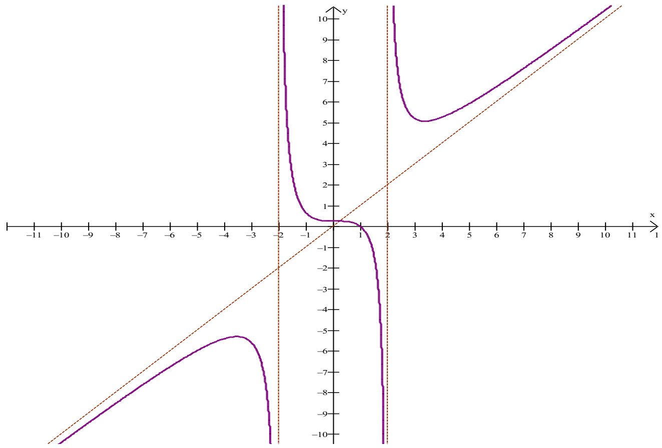
10. Domain: $x \neq -2, 2$ Range: $y \in (-\infty, \infty)$

Zeros: $(1, 0)$ y-int: $\left(0, \frac{1}{4}\right)$

VA: $x = -2, 2$ POE: None

SA: $y = x$

Extreme Points: $(-3.544,\ -5.316), (3.378,\ 5.066),\ (0,\ 0.25), (0.167,\ 0.251)$



11. Domain: $x \neq 2$

Range: $y \in (-\infty,\ \infty)$

Zeros: $(0,\ 0),\ (4,\ 0)$

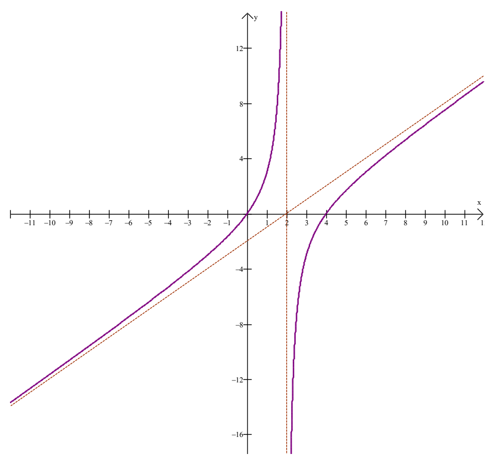
y-int: $(0,\ 0)$

VA: $x = 2$

POE: None

SA: $y = x - 2$

Extreme Points: None



12. Domain: $x \neq 2$

Range: $y \in \left[-\frac{25}{24},\ \infty\right)$

Zeros: $(0,\ 0),\ (5,\ 0)$

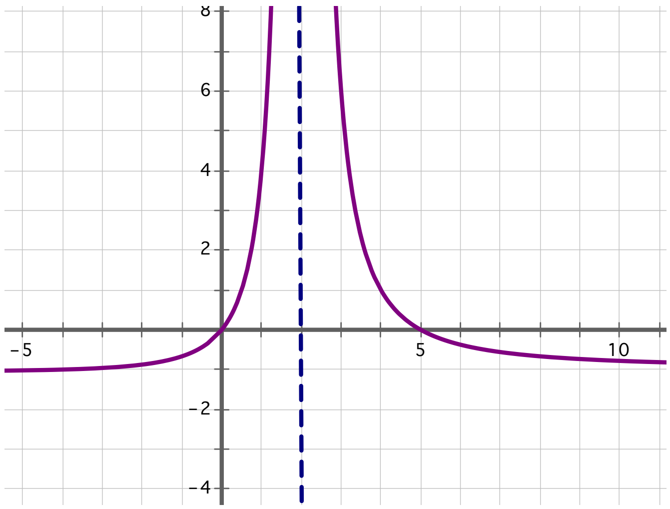
y-int: $(0,\ 0)$

VA: $x = 2$

POE: None

EB: $y = -1$

Extreme Points: $\left(-10,\ -\frac{25}{24}\right)$

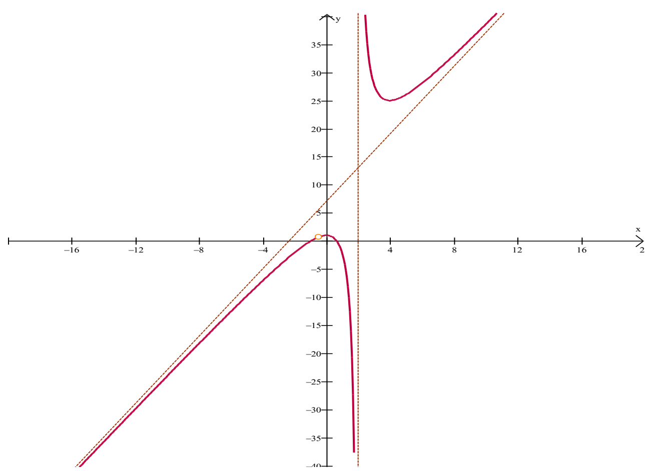


13. Domain: $x \neq -\frac{1}{2},\ 2$ Range: $y \in (-\infty,\ 1] \cup [25,\ \infty)$

Zeros: $\left(\frac{2}{3},\ 0\right),\ (-1,\ 0)$ y-int: $(0, 1)$

VA: $x = 2$ POE: $\left(-\frac{1}{2},\ \frac{7}{10}\right)$

SA: $y = 3x + 7$ Extreme Points: $(0,\ 1),\ (4,\ 25)$

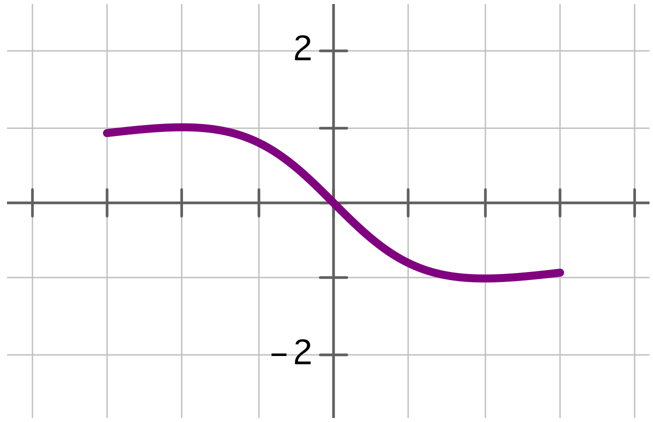


14. Domain: $x \in [-3, 3]$ Range: $y \in [-1, 1]$

Zeros: $(0,\ 0)$ y-int: $(0,\ 0)$

VA: None POE: None EB: None

Extreme Points: $(2,\ -1), (-2,\ 1)\left(3,\ -\frac{12}{13}\right), \left(-3,\ \frac{12}{13}\right)$



15. Domain: $x \in [-5, 5]$

Range: $y \in \left[-\frac{9}{16}, \frac{16}{41}\right]$

Zeros: $(\pm 3,\ 0)$

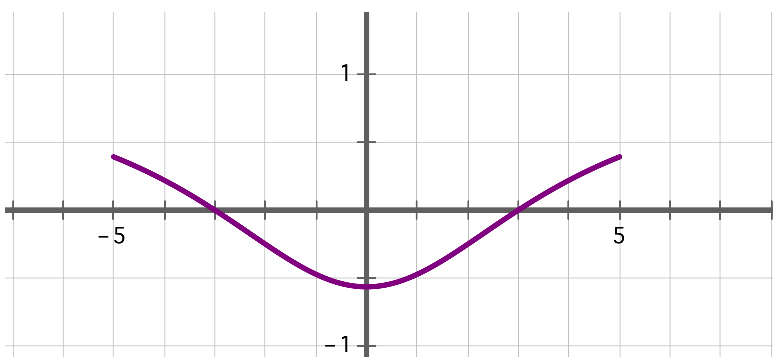
y-int: $\left(0,\ -\frac{9}{16}\right)$

VA: None

POE: none

EB: None

Extreme Points: $\left(0,\ -\frac{9}{16}\right)$ and $\left(\pm 5,\ \frac{16}{41}\right)$



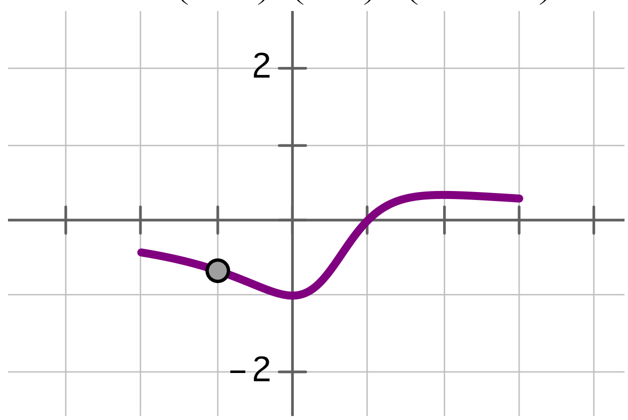
16. Domain: $x \in [-2, -1) \cup (-1, 3]$ Range: $y \in \left[-1, \frac{1}{3}\right]$

Zeros: $(1,\ 0)$ y-int: $(0,\ -1)$

VA: None POE: $\left(-1,\ -\frac{2}{3}\right)$

EB: None

Extreme Points: $(0,\ -1)$, $\left(2,\ \frac{1}{3}\right)$, $\left(3, \frac{2}{7}\right)$, $\left(-2,\ -\frac{3}{7}\right)$

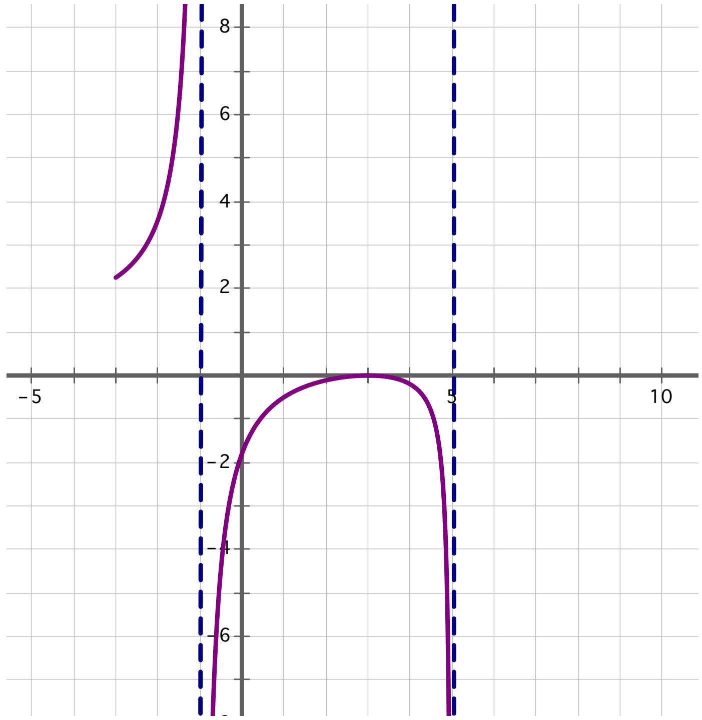


17. Domain: $x \in [-3, -1) \cup (-1, 5)$ Range: $y \in (-\infty,\ 0] \cup \left[\frac{9}{4},\ \infty\right)$

Zeros: $(3,\ 0)$ y-int: $\left(0,\ -\frac{9}{5}\right)$

VA: $x = -1,\ 5$ POE: None

EB: None Extreme Points: $\left(-3, \frac{9}{4}\right)$, $(3,\ 0)$

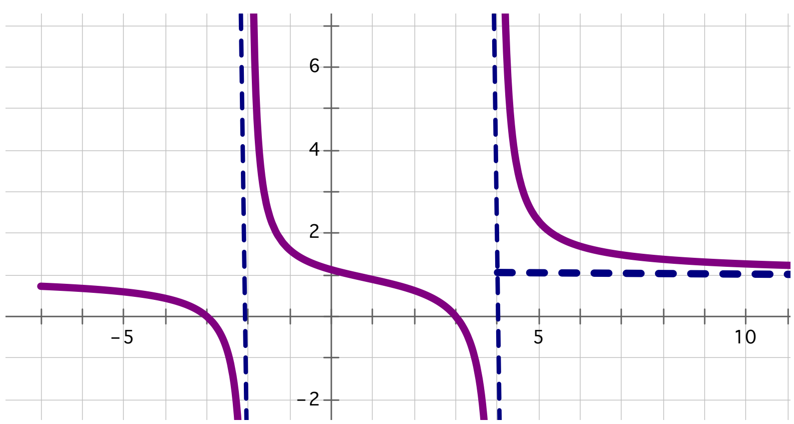


18. Domain: $x \in [-7, -2) \cup (-2,\ 4) \cup (4,\ \infty)$ Range: $y \in$ All Reals

Zeros: $(\pm 3,\ 0)$ y-int: $\left(0,\ \frac{9}{8}\right)$ VA: $x = -2, 4$

POE: None EB (left): None EB (right): $y = 1$

Extreme Points: $\left(-7, \frac{8}{11}\right)$

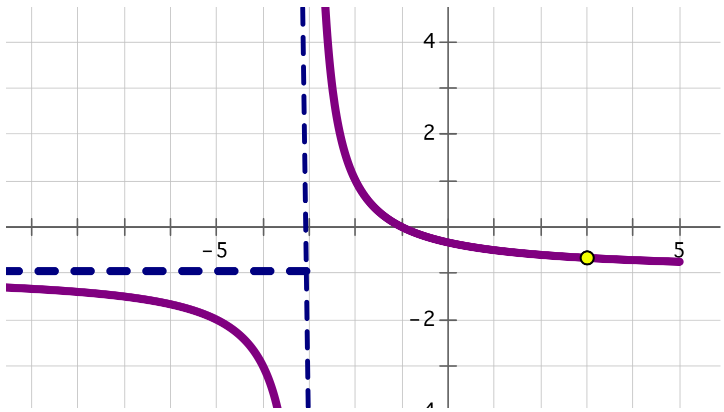


19. Domain: $x \in (-\infty, -3) \cup (-3, 3) \cup (3, 5]$

Zeros: $(-1, 0)$ y-int: $\left(0, -\frac{1}{3}\right)$ VA: $x = 3$ POE: $\left(3, -\frac{2}{3}\right)$

EB (left): $y = -1$ EB (right): None

Extreme Points: $(5, -0.75)$



Range: $y \in (-\infty, -1) \cup \left[-\frac{3}{4}, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$

20. $y = \dfrac{3x^2 - 12x}{x^3 - 4x^2 + 9x - 36}$ on $x \in (-\infty, 4]$

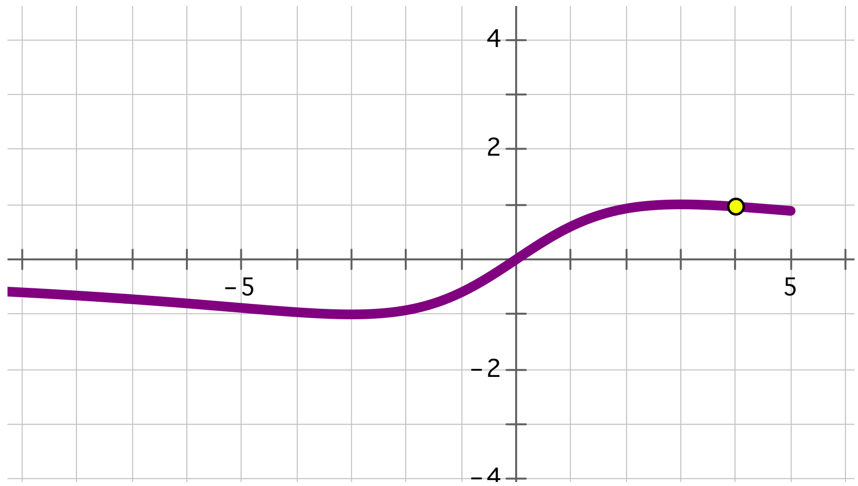
Domain: $x \in (-\infty, 4)$ Range: $y \in [-0.5, 0.5]$

Zeros: $(0, 0)$ y-int: $(0, 0)$

VA: none POE: $\left(4, \frac{12}{25}\right)$

EB (left): $y = 0$ EB (right): None

Extreme Points: $(-3, -0.5)$, $(3, 0.5)$



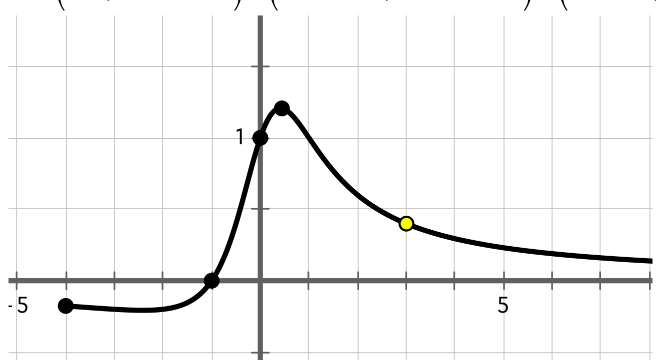
21. Domain: $x \in [-4, 3) \cup (3, \infty)$ Range: $y \in [-0.207, 1.207]$

Zeros: $(-1, 0)$ y-int: $(0, 1)$

VA: none POE: $(3, .4)$

EB (left): None EB (right): $y = 0$

Extreme Points: $(-4, -0.177)$, $(-2.414, -0.207)$, $(0.414, 1.207)$



22. $y = \dfrac{2(x-2)(x+1)}{(2x-1)(x-3)}$

23. $y = \dfrac{-x}{x(3x-1)}$

24. $y = \dfrac{(x-3)}{x^2+1}$

25. $y = \dfrac{2(x+1)(x+5)}{(x-1)}$

7-6 Multiple Choice Homework

1. A 2. E 3. A 4. B 5. D 6. D

7-7 Free Response Homework

1. Min point at $x=1$; Max point at $x=9$

2. Min point at $x=5-\sqrt{34}$; Max point at $x=5+\sqrt{34}$

3. Min point at $x=-0.721$; Max point at $x=1.387$

4. No critical values

5. Min point at $x=2$; Max point at $x=-2$

6. Min point at $x=0$

7. Min point at $x=0$; Max point at $x=2$

8. Min point at $x=11$; Max point at $x=3$

9. At $t=0.414$, it stops 0.828 units right of the origin.

10. At $t=-\sqrt{3}$, the object stops 0.144 units left of the origin. At $t=\sqrt{3}$, the object stops 0.144 units right of the origin.

11. $t\in(2,\ 3)$

12. $t\in(0,\ 4)$

7-7 Multiple Choice Homework

1. A 2. A 3. C 4. D 5. D

Rational Functions Practice Test Answer Key

Multiple Choice

1.	C	2.	B	3.	D	4.	B	5.	C
6.	A	7.	D	8.	B	9.	D	10.	B

Free Response

1. Zero: $(0,\ 0)$; VA: None; POE: $\left(4,\ \frac{16}{17}\right)$

2. $(-2,-1)$ and $(2,1)$

3. Tangent: $y+1=\frac{1}{3}(x-0)$
   Normal: $y+1=-3(x-0)$

4. Zero: $(\pm 4,\ 0)$; VA: $x=\pm 5$; POE: None; EB: $y=-1$

5. $(\pm 4,\ 0)$ and $\left(0,\ -\frac{16}{25}\right)$

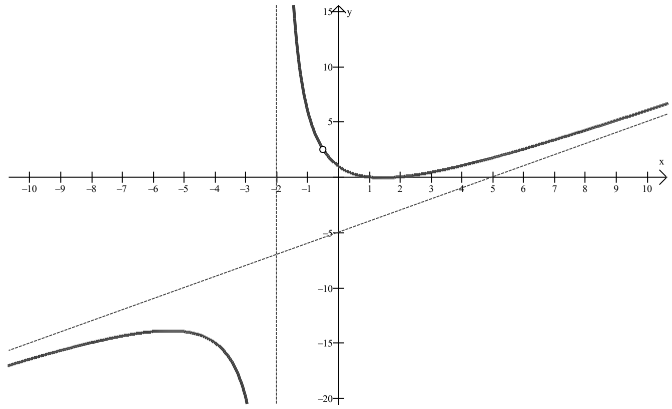
6. Domain: $x \neq -\frac{1}{2},\ -2$ Range: $y \in (-\infty,\ -13.928] \cup [-0.072,\ \infty)$

   Zeros: $(1,\ 0),\ (2,\ 0)$ y-int: $(0,\ 1)$

   VA: $x=-2$ POEs: $\left(-\frac{1}{2},\ \frac{5}{2}\right)$

   Extreme Points: $(-5.464,\ -13.928),\ (1.464,\ -0.072)$

   End Behavior: HA: None; SA: $y=x-5$



7. $y = \dfrac{3(x+2)(x+3)}{7(x-4)(x+3)}$

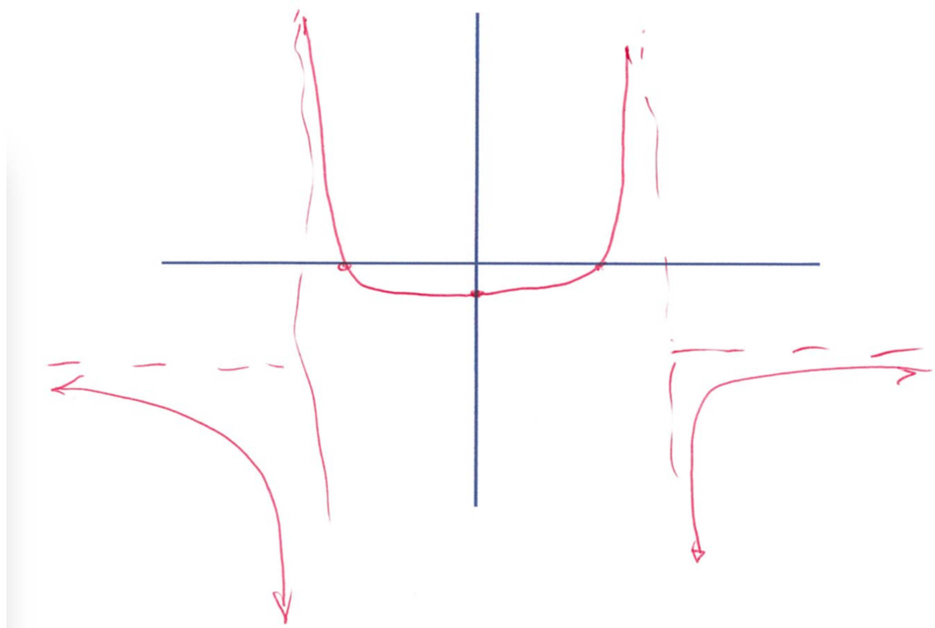
8. $x \in (-\infty, -1) \cup \left(-1, \dfrac{3}{2}\right] \cup (3, \infty)$

9. Domain: $x \neq \pm 5$ Range: $y \in (-\infty, -1) \cup \left[-\dfrac{16}{25}, \infty\right)$

Zero: $(\pm 4,\ 0)$ y-int: $\left(0,\ -\dfrac{16}{25}\right)$

VA: $x = \pm 5$ POEs: None

Extreme Points: $\left(0,\ -\dfrac{16}{25}\right)$ End Behavior: HA: $y = -1$



10. Domain: $x \neq 4$ Range: $y \in$ All Reals

Zero: $(0,\ 0)$ y-int: $(0,\ 0)$

VA: None POEs: $\left(4,\ \dfrac{16}{17}\right)$

Extreme Points: $(-2,-1)$ and $(2,1)$ End Behavior: HA: $y=0$