AP Calculus AB '20-21

Spring Final Part IIA v2

Calculator Allowed

Name:

Solution Key

1. More than 30% of observed star systems have multiple stars, and 70% of those have more than two stars. When stars are close together, they exchange mass in a process known as accretion. Consider a trinary system where S_1 is larger than S_2, and S_2 is larger than S_3. S_3 will lose mass to S_2, and S_2 will lose mass to S_1 While scientific readings are not available because of the time scale, let us suppose that S_2 loses mass to the larger S_1 at a rate of $L(t)=1+(.01t)^2+.23\sin\left(\frac{\pi}{25}t\right)$ and gains mass from the smaller S_3 at a rate of $G(t)=0.2+0.15\sqrt{t}$ where $0\le t\le 100$ years. $L(t)$ and $G(t)$ are measured in yottatons per year $\left(\frac{Y}{yr}\right)$. (A yottaton is 10^{26} tons, or 10^{-7} solar masses.)

(a) How much mass does S_2 lose to S_1 on $0\le t\le 100$? State the units. 1PT

$\int_0^{100} L(t)\,dt = 133.333$ yottatons

(b) At $t=50$, is the mass S_2 is gaining from S_3 increasing at an increasing rate? Using the correct units, ~~justify~~ explain your answer. 2PTS

$G'(50) = .0106$

THE RATE AT WHICH S_2 IS GAINING MASS FROM S_3 IS INCREASING BY $.011\ Y/YR^2$ AT $t=50$ YEARS

(c) At what times on $0 \le t \le 100$ is S_2 losing as much mass to S_1 as it is gaining from S_3? Justify your answer

GRAPH $L - G = 0$

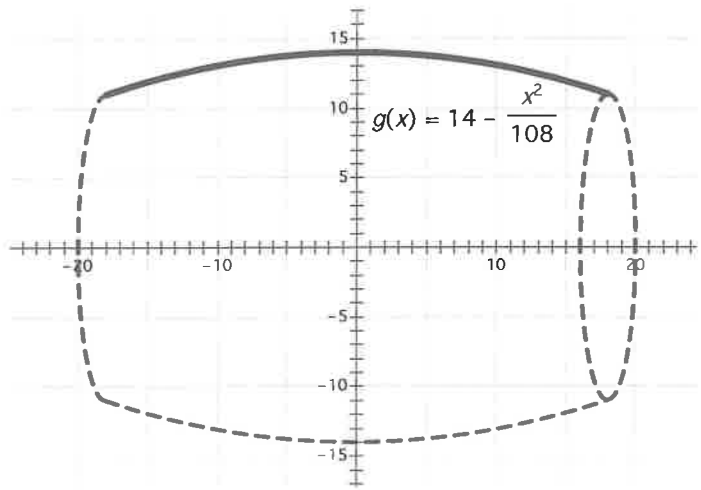
$t = 28.001,\ 50.376,\ 77.948,\ 91.465$

(d) If the mass of S_2 is 100 yottatons at $t = 0$, find the minimum mass of S_2 on the time interval $0 \le t \le 100$.

t	$w(t) = 100 + \int_0^t G - L\, dt$
0	100
28.01	88.152
50.376	91.191
77.948	87.137
91.465	87.847
100	86.667

THE ABSOLUTE ~~MAX~~ MINIMUM WEIGHT IS 86.667 YOTTATONS

2. Johannes Kepler provided fundamental advances before Newton or Leibnitz ever developed the Calculus. He did this when he remarried after his first wife's death and, at the wedding party, became incensed because he believed the wine merchant was cheating him on the price by volume of the wine. (He thought the way of measuring was inaccurate, but Kepler was wrong.)



A 53-gallon wine barrel has a lid diameter of 22 inches, a bilge diameter (the diameter of the widest part of the barrel) of 28, and a height of 36 inches. If the barrel were lying on its side, the same shape could be created by rotating the parabola $y = 14 - \frac{1}{108}x^2$ on $x \in [-18, 18]$ about the x-axis.

(a) Find the volume, in cubic inches of the shape formed by the rotation. Show the antiderivative steps. 3pts

$$V = \pi \int_{-18}^{18} \left(14 - \frac{1}{108}x^2\right)^2 dx =$$

$$= 2\pi \int_0^{18} 196 - \frac{7}{27}x^2 + \frac{1}{108^2}x^4 \, dx$$

$$= 2\pi \left[196x - \frac{7}{81}x^3 - \frac{1}{108^2 \cdot 5}x^5\right]_0^{18} = \cancel{193\,928.244} \text{ IN}^3$$

19203.928 IN^2

(b) The surface area of a solid of rotation is found by the equation $A=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$. Find the surface area of the sides of wine barrel, including both end lids. Show the set up and indicate the correct units.

$f'(x) = \frac{1}{54}x$

$$A = 2\pi \int_{-18}^{18} \left(14 - \frac{1}{108}x^2\right)\sqrt{1+\left(\frac{1}{54}x\right)^2}\,dx$$

$= 3751.106 \text{ in}^2$

(c) Assume that the volume of wine in the barrel can be approximated by a cylinder with a diameter of 25" $(V_{cyl} = \pi r^2 h)$. The barrel is placed upright and wine is scooped out in pitchers to serve to the wedding guests. If the wine is removed from the barrel at $4800\,in^3/hr$, how fast is the level of the wine in the barrel is dropping? Indicate units.

$V = \pi r^2 h = (12.5)^2 \pi h$

$-4800 = \frac{dV}{dt} = (12.5)^2 \pi \frac{dh}{dt}$

$-4800 = (12.5)^2 \pi \frac{dh}{dt}$

$\frac{dh}{dt} = -9.778 \text{ in}/\text{hr}$

3. In 1902, Mrs. Quattrin's great-grandparents moved from Carbon County, Wyoming, to Douglas, Arizona. They traveled by wagon, and the 1200-mile trip took 99 days. Mrs. Frost, being a school teacher, kept a journal of the trip, the data from which is in the table below where $R(t)$ is measured in miles per day and t is measured in days.

t	0	14	27	40	54	70	85	99
$R(t)$	0	12.9	5.0	13.3	18.0	14.9	9.9	22

2PTS a) Approximate $R'(21)$. Using the correct units, explain the meaning of $R'(21)$.

$$R'(21) \approx \frac{5-12.9}{27-14} = \frac{-7.9}{13} \approx -.608 \ \frac{\text{mi}}{\text{DAY}^2}$$

THE RATE AT WHICH THE FAMILY IS TRAVELING IS DECREASING BY .608 MILES PER DAY PER DAY ON DAY 21

b) Set up a right-hand Reimann Sum to approximate $\int_0^{99} R(t)dt$. Using the correct units, explain the meaning of $\frac{1}{99}\int_0^{99} R(t)dt$.

$$\int_0^{99} (R(t))\,dt \approx 14(12.9) + 13(5) + 13(13.3) + 14(18) + 16(14.9) + 15(9.9) + 14(22) =$$

$\frac{1}{99}\int_0^{99} R(t)\,dt$ IS THE APPROXIMATE AVERAGE MILES PER DAY THAT THE FAMILY TRAVELED OVER THE 99-~~DAY~~ DAY TRIP

c) Set up a Midpoint Reimann Sum to approximate $\int_{14}^{99} R(t)dt$. Using the correct units, explain the meaning of $\frac{1}{85}\int_{14}^{99} R(t)dt$. 3PTS

$$\int_{14}^{99} = 26(5) + 29(9.9) + 30(18) = 957.1$$

$\frac{1}{85}\int_{14}^{99} R(t)\,dt$ IS THE APPROXIMATE AVERAGE RATE THE FAMILY TRAVELED IN MILES PER DAY DURING THE LAST 85 DAYS OF THE TRIP

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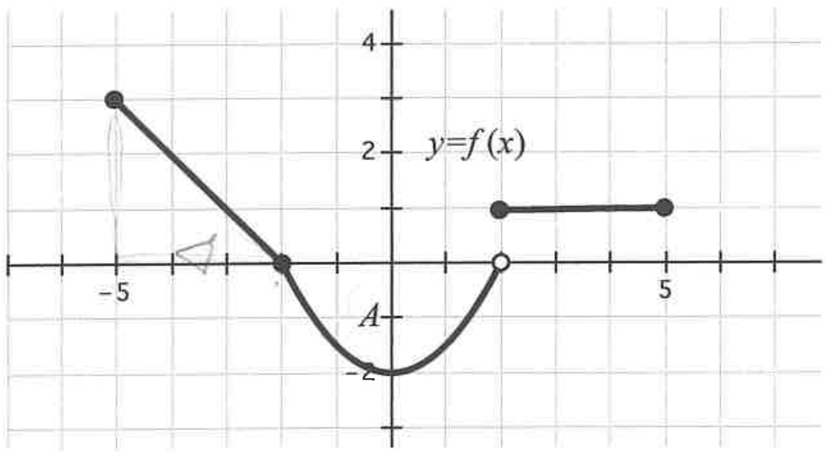
d) Assume there is a continuous and differentiable equation $S(t)$ which would model the data from the diary. Set up, but do not solve, an integral equation which would determine when the family had traveled 1000 miles. 1PT

$$\int_0^t S(x)\,dx = 1000$$



4. The graph above, $f(t)$ on $-5 \le x \le 5$, is comprised of two line segments and the graph of a parabola. Let $g(x) = 4 + \int_{-2}^{x} f(t)\,dt$, and let Area A equal 5.7.

(a) Find $g(-5)$ and $g'(-5)$.

$g(-5) = 4 + \int_{-2}^{-5} f(t)\,dt = 4 + (-4.5) = -\frac{1}{2}$

$g'(-5) = f(-5) = 3$

(b) Find $g(2)$.

$g(2) = 4 + \int_{-2}^{2} f(t)\,dt = 4 + (-5.7) = -1.7$

(c) At what x-value, on $-5 \le x \le 5$, does $g(x)$ have the absolute maximum? Explain. 3PTS

$g' = f$ AND ~~g~~

THE CRITICAL VALUES ARE ~~g~~ AT $x = -5, -2, 2, 5$

x	g
-5	$-1/2$
-2	4
2	-1.7
5	12.7

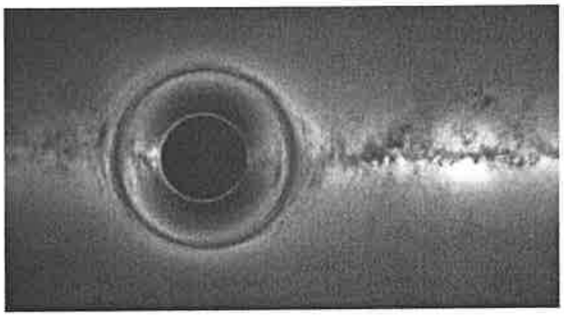
THE ABSOLUTE MAXIMUM IS AT $x = 5$

(d) On what interval(s) is $g(x)$ both decreasing and concave down? Explain why. 3PTS

g IS DECREASING AND CONCAVE DOWN

WHEN $g' = f$ IS NEGATIVE AND DECREASING

$\therefore x \in (-2, 0)$



5. Research into supermassive black holes (SMBH) seems to indicate that their accretion rate (rate at which they gain mass) follows the differential equation $\frac{dM}{dt} = .256(1000 - M)$, where M is the mass of the SMBH in what we will call Kellartons and t is measured in days.

$M = f(t)$ would be the particular solution to the differential equation where $M(0)=600$.

(a) Find the equation of the line tangent to $M = f(t)$ at $M(0)=600$.

$$\left.\frac{dM}{dt}\right|_{(0,600)} = .256(1000-600) = 102.4$$

$$M - 600 = 102.4t$$

(b) Use the line tangent found in (a) to approximate the amount of the mass at time $t = 10$ days.

$$M(10) \approx f(10) = 102.4(10) + 600$$

$$= 1624 \text{ KELLARTONS}$$

(c) Find the particular solution $M = f(t)$ by solving the differential equation $\frac{dM}{dt} = .256(1000 - M)$ with the initial condition $M(0) = 600$.

$$\int \frac{1}{1000-m}\, dm = \int .256\, dt$$

$$-\ln|1000-m| = .256t + C$$

$$|1000-m| = e^{-.256t+C} \rightarrow 1000 - m = Ke^{-.256t}$$

$$(0, 600) \rightarrow 400 = Ke^{0}$$

$$1000 - m = 400e^{-.256t}$$

$$m = 1000 - 400e^{-.256t}$$

(d) Find $\lim_{t \to \infty} M(t)$.

$$= 1000 - 400e^{-\infty} = 1000$$

6. Let $f(x)=\begin{cases} e^x, \text{ if } x\leq 0 \\ \cos x, \text{ if } 0<x \end{cases}$.

(a) Is $f(x)$ continuous at $x=0$? Justify your answer. 3 PTS

i) $f(0)$ EXISTS BECAUSE $x=0$ IS IN THE DOMAIN

ii) $\lim_{x\to 0} f(x)$ EXISTS BECAUSE $\lim_{x\to 0^-} f(x) = e^0 = 1 = \cos 0 = \lim_{x\to 0^+} f(x)$

iii) $f(0) = \lim_{x\to 0} f(x)$

$\therefore$ $f(x)$ IS CONTINOUOS AT $x=0$

(b) Find $f'(-1)$ and $f'\left(\frac{\pi}{4}\right)$. 2 PTS

$f'(-1) = e^{-1}$

$f'(\pi/4) = -\sin \frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

(c) Express $f'(x)$ as a piecewise-defined function. Explain why $f'(0)$ does not exist. (2 PTS)

$$f'(x) = \begin{cases} e^x & \text{IF } x < 0 \\ -\sin x & \text{IF } x > 0 \end{cases}$$

$f(x)$ IS CONTINUOUS BUT $\lim_{x \to 0^-} f'(x) = e^0 = 1 \neq 0 = -\sin 0 = \lim_{x \to 0^+} f'(x)$

(d) Find $\lim_{x \to \cancel{\pi}\ \pi/2} \dfrac{f'(x) \cancel{-2} +1}{\frac{\pi}{2} - x}$. Justify your answer. (2 PTS)

$f'(\pi/2) = -1 \quad \therefore$

$$\overset{L'H}{=} \lim_{x \to \pi/2} \frac{f''(x)}{-1} = \lim_{x \to \pi/2} \frac{-\cos \pi/2}{-1} = 0$$
