AP Calculus AB '20-21

Spring Final Part IIA v3

Calculator Allowed

Name:

Solution Key

1. A young novelist enters the Novel-for-November contest, where amateur writers attempt to write a full-length novel over the course of the month. After the month ends, she realizes that the rate at which she wrote varied over the month. She determines that a good model of the daily rate of her writing would be $W(t) = 7.5 + 7.5\cos\left[\frac{\pi}{15}(t-1)\right]$, where t is measured in days and $t=1$ is the morning of November 1st and $t=31$ is the end of the day on November 30th. Furthermore, let $E(t) = 6 + 8\cos\left[\frac{\pi}{30}(t-10)\right]$ model the rate at which her editor goes through the manuscript, beginning on November 10th ($t=10$).

(a) How many pages does the writer complete during the month of November? 1 PT

$\int_1^{31} w(t)\,dt = 225$ PAGES

(b) Find the value of $W(17)$ and $W'(17)$. Using the correct units, explain the meaning of both. 3 PTS

$w(17) = .164$ PAGES/DAY

$w'(17) = .367$ $\frac{\text{PAGES}}{\text{DAY}^2}$

$w(17)$ IS HOW MANY PAGES PER DAY ARE BEING WRITTEN ON NOV 16

$w'(17)$ IS THE RATE AT WHICH THE RATE OF THE # OF PAGES PER DAY IS CHANGING ON NOV 16 IN PAGES PER DAY PER DAY

(c) Find the number of pages which still need to be edited at the end of the day on November 30th. 3 PTS

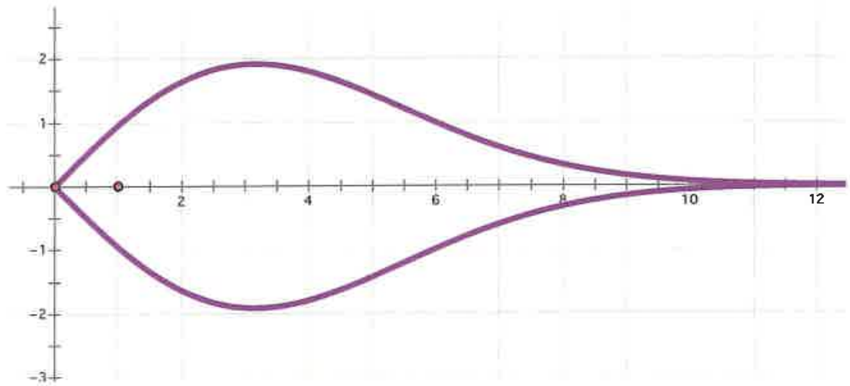
$$225 - \int_{10}^{31} E(t)\,dt = 37.196 \text{ PAGES}$$

(d) Set up, but do not solve, an integral equation that would determine how many more days would be needed to finish editing the manuscript.

2 PTS

$$\int_{31}^{t} E(t)\,dt = 37.196$$

2. Xenobiologists visiting another planet find sea creatures whose silhouette (outline) resembles the region bouned by $f(x)=xe^{-.05x^2}$ and $g(x)=-xe^{-.05x^2}$ from $x=0$ to $x=12$



(a) Find the area of the silhouette. Show the antiderivative. 3 PTS

$\int_0^{12} xe^{-.05x^2} - (-xe^{-.05x^2})\,dx = \int_0^{12} 2xe^{-.05x^2}\,dx =$

$= -20 \int_0^{12} e^{-.05x^2}(-.1x\,dx) = -20e^{-.05x^2}\Big]_0^{12} = 19.985$

(b) If the creature's body shape was created by revolving $f(x)$ about the x-axis, find the volume of said creature. 2 PTS

$V = \pi \int_0^{12} (xe^{-.05x^2})^2\,dx = \cancel{31.392}\ 44.021$

(c) Dissection reveals that the creature has a cross section perpendicular to the x-axis which is a regular hexagon with the diameter in the silhouette region. Given that the area of a regular hexagon is $A = \frac{3\sqrt{3}}{2} r^2$, find the volume of the creature. Show the setup. 2 PTS

$$V = \int_0^{12} \frac{3\sqrt{3}}{2} \left(x e^{-.05 x^2} \right)^2 dx$$

$$= 36.405$$

(d) If the live creature in part c) swam by at $2 \, {}^{in}/_{sec}$, how fast would the diameter be changing when $x = 3$? Show the derivative work.

$$\frac{d}{dt} \left[\text{DIAMETER} = 2r = 2x e^{-.05x^2} \right]$$

$$\frac{dD}{dt} = 2x e^{-.05x^2} (-.1x) \frac{dx}{dt} + 2 e^{-.05x^2} \frac{dx}{dt}$$

$$\left. \frac{dD}{dt} \right|_{x=3} = 0.255 \, \frac{\text{IN}}{\text{SEC}}$$

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3. Dr. Quattrin's paternal grandmother's family originated in the Alpine town of Sauris, Italy, where the temperature in January changes at a rate of $W(t)$ degrees Celsius per hour. $W(t)$ is a twice-differentiable, increasing and concave up function with selected values in the table below. At midnight ($t=0$), the temperature in Sauris is $-8°C$.

t (in hours after midnight)	0	1	3	6	8
$W(t)$ (in degrees Celsius per hour)	-2.6	-3.1	-1.2	1.9	2.5

a) At approximately what rate is the rate of change of the temperature changing at 2am ($t=2$)? Include units. 2 PTS

$$W'(2) = \frac{-1.2-(-3.1)}{3-1} = \frac{1.9}{2} = .95 \frac{°C}{Hr^2}$$

b) Set up, but do not solve, an integral equation which would determine the temperature in Sauris at 1pm. 2 PT

$$T(1) = -8 + \int_0^{13} W(t)\, dt$$

INSTANTANEOUS

c) Is there a time when the rate of change of the temperature equals 0.3? Justify your answer. 2 PTS

W (t) IS TWICE DIFFERENTIABLE SO THE MVT APPLIES.

AVE VALUE BETWEEN $t=6$ & $t=8$ $= \dfrac{2.5-1.9}{8-6} = .3$

$\therefore$ YES, THERE MUST BE A TIME BETWEEN $t=6$ & $t=8$

WHEN THE INSTANTANEOUS RATE OF CHANGE EQUALS

THE AVERAGE RATE OF CHANGE

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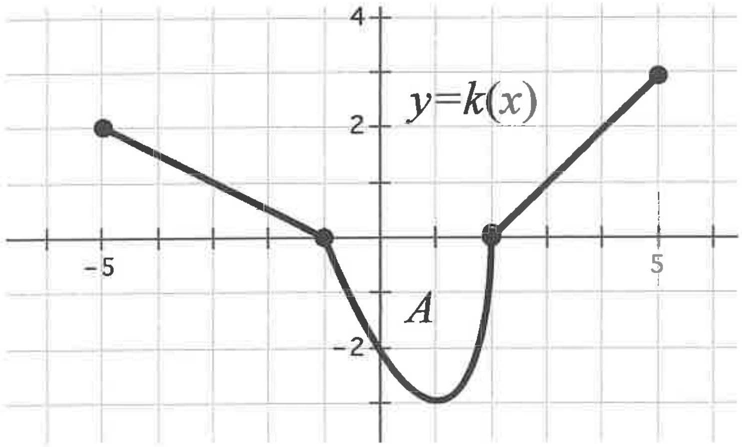
b) Use a right Riemann sum with subdivisions indicated by the table to approximate $\int_0^8 W(t)\,dt$. Using correct units, explain the meaning of this value in the context of this problem. 3 PTS

$$\int_0^8 W(t)\,dt \approx 1(-3.1)+2(-1.2)+3(1.9)+2(2.5)$$

$$= -3.1 - 2.4 + 5.7 + 5 = 5.2$$

THE TEMPERATURE RISES 5.2 °C BETWEEN MIDNIGHT

AND 8 AM



4. The graph above, $k(x)$ on $-5 \le x \le 5$, is comprised of two line segments and the graph of a radial function. Let $g(x) = 4 + \int_{-1}^{x} k(t)\,dt$. The area A is 5.

(a) Find $g(-5)$ and $g'(-5)$. 2pts

$g(-5) = 4 + \int_{-1}^{-5} k(t)\,dt = 4 + (-4) = 0$

$g'(-5) = h(-5) = 2$

(b) Find $g(2)$.

$g(2) = 4 + \int_{-1}^{2} k(t)\,dt = 4 + (-5) = -1$

(c) At what x-value, on $-5 \le x \le 5$, does $g(x)$ have the absolute maximum? Explain. 4PTS

x	$g(x)$
-5	0
-1	4
2	-1
5	3.5

THE ABSOLUTE MAXIMUM OCCURS WHEN $x = -1$

(d) On what interval(s) is $g(x)$ both increasing and concave down? Explain why. 2PTS

$g' = f$

g IS INCREASING AND CONCAVE DOWN WHEN g' IS POSITIVE AND DECREASING

$x \in [-5, -1)$

5. Research at the University of Tennessee Anthropological Research Facility, (aka The Body Farm) indicates that the decomposition rate of a certain body might follow the differential equation $\frac{dN}{dt}=0.02(200-N)$, where N is the number of pounds of flesh which has decomposed in t days. $N=f(t)$ is the particular solution to the differential equation where $N(0)=20$.

(a) Find the equation of the line tangent to $N=f(t)$ at $N(0)=20$.

$$\frac{dN}{dt} = .02(200-20) = .02\,(180) = 3.6$$

$$N - 20 = 3.6\,(t-0)$$

(b) Use the line tangent found in (a) to approximate the amount of the substance remaining at time $t=5$ days.

$$N(5) \approx 3.6\,(5-0) + 20$$

$$= 38 \text{ LBS}$$

(c) Find the particular solution $N=f(t)$ by solving the differential equation $\frac{dN}{dt}=.02(200-N)$ with the initial condition $N(0)=20$.

$$\int \frac{1}{200-N}\,dN=\int .02\,dt$$

$$-\ln|200-N| = .02t+C$$

$$|200-N| = e^{-.02t+C} \Rightarrow 200-N = ke^{-.02t}$$

$$(0,20) \Rightarrow k=180$$

$$N = 200-180e^{-.02t}$$

(d) Determine whether the amount of the flesh is decomposing at an increasing or a decreasing rate at time $t=5$ days. Explain your reasoning.

$$\frac{d}{dt}\left[\frac{dN}{dt} = .02(200-N) = 4-.02N\right]$$

$$\frac{d^2N}{dt^2} = -.02\frac{dN}{dt} = -.02(4-.02N)$$

$$= -.08+.0004N$$

$$\left.\frac{d^2N}{dt^2}\right|_{t=5} = -.08+.002<0 \;\therefore$$

THE RATE OF DECOMPOSITION IS INCREASING AT A DECREASING RATE

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6. Let $f(x)=\begin{cases} x-1, \text{ if } x<1 \\ \sin[\pi(x-1)], \text{ if } 1\le x \end{cases}$.

(a) Is $f(x)$ continuous at $x=1$? Justify your answer. 3PTS

i) $f(1)$ EXISTS BECAUSE $x=1$ IS IN THE DOMAIN

ii) $\lim_{x\to 1} f(x)$ EXISTS BECAUSE $\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(x-1) = 0$

$= \sin 0 = \lim_{x\to 1^+} f(x)$

iii) $\lim_{x\to 1} f(x) = f(1)$

$\therefore$ $f(x)$ IS CONTINUOUS AT $x=1$

(b) Find $f'(-1)$ and $f'(2)$. 2PTS

$f'(-1) = 1$

$f'(2) = \cos[\pi(1)]\,(\pi) = -\pi$

(c) Express $f'(x)$ as a piecewise-defined function. Explain why $f'(1)$ does not exist. 2 PTS

$$f'(x) = \begin{cases} 1 & \text{IF } x<1 \\ \pi\cos(\pi(x-1)) & \text{IF } x>1 \end{cases}$$

$$\lim_{x\to 1^-} f'(x) = 1 \neq \pi = \lim_{x\to 1^+} f'(x)$$

$\therefore \lim_{x\to 1} f'(x)$ DOES NOT EXIST

(d) Find $\lim_{x\to 1^-} \frac{f(x)}{x^2-x}$. Justify your answer. 2PTS

$$\overset{L'H}{=} \lim_{x\to 1^-} \frac{f'(x)}{2x-1} = \frac{1}{1} = 1$$
