Chapter 3 Overview: Derivatives Applications

Most of last year’s derivative work (and the first chapter) concentrated on tangent lines and extremes—the geometric applications of the derivatives. The broader context for derivatives is that of “change.” Slope is, indeed, a measure of change, but only one kind of change. If the axes of the graph are labeled as time and distance, slope become velocity (distance/time). There are two main contexts for derivatives in this chapter: motion and graphing.

In this first portion of this chapter we will concentrate on the applications of the derivative to motion. Much of this was covered last year. The underlying idea is that the independent variable provides context to the problem. The mechanics of differentiating in terms of $x$ and differentiating in terms of time $t$ are the same but the interpretations of the answers are very different. The Chain Rule takes on an added dimension here with Implicit Differentiation and Related Rates problems.

In the latter half of the chapter, we will consider the graphical applications of the derivative. Much of this is also a review of material covered last year, but with significant pieces added to emphasize the Calculus. Key topics include:

- Finding extremes
- The Mean Value Theorem
- The First Derivative Test
- The Second Derivative Test
- Optimization
- Graphing from the derivatives
- Making inferences regarding the original graph from the graph of its derivative

Several multiple-choice questions and at least one full free response question (often parts of others), have to do with this general topic.
3.1: Extrema and the Mean Value Theorem

One of the most valuable aspects of Calculus is that it allows us to find extreme values of functions. The ability to find maximum or minimum values of functions has wide-ranging applications. Every industry has uses for finding extremes, from optimizing profit and loss, to maximizing output of a chemical reaction, to minimizing surface areas of packages. This one tool of Calculus eventually revolutionized the way the entire world approached every aspect of industry. It allowed people to solve formerly unsolvable problems.

OBJECTIVES

Find critical values and extreme values for functions.
Understand the connection between slopes of secant lines and tangent lines
Apply the Mean Value Theorem to demonstrate that extremes exist within an interval.

It will be helpful to keep in mind a few things from last year for this chapter (and all other chapters following).

REMEMBER: Vocabulary:
1. Critical Value--The x-coordinate of the extreme
2. Maximum Value--The y-coordinate of the high point.
3. Minimum Value--The y-coordinate of the low point.
4. Relative Extremes--the highest or lowest points in any section of the curve.
5. Absolute Extremes--the highest or lowest points of the whole curve.
6. Interval of Increasing--the interval of x-values for which the curve is rising from left to right.
7. Interval of Decreasing--the interval of x-values for which the curve is dropping from left to right.

Critical Values of a function occur when
i. \( \frac{dy}{dx} = 0 \)
ii. \( \frac{dy}{dx} \) does not exist
iii. At the endpoints of its domain.
It is also helpful to remember that a critical value is referring specifically to the value of the $x$, while the extreme value refers to the value of the $y$.

Ex 1 Find the critical values of $y = x^3 - 9x$ on $x \in [-1, 6]$

i. \[ \frac{dy}{dx} = 3x^2 - 9 = 0 \]
   \[ x^2 = 3 \]
   \[ x = \pm \sqrt{3} \]
   but $-\sqrt{3}$ is not in the given domain, so $x = \sqrt{3}$.

ii. $\frac{dy}{dx}$ always exists

iii. $x = -1, 6$

Therefore the critical values are $x = -1, -\sqrt{3},$ and $6$

Ex 2 Find the extremes points for $y = -\sqrt{x^4 - 6x^2 + 8}$

\[ \frac{dy}{dx} = -\frac{1}{2} (x^4 - 6x^2 + 8)^{-1/2} (4x^3 - 12) \]
\[ = \frac{6x - 2x^3}{(x^4 - 6x^2 + 8)^{1/2}} \]

i) \[ \frac{dy}{dx} = \frac{6x - 2x^3}{(x^4 - 6x^2 + 8)^{1/2}} = 0 \]
\[ 6x - 2x^3 = 0 \]
\[ 6x - 2x^3 = 0 \]
\[ 2x(3 - x^2) = 0 \]
\[ x = 0, \pm \sqrt{3} \] but $x = \pm \sqrt{3}$ are not in the domain so \[ (0, -\sqrt{8}) \text{ or } (0, -2\sqrt{2}) \]
ii) \[ \frac{dy}{dx} = \frac{6x - 2x^3}{\left(x^4 - 6x^2 + 8\right)^{1/2}} \text{ dne} \]

\[ \left(x^4 - 6x^2 + 8\right)^{1/2} = 0 \]

\[ x = \pm 2, \quad \pm \sqrt{2} \]

\[ y = 0 \]

iii) There is no arbitrary domain.

So the extreme points are \((0, -2\sqrt{2}), (\pm 2, 0)\), and \((\pm \sqrt{2}, 0)\).

Ex 3 Find the extreme values of \( y = 2xe^x \)

i. \[ \frac{dy}{dx} = 2xe^x + e^x (2) \]

\[ 2xe^x + e^x (2) = 0 \]

\[ 2e^x (x + 1) = 0 \]

\[ e^x = 0 \quad \text{ or } \quad (x + 1) = 0 \]

\[ x = \text{no solution} \quad \text{or} \quad x = -1 \]

\[ y = -0.736 \]

ii. \( \frac{dy}{dx} \) always exists

iii. There is no arbitrary domain.

So \( y = -0.736 \) is the extreme value.
The Mean Value and Rolle’s Theorems
The Mean Value Theorem is an interesting piece of the history of Calculus that was used to prove a lot of what we take for granted. The Mean Value Theorem was used to prove that a derivative being positive or negative told you that the function was increasing or decreasing, respectively. Of course, this led directly to the first derivative test and the intervals of concavity.

**Mean Value Theorem**

If $f$ is a function that satisfies these two hypotheses
1. $f$ is continuous on the closed interval $[a,b]$
2. $f$ is differentiable on the closed interval $(a,b)$

Then there is a number $c$ in the interval $(a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$.

Again, translating from math to English, this just says that, if you have a smooth, continuous curve, the slope of the line connecting the endpoints has to equal the slope of a tangent somewhere in that interval. Alternatively, it says that the secant line through the endpoints has the same slope as a tangent line.

Rolle’s Theorem is a specific case of the Mean Value theorem, though Joseph-Louis Lagrange used it to prove the Mean Value Theorem. Therefore, Rolle’s Theorem was used to prove all of the rules we have used to interpret derivatives for the last couple of years. It was a very useful theorem, but it is now something of a historical curiosity.
Rolle’s Theorem

If $f$ is a function that satisfies these three hypotheses

1. $f$ is continuous on the closed interval $[a, b]$
2. $f$ is differentiable on the open interval $(a, b)$
3. $f(a) = f(b)$

Then there is a number $c$ in the interval $(a, b)$ such that $f'(c) = 0$.

Written in this typically mathematical way, it is a bit confusing, but it basically says that if you have a continuous, smooth curve with the initial point and the ending point at the same height, there is some point in the curve that has a derivative of zero. If you look at this from a graphical perspective, it should be pretty obvious.
Examples of Functions that satisfy Rolle’s Theorem

Ex 4 Show that the function \( f(x) = x^2 - 4x + 1 \), \([0, 4]\) satisfies all the conditions of the Mean Value Theorem and find \( c \).

Polynomials are continuous throughout their domain, so the first condition is satisfied.
Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

According to the Mean Value Theorem

\[
f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{(4^2 - 4(4) + 1) - (0^2 - 4(0) + 1)}{4} = 0
\]

\[
f'(x) = 2x - 4
\]
\[
f'(c) = 2c - 4
\]

Since, by applying the Mean Value Theorem, we found that \( f'(c) = 0 \), so

\[
2c - 4 = 0
\]
\[
c = 2
\]
Ex 5 Suppose that $f(0) = -3$, and $f'(x) \leq 5$ for all values of $x$. What is the largest possible value for $f(2)$?

We know that $f$ is differentiable for all values of $x$ and must therefore also be continuous. So we just make up an interval to look at using the mean value theorem. The interval will be $[0, 2]$ because we are looking at $f(0)$ and $f(2)$.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$f'(c) = \frac{f(2) - (-3)}{2}$$

$$2f'(c) = f(2) + 3$$

$$2f'(c) - 3 = f(2)$$

Since we know the maximum value for $f'(c)$ is 5, plug in 5 for $f'(c)$, and we get

$$2(5) - 3 = f(2)$$

$$f(2) = 7$$

Therefore the maximum possible value for $f(2)$ is 7.
3.1 Homework

Find the critical values and extreme values for each function.

1. \( y = 2x^3 + 9x^2 - 168x \)  
2. \( y = 3x^4 + 2x^3 + 12x^2 + 12x - 42 \)

3. \( y = \frac{x^2 + 1}{x^3 - 4x} \)  
4. \( y = \frac{x - 5}{x^2 + 9} \)

5. \( y = \sqrt[3]{9x^3 - 4x^2 - 27x + 12} \)  
6. \( y = \sqrt[3]{\frac{3x}{9 - x^2}} \)

7. \( y = \left( x^2 \right)^{\frac{3}{\sqrt{9 - x^2}}} \)  
8. \( y = \left( x - x^2 \right)e^x \)
9. \[ y = \frac{\sqrt{3}}{2} x + \cos x \text{ on } x \in [0, 2\pi] \]

10. \[ y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x \]

11. Find critical values and extreme values for \( f(x) \), pictured below
Verify that the following functions fit all the conditions of Rolle’s Theorem, and then find all values of $c$ that satisfy the conclusion of Rolle’s Theorem.

12. $f(x) = x^3 - 3x^2 + 2x + 5, \ [0,2]$

13. $f(x) = \sin(2\pi x), \ [-1,1]$

14. $g(t) = t\sqrt{t+6}, \ [-6,0]$
Given the graphs of the functions below, estimate all values of $c$ that satisfy the conclusion of the Mean Value Theorem for the interval $[1,7]$.

15.

![Graph 1](image1)

16.

![Graph 2](image2)
17. Given that \( f(x) \) is a twice differentiable function with \( f(2) = 6 \). The derivative of \( f(x) \), \( f'(x) \), is pictured below on the closed interval \(-1 \leq x \leq 5\). The graph of \( f'(x) \) has horizontal tangent lines at \( x = 1 \) and at \( x = 3 \). Use this information to answer the questions below.

![Graph of f'(x) with horizontal tangents at x = 1 and x = 3.]

a) Find the \( x \)-coordinate of each point of inflection on \( f \). Explain your reasoning.

b) Find where the function \( f \) attains its absolute maximum value and its absolute minimum value on the closed interval \(-1 \leq x \leq 5\). Show the work that leads to this conclusion.

c) Let \( g \) be defined as the function \( g(x) = x \cdot f(x) \). Find the equation of the graph of the tangent line to \( g \) at \( x = 2 \).
Answers: 3.1 Homework

1. \[ y = 2x^3 + 9x^2 - 168x \]
   C.V. at \( x = 4 \) and \(-7\)
   E.V. at \( y = -400 \) and \( 931 \)

2. \[ y = 3x^4 + 2x^3 + 12x^2 + 12x - 42 \]
   C.V. at \( x = -0.5 \)
   E.V. at \( y = -45.063 \)

3. \[ y = \frac{x^2 + 1}{x^3 - 4x} \]
   C.V. at \( x = -0.729 \) and \( 0.729 \)
   E.V. at \( x = 0.606 \) and \(-0.606 \)

4. \[ y = \frac{x - 5}{x^2 + 9} \]
   C.V. at \( x = -0.831 \) and \( 10.831 \)
   E.V. at \( x = 0.046 \) and \(-0.602 \)

5. \[ y = \sqrt{9x^3 - 4x^2 - 27x + 12} \]
   C.V. at \( x = 4 \) and \(-7\)
   E.V. at \( y = 0 \) and \( 5.151 \)

6. \[ y = \sqrt{\frac{3x}{9 - x^2}} \]
   C.V. at \( x = 0 \) and \( 2\pi \)
   E.V. at \( y = 0 \) and \( 8.845 \)

7. \[ y = (x^2)^{\frac{3}{2}} \]
   C.V. at \( x = 0 \) and \( \pm 2.598 \)
   E.V. at \( y = 0 \) and \( 8.845 \)

8. \[ y = (x - x^2)e^x \]
   C.V. at \( x = -1.618 \) and \( .618 \)
   E.V. at \( y = -0.840 \) and \( 0.438 \)

9. \[ y = \frac{\sqrt{3}}{2}x + \cos x \] on \( x \in [0, 2\pi] \)
   C.V. at \( x = 0 \) and \( 2\pi \) and \( 1.047 \) and \( 2.094 \)
   E.V. at \( y = 1 \) and \( 6.441 \) and \( 1.407 \) and \( 1.314 \)

10. \[ y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1}x \]
    None

11. Find critical values and extreme values for \( f(x) \), pictured below
    C.V. at \( x = -2 \) and \(-1 \) and \( 2 \) and \( 5 \) and \( 6 \)
    E.V. at \( y = -6 \) and \( 1 \) and \(-8 \) and \( 3 \)

12. \[ f(x) = x^3 - 3x^2 + 2x + 5, \ x \in [0, 2] \]
    Function is continuous and differentiable on the interval.
    \( f(0) = f(2) \)
    \( c = 0.423 \)
13. \( f(x) = \sin(2\pi x), \ x \in [-1,1] \)
Function is continuous and differentiable on the interval.
\( f(-1) = f(1) \)
\( c = \pm \frac{3}{4} \) and \( \pm \frac{1}{4} \)

14. \( g(t) = t\sqrt{t+6}, \ t \in [-6,0] \)
Function is continuous and differentiable on the interval.
\( f(-6) = f(0) \)
\( c = -4 \)

15. \[ \text{Graph of function} \]

\[ c \approx 3.4 \]

16. \[ \text{Graph of function} \]

\[ c \approx 2.4, 4.6, 7 \]
17. Given that \( f(x) \) is a twice differentiable function with \( f(2) = 6 \). The derivative of \( f(x) \), \( f'(x) \), is pictured below on the closed interval \(-1 \leq x \leq 5\). The graph of \( f'(x) \) has horizontal tangent lines at \( x = 1 \) and at \( x = 3 \). Use this information to answer the questions below.

\[ \text{Graph of } f'(x) \]

a) Find the \( x \)-coordinate of each point of inflection on \( f \). Explain your reasoning.

\( x = 1 \) and \( x = 3 \) because these are places where \( f' \) has a slope of 0. This means that \( f'' = 0 \) at these points. In addition, \( f'' \) changes from positive to negative or negative to positive at these values. Therefore, \( f \) has concavity changes at these \( x \) values.

b) Find where the function \( f \) attains its absolute maximum value and its absolute minimum value on the closed interval \(-1 \leq x \leq 5\). Show the work that leads to this conclusion.

\( f \) has critical values at \( x = -1, 4, \) and 5, (endpoints of the interval or 0 of \( f' \)). Maxima occur at \( x = -1 \) and 5, with the absolute maximum occurring at \( x = -1 \). This is because \( f \) decreases from \(-1 \) (because \( f' \) is negative) and most of the area under \( f' \) is negative, meaning the value of \( f(5) < f(-1) \). The absolute minimum occurs at \( x = 4 \) because it is the only minimum on the function (\( f \) switches from decreasing to increasing at this point).

c) Let \( g \) be defined as the function \( g(x) = x \cdot f(x) \). Find the equation of the graph of the tangent line to \( g \) at \( x = 2 \).

\( y - 12 = -2(x - 2) \)
3.2: Rate of Change

As you read, one of the original concepts for the derivative comes from the slope of a line. Since we also know that slopes represent rates in algebra, it is just a small leap from there to recognizing that derivatives can represent the rate of change of any function.

Any time a dependent variable changes in comparison to an independent variable, you have a rate of change of the dependent variable with respect to the independent variable. The most common time you may have seen this is with the rate of change of position with respect to time. Hopefully, you recognize that change in position over change in time is velocity.

OBJECTIVES

- Recognize and evaluate derivatives as rates of change.
- Interpret derivatives as rates of change.
- Utilize the language of rates of change with respect to derivatives.

Ex 1 For the function $S = 2\pi r^2 + \pi rL$, find the rate of change of $S$ with respect to $r$, assuming $L$ is constant.

The first thing we should notice is that when we are asked for the change of $S$ with respect to $r$, that we are really being asked for $\frac{dS}{dr}$, so we simply take the derivative.

$$\frac{d}{dr} \left[ S = 2\pi r^2 + \pi rL \right]$$

$$\frac{dS}{dr} = 4\pi r + \pi L$$

The rate of change of $S$ with respect to $r$ is simply $\frac{dS}{dr}$. 

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Ex. 2 A restaurant determines that its revenue from hamburger sales is given by 
\[ R = x \left( \frac{60,000 - x}{20,000} \right), \]
where \( x \) is the number of hamburgers. Find the increase in revenue (marginal revenue) for monthly sales at 20,000 hamburgers.

Since we want to find an increase in revenue, we must be looking at a derivative, so we will take the derivative of the function. It would help to simplify the function first.

\[ R = x \left( \frac{60,000 - x}{20,000} \right) = 3x - \frac{x^2}{20,000} \]

\[ \frac{dR}{dx} = 3 - \frac{x}{10,000} \]

\[ \left. \frac{dR}{dx} \right|_{x=20,000} = 3 - \frac{20,000}{10,000} = \$1/\text{hamburger} \]

Ex 3 The energy in joules of a photon varies with the wavelength of the light, \( \lambda \), according to the function \[ E = \frac{1.9864 \times 10^{-25}}{\lambda} \]. Find how fast the energy is changing for a photon of light from Superman’s X-ray vision (light with \( \lambda = 1.0 \times 10^{-9} \text{ m} \)). What does this number mean?

\[ \frac{d}{d\lambda} \left[ E = \frac{1.9864 \times 10^{-25}}{\lambda} \right] \]

\[ \frac{dE}{d\lambda} = -1.9864 \times 10^{-25} \lambda^{-2} \]

But we want \[ \left. \frac{dE}{d\lambda} \right|_{\lambda=1.0 \times 10^{-9}} \], so we plug in \( \lambda = 1.0 \times 10^{-9} \text{ m} \) and find that

\[ \left. \frac{dE}{d\lambda} \right|_{\lambda=1.0 \times 10^{-9}} = -1.9864 \times 10^{-7} \text{ joules/meter} \]

This means that at the particular wavelength, \( \lambda = 1.0 \times 10^{-9} \text{ m} \), energy goes down by \( 1.9864 \times 10^{-7} \) joules for every 1 meter increase in the wavelength.
Ex 4  For the equation in example 2, find the rate of change of the wavelength with respect to the energy when \( E = 1.9878 \times 10^{-16} \).

Notice, we are being asked for \( \frac{d \lambda}{dE} \) this time, so we have one of two approaches. We could solve the equation for \( \lambda \), and then take the derivative.

\[
E = \frac{1.9864 \times 10^{-25}}{\lambda}
\]

\[
E\lambda = 1.9864 \times 10^{-25}
\]

\[
\lambda = \frac{1.9864 \times 10^{-25}}{E}
\]

\[
\frac{d}{dE} \left[ \lambda = \frac{1.9864 \times 10^{-25}}{E} \right]
\]

\[
\frac{d \lambda}{dE} = -1.9864 \times 10^{-25} E^{-2}
\]

\[
\frac{d \lambda}{dE} \bigg|_{E=1.9878 \times 10^{-16}} = -5.0271 \times 10^6 \text{ meters/Joule}
\]

Alternatively, we could find the value of \( \frac{dE}{d \lambda} \) for this value of \( E \) and then take the reciprocal to find \( \frac{d \lambda}{dE} \).

\[
\frac{dE}{d \lambda} = -1.9864 \times 10^{-25} \lambda^{-2}, \text{ and when } E = 1.9878 \times 10^{-16}, \lambda = 9.99296 \times 10^{-10}
\]

And at this value for \( \lambda \), \( \frac{dE}{d \lambda} = -1.9892 \times 10^{-7} \text{ Joules/meter} \).

The reciprocal of this is \( \frac{d \lambda}{dE} = -5.0271 \times 10^6 \text{ meters/Joule} \).

Clearly, in this case, both ways are relatively simple, but there may be times when it is much easier to find the reciprocal of the derivative if it is too difficult to isolate one of the variables.
Ex 5  The height of a person diving from a high dive is given by the equation 
\[ y = -4.9t^2 + 1.2t + 10 \] 
where \( y \) is in meters and \( t \) is in seconds. Use this fact to answer each of the following.

a) Find the rate of change of height with respect to time at \( t = 0.5 \)

b) Interpret the meaning of the value from a)

c) Find when the value of \( \frac{dy}{dt} = -2.5\text{ meters/second} \). Use this to find the height of the diver at this time. Interpret the meaning of all of this.

a) The rate of change is the derivative, so

\[ \frac{dy}{dt} = -9.8t + 1.2 \]

\[ \frac{dy}{dt} \bigg|_{t=0.5} = -9.8(0.5) + 1.2 = -3.7\text{ meters/second} \]

b) \(-3.7\text{ meters/second}\) means that the person is traveling downward at \(3.7\text{ meters/second}\) at 0.5 seconds. You may notice that this is a velocity (because it is a rate of change of height versus time).

c) We just need to set the derivative equal to the value given and solve for \( t \).

\[ \frac{dy}{dt} = -9.8t + 1.2 = -2.5 \]

\[-9.8t = -3.7\]

\[ t = 0.378 \text{ seconds} \]

We need the height, so we go back to our initial equation we were given.

\[ y(0.378) = -4.9(0.378)^2 + 1.2(0.378) + 10 \]

\[ y(0.378) = 9.755 \text{ meters} \]

So at 0.378 seconds, the diver is 9.755 meters high and is traveling downwards at \(2.5\text{ meters/second}\).

What you may have noticed from the previous example, an interesting application of rates of change comes from basic physics. For a displacement function, the rate of change is clearly the velocity, and the rate of change of the velocity is
acceleration. Of course this means that the derivative of displacement is velocity
and the derivative of velocity is acceleration.

Derivatives are used extensively in Physics to describe particle motion (or any
other linear motion). If the variables represent time and distance, the derivative
will be a rate or velocity of the particle. Because of their meaning, the letters $t$ and
either $s$ or $x$ are used to represent time and distance, respectively.

**Rectilinear Motion**--Defn: Movement that occurs in a straight line.

**Parameter**--Defn: A dummy variable that determines $x$- and $y$-coordinates
independent of one another.

**Parametric Motion**--Defn: Movement that occurs in a plane.

**Velocity**--Defn: Directed speed.

Means: How fast something it is going and whether it is moving right
or left.

**Average Velocity**--Defn: Distance traveled/time or $\frac{x_2-x_1}{t_2-t_1}$

Means: The average rate, as we used it in Algebra.

**Instantaneous Velocity**--Defn: Velocity at a particular time $t$.

Means: $\frac{ds}{dt}$, $\frac{dx}{dt}$, or $\frac{dy}{dt}$, or the rate at any given instant.

**Acceleration**--Defn: The rate of change of the velocity, or $\frac{dv}{dt}$.

**OBJECTIVE**

Given a distance function of an object in rectilinear or parametric motion, find
the velocity and acceleration functions or vectors.
Use the velocity and acceleration functions to describe the motion of an
object.
We will consider rectilinear motion first and then parametric motion. In either case, though, there are three things implied in the definitions above:

1. If we have a distance equation, its derivative will be the velocity equation.
2. The derivative of velocity is acceleration.
3. If the velocity is 0, the particle is stopped—usually paused in order to switch directions.

**Rectilinear Motion**

For rectilinear motion, it is helpful to remember some things from Precalculus.

\[
h = \frac{1}{2}at^2 + v_0t + h_0
\]

where \( a \) = the gravitational constant

\( a = 32 \text{ ft/sec}^2 \) or \( 9.8 \text{ m/sec}^2 \) on Earth,

\( v_0 \) is the initial velocity and \( h_0 \) is the initial height

**EX 6** A gun is fired up in the air from a 1600 foot tall building at 240 ft/second. How fast is the bullet going when it hits the ground?

\[
h(t) = -16t^2 + 240t + 1600 = 0
\]

\( t^2 - 15t - 100 = 0 \)

\( (t - 20)(t + 5) = 0 \)

\( t = 20 \) seconds is when it hits the ground.

\[
v(t) = -32t + 240
\]

\[
v(20) = -32(20) + 240 = -400 \frac{\text{ft}}{\text{sec}} \text{ is the velocity.}
\]

The velocity is negative because the bullet is coming down. The speed at which it is going is \( 400 \frac{\text{ft}}{\text{sec}} \).
EX 7 What is the acceleration due to gravity of any falling object?

We know from before that any object launched from height \( h_0 \) feet with initial velocity \( v_0 \) ft/sec follows the equation

\[
h = -16t^2 + v_0 t + h_0
\]

\[
v = h' = -32t + v_0
\]

\[
a = v' = -32 \text{ ft/sec}^2
\]

This number is known as the Gravitational Constant.

EX 8 A particle's distance \( x(t) \) from the origin at time \( t \geq 0 \) is described by

\[
x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1
\]

How far from the origin is it when it stops to switch directions?

\[
x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1
\]

\[
v(t) = 4t^3 - 6t^2 - 22t + 12 = 0
\]

\[
= 2t^3 - 3t^2 - 11t + 6 = 0
\]

\[
= (2t-1)(t+2)(t-3) = 0
\]

\[
t = \frac{1}{2}, -2, \text{ and } 3
\]

But the problem specifies that \( t \geq 0 \), so \( t = \frac{1}{2} \) and 3.

\[
x\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 - 2 \left(\frac{1}{2}\right)^3 - 11 \left(\frac{1}{2}\right)^2 + 12 \left(\frac{1}{2}\right) + 1 = 4.0625
\]

\[
x(3) = (3)^4 - 2(3)^3 - 11(3)^2 + 12(3) + 1 = -35
\]

Therefore, this particle stops 4.0625 units to the right of the origin (at 1/2 seconds) and 35 units left of the origin (at 3 seconds).
EX 9  A particle’s distance \( x(t) \) from the origin at time \( t \geq 0 \) is described by \( x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1 \). What is the acceleration when the particle stops to switch directions?

Since this is the same problem as EX 3 in 2-7, we already know

\[
    x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1 \\
    v(t) = 4t^3 - 6t^2 - 22t + 12 = 0 \\
    = 2t^3 - 3t^2 - 11t + 6 = 0 \\
    = (2t - 1)(t + 2)(t - 3) = 0 \\
    t = \frac{1}{2}, -2, \text{ and } 3
\]

and \( t = \frac{1}{2} \) and 3 because the time \( t \geq 0 \). All we need to do is substitute these times into the acceleration equation.

\[
    v(t) = 4t^3 - 6t^2 - 22t + 12 \\
    a(t) = 12t^2 - 12t - 22
\]

\[
    a\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) - 22 = -25
\]

Note that the interpretation of a negative acceleration is **not**, like velocity, the direction it is moving, but which way the acceleration is affecting the particle. The acceleration and velocity’s directions together determine if the particle is slowing down or speeding up.
Ex 10 A model rocket’s height is given by the equation
\[ x(t) = 100\tan\left(\frac{\pi}{14}t\right) - 4.9t^2 + 1 \] on \( t \in [0,5] \) where \( t \) is in seconds, and \( x \) is in meters. Use this to find each of the following:

a) \( v(2.7) \)

b) \( a(2.7) \)

c) Whether the rocket is speeding up or slowing down at \( t = 2.7 \) seconds.

a) \( x'(t) = v(t) = \frac{50\pi}{7} \sec^2\left(\frac{\pi}{14}t\right) - 9.8t \)

\[ v(2.7) = \frac{50\pi}{7} \sec^2\left(\frac{2.7\pi}{14}\right) - 9.8(2.7) \]

\[ v(2.7) = 6.751 \text{ meters/second} \]

b) \( x'(t) = v(t) = \frac{50\pi}{7} \sec^2\left(\frac{\pi}{14}t\right) - 9.8t \)

\[ v'(t) = a(t) = \frac{100\pi}{7} \sec^2\left(\frac{\pi}{14}t\right) \tan\left(\frac{\pi}{14}t\right) - 9.8 \]

\[ a(2.7) = \frac{100\pi}{7} \sec^2\left(\frac{2.7\pi}{14}\right) \tan\left(\frac{2.7\pi}{14}\right) - 9.8 \]

\[ a(2.7) = 10.326 \text{ meters/second}^2 \]

c) Since both \( v \) and \( a \) are positive at \( t = 2.7 \), the rocket is speeding up.

The question of speeding up or slowing down is very common on the AP test. Since these are questions of speed rather than velocity, they require a little bit of extra thought.

Something is speeding up if it’s speed is getting bigger (obviously), but since speed is non-directional, a large negative velocity is a large speed. If a negative velocity gets more negative (that is, the absolute value gets larger) then it is speeding up. If a positive value gets more positive, it is also speeding up.

Therefore, a particle speeds up when its velocity and acceleration have the same sign, and it slows down if they have opposite signs.
**Speeding Up/Slowing Down**

1. If the velocity and acceleration have the same sign (both positive or both negative), the particle is **speeding up**.

2. If the velocity and acceleration have the opposite signs (one positive and the other negative), the particle is **slowing down**.

EX 11 Describe the motion of a particle whose distance from the origin $x(t)$ is given by $x(t) = 2t^3 - 3t^2 - 12t + 1$.

$x(t) = 2t^3 - 3t^2 - 12t + 1$
$v(t) = 6t^2 - 6t - 12$

Since the particle stops at $v = 0$, we have:

$v(t) = 6t^2 - 6t - 12 = 0$
$v(t) = t^2 - t - 2 = 0$
$v(t) = (t + 1)(t - 2) = 0$
$t = -1 \text{ or } 2$

$x(-1) = 8 \text{ and } x(2) = -19$

The sign pattern shows:

\[
\begin{array}{c|ccc}
\text{v} & + & 0 & - & 0 & + \\
\hline
\text{t} & -1 & 2
\end{array}
\]

which means the particle was moving right. One second before we started timing, it stopped 8 units to the right of the origin and began moving left. At two seconds, it stopped again 19 units left of the origin and began moving right.

(For a review of sign patterns, check out appendix A, and we will look at more on interpreting them later in this chapter)
3.2 Homework Set A

1. The height of a particle is given by the function \( h(t) = t^3 - 4.5t^2 - 7t \).
   a. When does the particle reach a velocity of 7 m/s?
   b. Find the particle’s velocity at 5 seconds and 10 seconds.

2. Newton’s Law of Gravitation states that the magnitude of the force (\( F \)) exerted by a mass (\( M \)) on another mass (\( m \)) is given by \( F = \frac{GMm}{r^2} \).
   a. Find the rate of change of the force with respect to the radius assuming that the masses are constant. Explain what it means.
   b. Find the rate of change of the force with respect to mass (\( M \)) if all other values are held constant and explain its meaning.
   c. If you know that a force changes at a rate of 0.002 N/m at \( r = 2.0 \times 10^8 \) m, find how fast the force is changing when \( r = 1.0 \times 10^8 \) m.
3. An object in free fall with no forces acting on it other than gravity is governed by the equation \( y(t) = y_0 + v_0 t + \frac{1}{2} gt^2 \).

   a. Find the equations for the object’s velocity and acceleration.
   b. Find when the object’s height \( (y) \) is increasing
   c. Find when its velocity is increasing and when it is decreasing.

In problems 4 - 7, the motion of a particle is described by the following distance equations. For each, find
   a) when the particle is stopped,
   b) which direction it is moving at \( t = 3 \) seconds,
   c) where it is at \( t = 3 \), and
   d) \( a(3) \)

   e) Whether the particle is speeding up or slowing down at \( t = 3 \) seconds.

4. \( x(t) = 2t^3 - 21t^2 + 60t + 4 \)
5. \[ x(t) = t^3 - 6t^2 + 12t + 5 \]

6. \[ x(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48 \]

7. \[ x(t) = 12t^5 - 15t^4 - 220t^3 + 270t^2 + 1080t \]

Find maximum height of a projectile launched vertically from the given height and with the given initial velocity.

8. \[ h_0 = 25 \text{ feet}, \ v_0 = 64 \frac{\text{ft}}{\text{sec}} \]

9. \[ h_0 = 10 \text{ meters}, \ v_0 = 294 \frac{\text{m}}{\text{sec}} \]
10. The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are \( s = 1.86t^2 \) on Mars, \( s = 11.44t^2 \) on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec on each planet?

11. Find \( x(t) \) when \( v = 0.8 \). \( x(t) = t^2 - 5t + 4 \)

Find \( v(t) \) and \( a(t) \) for problems 12 and 13. Then find whether the particle is speeding up or slowing down at \( t = 1 \).

12. \( x(t) = t^3 - 6t^2 - 63t + 4 \)

13. \( x(t) = 6t^5 - 15t^4 - 8t^3 + 24t^2 + 12 \)
For problems 14 to 16, find $x(t)$ and $v(t)$ when $a(t) = 0$.

14. $x(t) = 2t^3 - 21t^2 + 60t + 4$

15. $x(t) = t^3 - 6t^2 + 12t + 5$

16. $x(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48$
3.2 Homework Set B

1. The distance of a particle from the origin is given by the function \( x(t) = \sqrt{9t - t^3} \).
   a. When does the particle reach a velocity of 5 m/s to the left? 2 m/s to the right?
   b. Find the particle’s velocity at 3 seconds.

2. A circular puddle is increasing in size as water drips into it (hint: remember the area of a circle)
   a. Find the rate of change of the area with respect to the radius when \( r = 25 \text{ cm}, r = 15 \text{ cm}, \) and \( r = 1 \text{ m} \).
   b. Find the rate of change of the radius with respect to the area when \( A = 4 \text{ meters}^2 \) and when \( A = 7 \text{ meters}^2 \).
   c. Assume that the puddle is actually a cylinder with a depth of 0.5 cm. Find the rate of change of the volume with respect to the radius when \( r = 25 \text{ cm}, r = 15 \text{ cm}, \) and \( r = 1 \text{ m} \). Is there any relationship between these answers and the answers in part a? Explain.
3. Coulomb’s Law states that the magnitude of the force (F) exerted by one charge (Q) on another charge (q) is given by $F = \frac{k_c Q q}{r^2}$, where $k_c = 8.99 \times 10^9 \text{ N m}^2/\text{C}^2$.

   a. Find the rate of change of the force with respect to the radius assuming that the charges are constant. Explain what it means.

   b. Find the rate of change of the force with respect to charge (Q) if all other values are held constant and explain its meaning.

   c. If you know that a force changes at a rate of 0.6 N/m at $r = 2.0 \times 10^{-2} \text{ m}$, find how fast the force is changing when $r = 1.0 \times 10^{-2} \text{ m}$.

4. An object flying in the $x$ direction with no forces acting on it is governed by the equation $x(t) = x_0 + v_0 t$.

   a. Find the equations for the object’s velocity and acceleration.

   b. Find when the object’s distance ($x$) is increasing; find when its velocity is increasing and when it is decreasing, assuming $v_0$ is positive.

   c. Explain why the answers you got in part b make sense.
5. Find the acceleration and velocity equations for a particle whose position is given by \( x(t) = \tan^{-1}(t+2) \). Then find the position, velocity, and acceleration of the particle when \( t = 3 \).

6. Find \( a(\pi) \) when \( y(t) = \sec(5t) \)

7. Find when the velocity of a particle is increasing if the particle’s position is given by \( x(t) = -t^4 + 9t^2 + 10 \).

8. Find when the particle whose position is given by the function in problem 7 has a decreasing acceleration.
9. Find the maximum distance from the origin for the particle whose position is given by \( y(t) = t^3 - 29t^2 - t + 29 \). Then find the acceleration of the particle at that point.

10. If the velocity of a particle moving along the \( x \) axis is given by \( v(t) = \sec(2t) \tan(2t) + t^2 \), find the acceleration of the particle when \( t = \pi \).

Then find the position equation of the particle if \( x(\pi) = \frac{\pi^3}{3} + \frac{1}{2} \).

11. If the position of a particle is given by \( y(t) = t^3 + 6t + \cos t \) on \( t \in [0, 2\pi] \), find the position of the particle when the acceleration is \( 3\pi \).
Answers: 3.2 Homework Set A
1. The height of a particle is given by the function \( h(t) = t^3 - 4.5t^2 - 7t \).
   a. When does the particle reach a velocity of 7 m/s?
   b. Find the particle’s velocity at 5 seconds and 10 seconds.
      a. \( t = -1.130, 4.130 \)  
      b. \( v(5) = 23, v(10) = 203 \)

2. Newton’s Law of Gravitation states that the magnitude of the force \( (F) \) exerted by a mass \( (M) \) on another mass \( (m) \) is given by \( F = \frac{GMm}{r^2} \).
   a. Find the rate of change of the force with respect to the radius assuming that the masses are constant. Explain what it means.
   b. Find the rate of change of the force with respect to mass \( (M) \) if all other values are held constant and explain its meaning.
   c. If you know that a force changes at a rate of 0.002 N/m\(^3\) at \( r = 2.0 \times 10^8 \) m, find how fast the force is changing when \( r = 1.0 \times 10^8 \) m.
      a. \( \frac{dF}{dr} = -2GMmr^{-3} \); this is how fast the force changes as the radius changes.
      b. \( \frac{dF}{dM} = \frac{Gm}{r^2} \); this is how fast the force changes as the mass \( (M) \) changes.
      c. 0.016 N/m\(^3\)

3. An object in free fall with no forces acting on it other than gravity is governed by the equation \( y(t) = y_0 + v_0t + \frac{1}{2}gt^2 \).
   a. Find the equations for the object’s velocity and acceleration.
   b. Find when the object’s height \( (y) \) is increasing, assuming \( g \) is negative and \( v_0 \) is positive.
   c. Find when its velocity is increasing and when it is decreasing, assuming \( g \) is negative.
      a. \( v(t) = v_0 + gt \), \( a(t) = g \)
      b. \( y \) is increasing when \( t < -\frac{v_0}{g} \).
      c. \( v \) is always decreasing.
4. $x(t) = 2t^3 - 21t^2 + 60t + 4$
   a) $t = 2, 5$
   b) moving left
   c) 49 units right of origin
   d) $a(3) = -6$
   e) speeding up

5. $x(t) = t^3 - 6t^2 + 12t + 5$
   a) $t = 2$
   b) moving right
   c) 14 units right of origin
   d) $a(3) = 2$
   e) speeding up

6. $x(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48$
   a) $t = -4, 3, 4/3$
   b) stopped
   c) 141 units right of origin
   d) $a(3) = 420$
   e) stopped

7. $x(t) = 12t^5 - 15t^4 - 220t^3 + 270t^2 + 1080t$
   a) $t = -3, -1, 2, 3$
   b) stopped
   c) 1461 units right of origin
   d) $a(3) = 1440$
   e) stopped

8. $h_0 = 25 \text{ feet, } v_0 = 64 \frac{\text{ft}}{\text{sec}}$
   89 feet

9. $h_0 = 10 \text{ meters, } v_0 = 294 \frac{\text{m}}{\text{sec}}$
   3520 meters

10. The equations for free fall at the surfaces of Mars and Jupiter ($s$ in meters, $t$ in seconds) are $s = 1.86t^2$ on Mars, $s = 11.44t^2$ on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec on each planet?
   7.473 seconds on Mars; 1.215 seconds on Jupiter

11. Find $x(t)$ when $v = 0.8$. $x(t) = t^2 - 5t + 4$
   $x(2.9) = -2.09$

12. $x(t) = t^3 - 6t^2 - 63t + 4$
   $v(t) = 3t^2 - 12t - 63$
   $a(t) = 6t - 12$
   Speeding up

13. $x(t) = 6t^5 - 15t^4 - 8t^3 + 24t^2 + 12$
   $v(t) = 30t^4 - 60t^3 - 24t^2 + 48t$
   $a(t) = 120t^3 - 180t^2 - 48t + 48$
   Speeding up
14. \( x(t) = 2t^3 - 21t^2 + 60t + 4 \)
\[ v(3.5) = -13.5 \]
\[ x(3.5) = 42.5 \]
15. \( x(t) = t^3 - 6t^2 + 12t + 5 \)
\[ v(2) = 0 \]
\[ x(2) = 13 \]
16. \( x(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48 \)
\[ v(-2) = 1200 \]
\[ v(\%2) = -143.210 \]
\[ x(-2) = 1984 \]
\[ x(\%2) = 188.840 \]

### 3.2 Homework Set B

1. The distance of a particle from the origin is given by the function \( x(t) = \sqrt{9t - t^3} \).
   a. When does the particle reach a velocity of 5 m/s to the left? 2 m/s to the right?
   b. Find the particle’s velocity at 3 seconds.
      a. \( t = -3.255, 2.857; \ t = 0.489 \)
      b. \( v(3) = \text{DNE} \)

2. A circular puddle is increasing in size as water drips into it (hint: remember the area of a circle)
   a. Find the rate of change of the area with respect to the radius when \( r = 25 \) cm, \( r = 15 \) cm, and \( r = 1 \) m.
   b. Find the rate of change of the radius with respect to the area when \( A = 4 \) meters\(^2\) and when \( A = 7 \) meters\(^2\).
   c. Assume that the puddle is actually a cylinder with a depth of 0.5 cm. Find the rate of change of the volume with respect to the radius when \( r = 25 \) cm, \( r = 15 \) cm, and \( r = 1 \) m. Is there any relationship between these answers and the answers in part a? Explain.
      a. \( \frac{dA}{dr} \bigg|_{r=25} = 50\pi \text{ cm}, \frac{dA}{dr} \bigg|_{r=15} = 30\pi \text{ cm}, \frac{dA}{dr} \bigg|_{r=100} = 200\pi \text{ cm} \)
      b. \( \frac{dr}{dA} \bigg|_{A=4} = \frac{\sqrt{\pi}}{4}, \frac{dr}{dA} \bigg|_{A=7} = \frac{1}{2\sqrt{7}} \)
      c. \( \frac{dV}{dr} \bigg|_{r=25} = 25\pi \text{ cm}^2, \frac{dV}{dr} \bigg|_{r=15} = 15\pi \text{ cm}^2, \frac{dV}{dr} \bigg|_{r=100} = 100\pi \text{ cm}^2 \)

They are directly related, since depth is a constant value.
3. Coulomb’s Law states that the magnitude of the force (F) exerted by one charge (Q) on another charge (q) is given by \( F = \frac{k_c Q q}{r^2} \), where 
\[ k_c = 8.99 \times 10^9 \, \text{N m}^2/\text{C}^2. \]

a. Find the rate of change of the force with respect to the radius assuming that the charges are constant. Explain what it means.
b. Find the rate of change of the force with respect to charge (Q) if all other values are held constant and explain its meaning.
c. If you know that a force changes at a rate of 0.6 N/m at \( r = 2.0 \times 10^{-2} \, \text{m} \), find how fast the force is changing when \( r = 1.0 \times 10^{-2} \, \text{m} \).

\[
\text{a. } \frac{dF}{dr} = -2k_c Q r^{-3}; \text{ it is how fast force changes as radius changes}
\]
\[
\text{b. } \frac{dF}{dQ} = \frac{k_c q}{r^2}; \text{ it is how fast force changes as charge (Q) changes}
\]
\[
\text{c. } 4.8 \, \text{N/m}
\]

4. An object flying in the x direction with no forces acting on it is governed by the equation \( x(t) = x_0 + v_0 t \).

a. Find the equations for the object’s velocity and acceleration.
b. Find when the object’s distance (x) is increasing; find when its velocity is increasing and when it is decreasing, assuming \( v_0 \) is positive.
c. Explain why the answers you got in part b make sense.

\[
\text{a. } v(t) = v_0, \, a = 0
\]
\[
\text{b. } \text{Distance is always increasing, because } v_0 \text{ is always positive. Velocity is constant because acceleration is 0.}
\]
\[
\text{c. } \text{This makes sense, because if no force is acting, then there is no acceleration, and it should move at a constant velocity.}
\]

5. Find the acceleration and velocity equations for a particle whose position is given by \( x(t) = \tan^{-1}(t + 2) \). Then find the position, velocity, and acceleration of the particle when \( t = 3 \).

\[
v(t) = \frac{1}{t^2 + 4t + 5}
\]
\[
a(t) = -(t^2 + 4t + 5)^{-2} (2t + 4)
\]
\[
x(3) = \tan^{-1} 5 \quad v(t) = \frac{1}{26} \quad a(t) = -\frac{5}{338}
6. Find \( a(\pi) \) when \( y(t) = \sec(5t) \) \( a(\pi) = -25 \)

7. Find when the velocity of a particle is increasing if the particle’s position is given by \( x(t) = -t^4 + 9t^2 + 10 \).
   \[ t \in (-1.225, 1.225) \]

8. Find when the particle whose position is given by the function in problem 3 has a decreasing acceleration.
   \[ t < 0 \]

9. Find the maximum distance from the origin for the particle whose position is given by \( y(t) = t^3 - 29t^2 - t + 29 \). Then find the acceleration of the particle at that point.
   29.009 units from origin.
   \( a(-0.017) = -58.103 \)

10. If the velocity of a particle moving along the \( x \) axis is given by \( v(t) = \sec(2t)\tan(2t) + t^2 \), find the acceleration of the particle when \( t = \pi \).

    Then find the position equation of the particle if \( x(\pi) = \frac{\pi^3}{3} + \frac{1}{2} \)

    \( a(\pi) = 2 + 2\pi \)
    \( x(t) = \frac{1}{2} \sec(2t) + \frac{t^3}{3} \)

11. If the position of a particle is given by \( y(t) = t^3 + 6t + \cos t \) on \( t \in [0, 2\pi] \), find the position of the particle when the acceleration is \( 3\pi \).

    \( y\left(\frac{\pi}{2}\right) = \frac{\pi^3}{8} + 3\pi \)
3.3: Implicit Differentiation, Part 2

One of the more useful aspects of the chain rule that we reviewed earlier is that we can take derivatives of more complicated equations that would be difficult to take the derivative of otherwise. One of the key elements to remember is that we already know the derivative of $y$ with respect to $x$—that is, $\frac{dy}{dx}$. This can be a powerful tool as it allows us to take the derivative of relations as well as functions while bypassing a lot of tedious algebra.

OBJECTIVES

- Take derivatives of relations implicitly.
- Use implicit differentiation to find higher order derivatives.

Ex 1 Find the derivative of $\ln(y) = 3x^2 + 5x + 7$

Notice there are two different ways of doing this problem. First, we could simply solve for $y$ and then take the derivative.

\[ \ln(y) = 3x^2 + 5x + 7 \]
\[ y = e^{3x^2 + 5x + 7} \]

Now take the derivative
\[ \frac{dy}{dx} = (e^{3x^2 + 5x + 7})(6x + 5) \]

Implicit Differentiation allows us the luxury of taking the derivative without the first algebra step because of the chain rule

\[ \frac{d}{dx}\left[ \ln(y) = 3x^2 + 5x + 7 \right] \]
\[ \frac{1}{y}\left( \frac{dy}{dx} \right) = 6x + 5 \]

Notice when we took the derivative, we had to use the chain rule;
\[ \frac{d}{dx}[y] = \frac{dy}{dx} \]
\[ \frac{dy}{dx} = y(6x + 5) \]
You might not immediately recognize that the two answers are the same, but since \( y = e^{3x^2 + 5x + 7} \), a simple substitution can show you that they are actually the same.

In terms of functions, this may not be very interesting or important, because it is often simple to isolate \( y \). But consider a non-function, like this circle.

**Ex 2** Find if \( x^2 + y^2 = 25 \)

\[
\frac{d}{dx} \left[ x^2 + y^2 = 25 \right] \rightarrow \frac{d}{dx} \left( x^2 \right) + \frac{d}{dx} \left( y^2 \right) = \frac{d}{dx} \left( 25 \right)
\]

\[
2x + 2y \frac{dy}{dx} = 0
\]

We can now isolate \( \frac{dy}{dx} \)

\[
2y \frac{dy}{dx} = -2x
\]

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

But even with this function, we could have solved for \( y \) and then found \( \frac{dy}{dx} \).

\[
x^2 + y^2 = 25
\]

\[
y^2 = 25 - x^2
\]

\[
y = \pm \sqrt{25 - x^2}
\]

\[
\frac{dy}{dx} = \pm \frac{-x}{\sqrt{25 - x^2}}
\]

Notice that this is the exact same answer as we found with implicit differentiation. You could substitute \( y \) for \( \pm \sqrt{25 - x^2} \) in the denominator and come up with the same derivative, \( \frac{dy}{dx} = -\frac{x}{y} \).

The other thing that you may notice is that this is the differential equation we solved last chapter – and the solution to the differential was a circle.
Ex 3 Find \( \frac{dy}{dx} \) for the hyperbola \( x^2 - 3xy + 3y^2 = 2 \)

It would be very difficult to solve for \( y \) here, so implicit differentiation is really our only option.

\[
\frac{d}{dx} \left[ x^2 - 3xy + 3y^2 = 2 \right] \\
2x - 3x \frac{dy}{dx} - 3y + 6y \frac{dy}{dx} = 0 \\
2x - 3y = 3x \frac{dy}{dx} - 6y \frac{dy}{dx} \\
2x - 3y = (3x - 6y) \frac{dy}{dx} \\
\frac{dy}{dx} = \frac{2x - 3y}{3x - 6y}
\]

Of course if we want to find a second derivative, we can use implicit differentiation a second time.

Ex 4 Find \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \) for the hyperbola \( x^2 - 3y^2 + 4x - 12y - 2 = 0 \)

\[
\frac{d}{dx} \left[ x^2 - 3y^2 + 4x - 12y - 2 = 0 \right] \\
2x - 6y \frac{dy}{dx} + 4 - 12 \frac{dy}{dx} = 0 \\
2x + 4 = 6y \frac{dy}{dx} + 12 \frac{dy}{dx} \\
2x + 4 = (6y + 12) \frac{dy}{dx} \\
\frac{dy}{dx} = \frac{2x + 4}{6y + 12} \\
\frac{dy}{dx} = \frac{x + 2}{3y + 6}
\]
Now we just take the derivative again to find \( \frac{d^2 y}{dx^2} \).

\[
\frac{d}{dx}\left[ \frac{dy}{dx} \right] = \frac{x+2}{3y+6}
\]

\[
\frac{d^2 y}{dx^2} = \frac{(3y+6)-(x+2)3\frac{dy}{dx}}{(3y+6)^2}
\]

Since we already know \( \frac{dy}{dx} \), we can substitute

\[
\frac{d^2 y}{dx^2} = \frac{(3y+6)-(x+2)\left(3\frac{x+2}{3y+6}\right)}{(3y+6)^2}
= \frac{(3y+6)^2-3(x+2)^2}{(3y+6)^3}
\]

Ex 5 Find \( \frac{dy}{dx} \) and \( \frac{d^2 y}{dx^2} \) for \( \sin(y) = 2 \cos(3x) \)

\[
\frac{d}{dx}\left[ \sin(y) = 2 \cos(3x) \right]
\]

\[
\cos(y)\frac{dy}{dx} = -6 \sin(3x)
\]

\[
\frac{dy}{dx} = \frac{-6 \sin(3x)}{\cos(y)}
\]

\[
\frac{d^2 y}{dx^2} = \frac{-18 \cos(y) \cos(3x) - 6 \sin(3x) \sin(y) \left( \frac{dy}{dx} \right)}{\cos^2(y)}
\]

\[
\frac{d^2 y}{dx^2} = \frac{-18 \cos^2(y) \cos(3x) + 36 \sin^2(3x) \sin(y)}{\cos^3(y)}
\]

So \( \frac{dy}{dx} = \frac{-6 \sin(3x)}{\cos(y)} \) and \( \frac{d^2 y}{dx^2} = \frac{-18 \cos^2(y) \cos(3x) + 36 \sin^2(3x) \sin(y)}{\cos^3(y)} \)
Be Careful! There is a lot of algebraic simplification that happens in these problems, and it is easy to make mistakes. Take your time with the simplifications so that you don’t make careless mistakes.

Another issue that arises is the need to use both the Product Rule and the Quotient Rule. Make sure you look for these when you are working through a problem.

Ex 6 Find $\frac{dy}{dx}$ for $e^{x^2} + xy^2 - 16 = \frac{\tan y}{3x}$

$\frac{d}{dx} \left[ e^{x^2} + xy^2 - 16 = \frac{\tan y}{3x} \right]$

$2xe^{x^2} + 2xy \frac{dy}{dx} + y^2 = \frac{3x \sec^2 y \frac{dy}{dx} - 3 \tan x}{9x^2}$

$2xe^{x^2} + 2xy \frac{dy}{dx} + y^2 = \frac{x \sec^2 y \frac{dy}{dx} - \tan x}{3x^2}$

$6x^3 e^{x^2} + 6x^3 y \frac{dy}{dx} + 3x^2 y^2 = x \sec^2 y \frac{dy}{dx} - \tan x$

$6x^3 e^{x^2} + 3x^2 y^2 + \tan x = x \sec^2 y \frac{dy}{dx} - 6x^3 y \frac{dy}{dx}$

$6x^3 e^{x^2} + 3x^2 y^2 + \tan x = \frac{dy}{dx} \left( x \sec^2 y - 6x^3 y \right)$

$\frac{dy}{dx} = \frac{6x^3 e^{x^2} + 3x^2 y^2 + \tan x}{x \sec^2 y - 6x^3 y}$
3.3 Homework Set A

Find $\frac{dy}{dx}$ for each of these equations, first by implicit differentiation, then by solving for $y$ and differentiating. Show that $\frac{dy}{dx}$ is the same in both cases.

1. $xy + 2x + 3x^2 = 4$

2. $\frac{1}{x} + \frac{1}{y} = 1$

3. $\sqrt{x} + \sqrt{y} = 4$
Find $\frac{dy}{dx}$ for each of these equations by implicit differentiation.

4. $x^2 + y^2 = 1$

5. $x^3 + 10x^2 y + 7y^2 = 60$

6. $x^2 y^2 + x \sin(y) = 4$

7. $4 \cos(x) \sin(y) = 1$

8. $e^{x^2 y} = x + y$

9. $\tan(x - y) = \frac{y}{1 + x^2}$
Find the equation of the line tangent to each of the following relations at the given point.

10. \( x^2 - y^2 - 6y - 3 = 0 \) at \( (\sqrt{3}, 0) \)

11. \( 9x^2 + 4y^2 + 36x - 8y + 4 = 0 \) at \( (0, -2) \)

12. \( 12x^2 - 4y^2 + 72x + 16y + 44 = 0 \) at \( (-1, -3) \)

13. Find the equation of the lines tangent and normal to 
\[ y - \frac{4}{\pi} x^2 = 2e^{ysinx} + y^3 - 3 \] through the point \( \left( \frac{\pi}{2}, 0 \right) \).
14. Find the equation of the line tangent to \( x^3 + \frac{y}{x} + y^2 = 7 \), through the point (1,2).

15. Find the equation of the line tangent to \( x^2 + 3xy + y^2 = 8 \), through the point (1,2).

16. Find \( \frac{d^2y}{dx^2} \) if \( xy + y^2 = 1 \)  

17. Find \( \frac{d^2y}{dx^2} \) if \( 4x^2 + 9y^2 = 36 \)
3.3 Homework Set B

1. Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) if \( y^4 + 3xy = 11 \)

2. Use implicit differentiation to find \( \frac{dy}{dx} \) for \( 4xy^2 - \sin(3x) = 5\tan(y) + 7 \)

3. Find \( \frac{dy}{dx} \) for \( 5x^2 + x^3y - \ln y = 16 \)

4. Find \( \frac{dy}{dx} \) for \( \cos y + \sin x = \frac{x^2}{y^2} + 16 \)
5. Find $\frac{dy}{dx}$ if $y^4 + \frac{4}{y^2} + \ln(xy) = -12$

6. Find $\frac{dy}{dx}$ for $xy^2 - y^4 = 3x + 15$

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 16$

8. Find $\frac{dy}{dx}$ if $3x^3 - e^{xy^2} - 17 = \cos y + x$
Answers: 3.3 Homework Set A

1.  \( xy + 2x + 3x^2 = 4 \)
   Imp: \( \frac{dy}{dx} = -\frac{6x - y - 2}{x} \); Exp: \( \frac{dy}{dx} = -\frac{4 - 3x^2}{x^2} \)

2.  \( \frac{1}{x} + \frac{1}{y} = 1 \)
   Imp: \( \frac{dy}{dx} = -\frac{y^2}{x^2} \); Exp: \( \frac{dy}{dx} = -\frac{1}{(1-x)^2} \)

3.  \( \sqrt{x} + \sqrt{y} = 4 \)
   Imp: \( \frac{dy}{dx} = -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \); Exp: \( \frac{dy}{dx} = \frac{-4 + \sqrt{x}}{\sqrt{x}} \)

4.  \( x^2 + y^2 = 1 \)
   \( \frac{dy}{dx} = -\frac{x}{y} \)

5.  \( x^3 + 10x^2 y + 7y^2 = 60 \)
   \( \frac{dy}{dx} = \frac{-3x^2 - 20xy}{10x^2 + 14} \)

6.  \( x^2 y^2 + x\sin(y) = 4 \)
   \( \frac{dy}{dx} = \frac{-\sin(y) - 2xy^2}{2x^2 y + x\cos(y)} \)

7.  \( 4\cos(x)\sin(y) = 1 \)
   \( \frac{dy}{dx} = \tan(x)\tan(y) \)

8.  \( e^{x^2} = x + y \)
   \( \frac{dy}{dx} = \frac{1 - 2xye^{x^2}y}{x^2e^{x^2} - 1} \)

9.  \( \tan(x - y) = \frac{y}{1 + x^2} \)
   \( \frac{dy}{dx} = \frac{\sec(x - y)(1 + x^2)^2 + 2xy}{1 + x^2 + \sec(x - y)(1 + x^2)^2} \)

10. \( x^2 - y^2 - 6y - 3 = 0 \) at \( (\sqrt{3}, 0) \)
    \( y = \frac{\sqrt{3}}{3}(x - \sqrt{3}) \)

11. \( 9x^2 + 4y^2 + 36x - 8y + 4 = 0 \) at \( (0, -2) \)
    \( y + 2 = \frac{2}{3}x \)
12. \[ 12x^2 - 4y^2 + 72x + 16y + 44 = 0 \] at \((-1, -3)\)
\[ y + 3 = \frac{-6}{5}(x + 1) \]

13. Find the equation of the lines tangent and normal to
\[ y - \frac{4}{\pi^2}x^2 = 2e^{y\sin x} + y^3 - 3 \] through the point \(\left(\frac{\pi}{2}, 0\right)\).

Tangent: \(y - 0 = -\frac{4}{\pi(x - \frac{\pi}{2})}\)
Normal: \(y - 0 = \frac{\pi}{4(x - \frac{\pi}{2})}\)

14. Find the equation of the line tangent to \(x^3 + \frac{y}{x} + y^2 = 7\), through the point \((1,2)\).
\[ y - 2 = -\frac{10}{17}(x-1) \]

15. Find the equation of the line tangent to \(x^2 + 3xy + y^2 = 8\), through the point \((1,2)\).
\[ y - 2 = -\frac{8}{7}(x-1) \]

16. Find \(\frac{d^2y}{dx^2}\) if \(xy + y^2 = 1\)
\[ \frac{d^2y}{dx^2} = \frac{2y(x + y)}{(x + 2y)^3} \]

17. Find \(\frac{d^2y}{dx^2}\) if \(4x^2 + 9y^2 = 36\)
\[ \frac{d^2y}{dx^2} = \frac{-36y^2 - 16x^2}{81y^2} \]

3.3 Homework Set B

1. Find \(\frac{dy}{dx}\) and \(\frac{d^2y}{dx^2}\) if \(y^4 + 3xy = 11\)
\[ \frac{dy}{dx} = \frac{-3y}{4y^3 + 3x} \]
\[ \frac{d^2y}{dx^2} = \frac{-36y^4 + 56xy}{(4y^3 + 3x)^3} \]

2. Use implicit differentiation to find \(\frac{dy}{dx}\) for \(4xy^2 - \sin(3x) = 5\tan(y) + 7\)
\[ \frac{dy}{dx} = \frac{3\cos(3x)}{8xy - 3\cos(3x)} \]
3. Find $\frac{dy}{dx}$ for $5x^2 + x^3y - \ln y = 16$

$$\frac{dy}{dx} = \frac{-3x^2y^2 - 10xy}{x^3y + 1}$$

4. Find $\frac{dy}{dx}$ for $\cos y + \sin x = \frac{x^2}{y^2} + 16$

$$\frac{dy}{dx} = \frac{y^3 \cos x - 2xy}{y^3 \sin y - 2x^2}$$

5. Find $\frac{dy}{dx}$ if $y^4 + \frac{4}{y^2} + \ln(xy) = -12$

$$\frac{dy}{dx} = \frac{-xy^3}{4xy^6 - 8x + xy^2}$$

6. Find $\frac{dy}{dx}$ for $xy^2 - y^4 = 3x + 15$

$$\frac{dy}{dx} = \frac{3 - y^2}{2xy - 4xy^3}$$

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 16$

$$\frac{dy}{dx} = \frac{-x}{y} \quad \frac{d^2y}{dx^2} = \frac{-y^2 + x^2}{y^3}$$

8. Find $\frac{dy}{dx}$ if $3x^3 - e^{xy^2} - 17 = \cos y + x$

$$\frac{dy}{dx} = \frac{1 - 9x^2 + e^{xy^2}y^2}{\sin y - 2e^{xy^2}xy}$$
3.4: Related Rates

We have been looking at derivatives as rates of change for this chapter, and one of the principle ways that this comes up in Calculus is in the topic “related rates”. Hopefully, you recall this section from your Precalculus class – the topic in chapter 4 where everyone just skipped this problem on the test. Well, it is a very important topic for Calculus, so you don’t get to skip this topic anymore.

OBJECTIVE

Solve related rates problems.

As we saw in the previous section, parametric equations related an x and y to some other variable, t. This is very common in advanced mathematics; having some independent variable on which several other variables depend. In the case of related rates, the variable is time, and we seldom actually see that variable in the problem. This is very similar to both implicit differentiation in the previous section and motion in the section before that. With related rates, each of the variables given to us are actually in terms of another variable, t, (usually time). All of the variables are actually functions of time, but we never see t anywhere in the equations we use.

These problems are “rate” problems. This means that time is a factor, even though we never see it as a variable. When we see a rate with time in the denominator, it should be clear that we have a rate of change with respect to t (that is $\frac{d}{dt}$).

Suppose, for example, I am inflating a spherical balloon. It is obvious as you watch the balloon growing in size that the volume, radius, and surface area of the balloon are all increasing as you inflate the balloon. The problem arises in that I do not have an equation in terms of t for each of those variables. I do have an equation relating volume and radius, and another one relating surface area and radius.

$$V = \frac{4}{3} \pi r^3 \quad S = 4\pi r^2$$
Since I know that both of the equations have time as a component (even though I do not see \( t \) as a variable). I could take the derivative of each with respect to time.

\[
\frac{d}{dt} \left[ V = \frac{4}{3} \pi r^3 \right] \quad \frac{d}{dt} \left[ S = 4\pi r^2 \right]
\]

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}
\]

Notice that we had to use implicit differentiation and The Chain Rule for all of the variables. Since each was defined in terms of \( t \), when we took the derivative, The Chain Rule gave us the implicit fraction \( \frac{d \text{ whatever}}{dt} \).

For example, the derivative of \( V \) with respect to \( t \) must be \( \frac{dV}{dt} \), and the derivative of \( r^3 \) with respect to \( t \) must be \( 3r^2 \frac{dr}{dt} \).

Ex 1 Two cars approach an intersection, one traveling south at 20 mph and the other traveling west at 30 mph. How fast is the direct distance between them decreasing when the westbound car is 0.6 miles and the southbound car is 0.8 miles from the intersection?

As we can see in the picture, the distance between the two cars are related by the Pythagorean Theorem.

\[ x^2 + y^2 = r^2 \]
We know several pieces of information. The southbound car is moving at 20 mph; i.e. \( \frac{dy}{dt} = -20 \). By similar logic we can deduce each of the following:

\[
\begin{align*}
\frac{dy}{dt} &= -20 \\
\frac{dx}{dt} &= -30 \\
y &= 0.8 \\
x &= 0.6
\end{align*}
\]

And, by the Pythagorean Theorem, \( r = 1.0 \)

It is very important to notice that even though we solved for \( r \) using the Pythagorean Theorem, \( r \) is still a variable (the distance between the two cars is actually changing).

Now we take the derivative of the Pythagorean Theorem and get

\[
\frac{d}{dt} \left[ x^2 + y^2 = r^2 \right]
\]

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt}
\]

This is essentially an equation in six variables. But we know five of those six variables, so just substitute and solve.

\[
2(0.8)(-30) + 2(0.6)(-20) = 2(1.0) \frac{dr}{dt}
\]

\[
\frac{dr}{dt} = -36 \text{ miles/hour}
\]

It should make sense that \( \frac{dr}{dt} \) is negative since the two cars are approaching one another. We know the units based on the fraction, \( \frac{dr}{dt} \). Since \( r \) was in miles and \( t \) was in hours, our final units must be miles/hour.
Ex 2  Sand is dumped onto a pile at 30 ft$^3$/min. The pile forms a cone with a height equal to the base diameter. How fast is the base area changing when the pile is 10 feet high?

The units on the 30 tell us that it is the change in volume, or \( \frac{dV}{dt} \). We know that the volume of a cone is \( V = \frac{1}{3}\pi r^2 h \). But this equation has too many variables for us to differentiate it as it stands, and we are looking for \( \frac{dA}{dt} \), or how fast the base area is changing.

We should start by organizing what exactly we know, and what we want. It is also often helpful (but not always necessary) to draw a picture to help us identify what we know.

<table>
<thead>
<tr>
<th>What we know:</th>
<th>What I need to find:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dV}{dt} = 30 )</td>
<td>( \frac{dA}{dt} )</td>
</tr>
<tr>
<td>( h = d ) (the problem said height was the same as diameter)</td>
<td>when ( h = 10 ) feet</td>
</tr>
<tr>
<td>( V = \frac{1}{3}\pi r^2 h )</td>
<td></td>
</tr>
<tr>
<td>( A_{\text{base}} = \pi r^2 ) (since the base of a cone is a circle)</td>
<td></td>
</tr>
</tbody>
</table>
\[ V = \frac{1}{3} \pi r^2 (2r) \text{ and } A = \pi r^2 \]

At this point, you could differentiate both equations and substitute to solve for \( \frac{dA}{dt} \), or you could substitute \( A \) into the volume equation and differentiate to find \( \frac{dA}{dt} \). We will do the latter.

\[
V = \frac{2}{3} \pi \left( \frac{A}{\pi} \right)^3 \\
V = \frac{2}{3\sqrt{\pi}} A^3 \\
\frac{dV}{dt} = \frac{1}{\sqrt{\pi}} A^\frac{1}{2} \frac{dA}{dt}
\]

Since we know that \( \frac{dV}{dt} = 30 \text{ ft}^3/\text{min.} \) and \( A = 25\pi \text{ ft}^2 \) (because \( r = 5 \text{ feet} \)),

\[ \frac{dA}{dt} = 6 \text{ ft}^2/\text{min.} \]
Obviously, remembering formulas from algebra and geometry classes is really important, and we will expect that you have certain formulas committed to memory. Here is a short list that you should know.

<table>
<thead>
<tr>
<th>Common Formulas for Related Rates Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pythagorean Theorem:</strong></td>
</tr>
<tr>
<td>$x^2 + y^2 = r^2$</td>
</tr>
<tr>
<td><strong>Area Formulas:</strong></td>
</tr>
<tr>
<td>Circle $A = \pi r^2$</td>
</tr>
<tr>
<td>Rectangle $A = lw$</td>
</tr>
<tr>
<td>Trapezoid $A = \frac{1}{2} h(b_1 + b_2)$</td>
</tr>
<tr>
<td><strong>Volume Formulas:</strong></td>
</tr>
<tr>
<td>Sphere $V = \frac{4}{3} \pi r^3$</td>
</tr>
<tr>
<td>Right Prism $V = Bh$</td>
</tr>
<tr>
<td>Cylinder $V = \pi r^2 h$</td>
</tr>
<tr>
<td>Cone $V = \frac{1}{3} \pi r^2 h$</td>
</tr>
<tr>
<td>Right Pyramid $V = \frac{1}{3} Bh$</td>
</tr>
<tr>
<td><strong>Surface Area Formulas:</strong></td>
</tr>
<tr>
<td>Sphere $S = 4\pi r^2$</td>
</tr>
<tr>
<td>Cylinder $S = 2\pi r^2 + 2\pi rl$</td>
</tr>
<tr>
<td>Cone $S = \pi r^2 + \pi rl$</td>
</tr>
<tr>
<td>Right Prism $S = 2B + Ph$</td>
</tr>
</tbody>
</table>
Ex 3 If two resistors are connected in parallel, then the total resistance is given by
the formula \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \), where all values for R are in Ohms (\( \Omega \)). If \( R_1 \) and
\( R_2 \) are increasing at rates of \( 0.3 \frac{\Omega}{\text{sec}} \) and \( 0.2 \frac{\Omega}{\text{sec}} \), respectively, find
how fast R is changing when \( R_1 = 80 \Omega \) and \( R_2 = 100 \Omega \).

\[
\frac{d}{dt} \left[ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \right]
\]

\[-R^2 \left( \frac{dR}{dt} \right) = -R_1^2 \left( \frac{dR_1}{dt} \right) - R_2^2 \left( \frac{dR_2}{dt} \right)\]

\[-\left( \frac{400}{9} \right)^2 \left( \frac{dR}{dt} \right) = -(80)^2 (0.3) - (100)^2 (0.2)\]

\[
\frac{dR}{dt} = \frac{107}{810} \text{ or } 0.132 \frac{\Omega}{\text{sec}}
\]
3.4 Homework Set A

1. If $V$ is the volume of a cube with edge length, $s$, and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{ds}{dt}$.

2. A particle is moving along the curve $y = \sqrt{1 + x^3}$. As it reaches the point (2,3), the $y$-coordinate is increasing at a rate of 4 cm/sec. How fast is the $x$-coordinate changing at that moment?

3. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph flies directly over a radar station. Find the rate at which the horizontal distance is increasing when it is 2 miles from the station. Find the rate at which the distance between the station and the plane is increasing when the plane is 2 miles from the station.
4. If a snowball melts so that its surface area is decreasing at a rate of 1 \( \text{cm}^2/\text{min} \), find the rate at which the diameter is decreasing when it has a diameter of 10 cm.

5. A street light is mounted at the top of a 15 foot tall pole. A 6 foot tall man walks away from the pole at a speed of 5 ft/sec in a straight line. How fast is the tip of his shadow moving when he is 40 feet from the pole?

6. Two cars start moving away from the same point. One travels south at 60 mph, and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?
7. The altitude of a triangle is increasing at a rate of 1 cm/min, while the area of the triangle is increasing at a rate of 2 cm²/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm²?

3.4 Homework Set B

1. Water is leaking out of an inverted conical tank at a rate of 5000 cm³/min. If the tank is 8 m tall and has a diameter of 4 m. Find the rate at which the height is decreasing when the water level is at 3 m. Then find the rate of change of the radius at that same instant.

2. Water is leaking out of an inverted conical tank at a rate of 5000 cm³/min at the same time that water is being pumped in at an unknown constant rate. If the tank is 8 m tall and has a diameter of 4 m, and when the water level is 6 m the height of the water is increasing at 20 cm/min, find the rate at which the water is being added.
3. A man starts walking north at 5 feet/sec from a point P. 3 minutes later a woman starts walking east at a rate of 4 feet/sec from a point 500 feet east of point P. At what rate are the two moving apart at 12 minutes after the woman starts walking.

4. A particle moves along the curve \( x^2 + xy + y = 17 \). When \( y = 2 \), \( \frac{dx}{dt} = 10 \), find all values of \( \frac{dy}{dt} \).

5. The altitude of a triangle is increasing at a rate of 2 cm/sec at the same time that the area of the triangle is increasing at a rate of 5 cm\(^2\)/sec. At what rate is the base increasing when the altitude is 12 cm and the area is 144 cm\(^2\)?
6. The total resistance of a certain circuit is given by \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \). If \( R_1 \) is increasing at a rate of 0.4 \( \Omega/\text{sec} \) and \( R_2 \) is decreasing at a rate of 0.2 \( \Omega/\text{sec} \), how fast is \( R \) changing when \( R_1 = 20 \Omega \) and \( R_2 = 50 \Omega \)?
Answers: 3.4 Homework Set A

1. If $V$ is the volume of a cube with edge length, $s$, and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{ds}{dt}$.

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$$

2. A particle is moving along the curve $y = \sqrt{1 + x^3}$. As it reaches the point (2,3), the $y$-coordinate is increasing at a rate of 4 cm/sec. How fast is the $x$-coordinate changing at that moment? 2 cm/sec

3. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph flies directly over a radar station. Find the rate at which the horizontal distance is increasing when it is 2 miles from the station. Find the rate at which the distance between the station and the plane is increasing when the plane is 2 miles from the station.

$250\sqrt{3}$ mph

4. If a snowball melts so that its surface area is decreasing at a rate of 1 cm$^2$/min, find the rate at which the diameter is decreasing when it has a diameter of 10 cm.

$$\frac{-1}{20\pi} \text{ cm/min}$$

5. A street light is mounted at the top of a 15 foot tall pole. A 6 foot tall man walks away from the pole at a speed of 5 ft/sec in a straight line. How fast is the tip of his shadow moving when he is 40 feet from the pole?

$$\frac{25}{3} \text{ ft/min}$$

6. Two cars start moving away from the same point. One travels south at 60 mph, and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

65 mph

7. The altitude of a triangle is increasing at a rate of 1 cm/min, while the area of the triangle is increasing at a rate of 2 cm$^2$/min. At what rate is the base of the triangle increasing when the altitude is 10 cm and the area is 100 cm$^2$?

$$-1.6 \text{ cm/min}$$
3.4 Homework Set B

1. Water is leaking out of an inverted conical tank at a rate of 5000 cm$^3$/min. If the tank is 8 m tall and has a diameter of 4 m. Find the rate at which the height is decreasing when the water level is at 3 m. Then find the rate of change of the radius at that same instant.

\[
\frac{dh}{dt} = \frac{80,000}{27\pi} \text{ cm/min} \quad \text{and} \quad \frac{dr}{dt} = \frac{20,000}{27\pi} \text{ cm/min}
\]

2. Water is leaking out of an inverted conical tank at a rate of 5000 cm$^3$/min at the same time that water is being pumped in at an unknown constant rate. If the tank is 8 m tall and has a diameter of 4 m, and when the water level is 6 m the height of the water is increasing at 20 cm/min, find the rate at which the water is being added.

\[
\frac{dV}{dt} = 540\pi + 5,000 \text{ cm/min}
\]

3. A man starts walking north at 5 feet/sec from a point P. 3 minutes later a woman starts walking east at a rate of 4 feet/sec from a point 500 feet east of point P. At what rate are the two moving apart at 12 minutes after the woman starts walking?

4.641 feet/sec

4. A particle moves along the curve $x^2 + xy + y = 17$. When $y = 2$, $\frac{dx}{dt} = 10$, find all values of $\frac{dy}{dt}$.

\[
\frac{dy}{dt} = -20, -22.5
\]

5. The altitude of a triangle is increasing at a rate of 2 cm/sec at the same time that the area of the triangle is increasing at a rate of 5 cm$^2$/sec. At what rate is the base increasing when the altitude is 12 cm and the area is 144 cm$^2$?

\[
\frac{db}{dt} = -\frac{19}{6} \text{ cm/sec}
\]
6. The total resistance of a certain circuit is given by \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \). If \( R_1 \) is increasing at a rate of \( 0.4 \, \Omega/\text{sec} \) and \( R_2 \) is decreasing at a rate of \( 0.2 \, \Omega/\text{sec} \), how fast is \( R \) changing when \( R_1 = 20 \, \Omega \) and \( R_2 = 50 \, \Omega \)?

\[
\frac{dR}{dt} = 0.188 \, \Omega/\text{sec}
\]
3.5 The First and Second Derivative Tests

**OBJECTIVE**

Use the 1\textsuperscript{st} and 2\textsuperscript{nd} derivative tests to find maxima and minima.

While the first derivative is what allows us to algebraically find extremes, and BOTH derivative tests allow us to interpret critical values as maxima or minima. Since the sign pattern of the first derivative tells us when that function is increasing or decreasing, we can figure out if a critical value is associated with a maximum or minimum depending on the sign change of the derivative.

**The 1st Derivative Test**

As the sign pattern of the 1st derivative is viewed left to right, the critical value represents a

1) relative maximum if the sign changes from + to −

2) relative minimum if the sign changes from − to +

or 3) neither a max. or min. if the sign does not change.

Ex 1 Apply the first derivative test to the function \( y = 5x^4 - 10x^2 \)

\[
\frac{dy}{dx} = 20x^3 - 20x = 0
\]

\[
20x(x^2 - 1) = 0
\]

\[
x = -1, 0, 1
\]

\[
y' \begin{array}{cccc}
- & 0 & + & 0 & - & 0 & + \\
\hline
x & -1 & 0 & 1
\end{array}
\]

So there are critical values at \( x \) values of −1, 0, and 1. When we look at the sign pattern, we can see we have a minimum at −1, a maximum at 0, and another minimum at 1.
The second derivative can tell us about the concavity of a curve, in essence which way the curve is “bending”:

| If \( f''(x) > 0 \), then \( f(x) \) is concave up. |
| If \( f''(x) < 0 \), then \( f(x) \) is concave down. |

“Concave up” means the curve is turning in an upwards direction:

Looking either like this: \( \quad \) Or this: \( \quad \)

“Concave down” means the curve is turning in an downwards direction:

Looking either like this: \( \quad \) Or this: \( \quad \)

(Note that we would need the first derivative to tell if a section of a curve was increasing or decreasing, the second derivative only tells us which way the curve “bends”)

If you have a critical value on a section of curve that is concave up, the critical value has to give a minimum value for the function (because of the shape of the curve). Similarly, if you have a critical value on a section of a curve that is concave down, the critical value must give a maximum value for the function (again, because of the shape of the curve). This can be generalized into what is called “The Second Derivative Test” – a test to find maximum or minimum values using the second derivative (essentially, using a function’s concavity).

**The 2\(^{nd}\) Derivative Test**

For a function \( f \),

1) If \( f' (c) = 0 \) and \( f''(c) > 0 \), then \( f \) has a relative minimum at \( c \).

2) If \( f' (c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a relative maximum at \( c \).
Ex 2 Use the 2nd Derivative Test to determine if the critical values of 
$g(t) = 27t - t^3$ are at a maximum or minimum value

$g'(t) = 27 - 3t^2 = 0$

$t = \pm 3$

So $-3$ and $3$ are critical values.

$g''(t) = -6t$

$g''(-3) = -6(-3) = 18$

$g''(3) = -6(3) = -18$

Therefore, $g$ has a minimum value at $t = -3$ and has a maximum value at $t = 3$. (Note that the numerical value “18” is irrelevant.)

Ex 3 Find the maximum values for the function $g(x) = \sqrt{x^3 - 9x}$

Domain: $x^3 - 9x \geq 0$

$y' \xleftarrow{x} - 0 + 0 - 0 +$

$x^3 - 9x \geq 0 \rightarrow x \in [-3,0] \cup [3,\infty)$

$g'(x) = \frac{3x^2 - 9}{2\sqrt{x^3 - 9x}}$

$g'(x) = 0$ when $3x^2 - 9 = 0$, so $x = \pm \sqrt{3}$

$g'(x)$ does not exist when $x^3 - 9x = 0$, so $x = \pm 3, 0$

$y' \xleftarrow{x} - \text{dne} + 0 - \text{dne} \frac{\text{dne}}{3} +$

Since $x = \sqrt{3}$ is not in the domain of the function, our critical values are at $x = \pm 3, 0, -\sqrt{3}$. From our sign pattern, we can conclude that at $x = -\sqrt{3}$, we have a maximum value, and by substituting this value into the function, we find that the maximum value is at $y = 3.224$. 
3.5 Homework

Find the absolute maximum and minimum values of \( f \) on the given intervals.

1. \( f(x) = \frac{x}{x^2 + 1}, \ [0, 2] \)

2. \( f(t) = \sqrt[3]{t} (8 - t), \ [0, 8] \)

3. \( f(z) = ze^{-z}, \ [0, 2] \)

4. \( f(x) = \frac{\ln(x)}{x}, \ [1, 3] \)

5. \( f(x) = e^{-x} - e^{-2x}, \ [0, 1] \)
For each of the following functions, apply the 1st Derivative Test, then verify the results with the 2nd Derivative Test.

6. \( f(x) = x^2 \ln x \)

7. \( h(f) = f^3 - 12f + 21 \)

8. \( f(x) = xe^{-x^2} \)
Answers: 3.6 Homework

1. \( f(x) = \frac{x}{x^2 + 1}, \quad [0, 2] \)
   \[
   f(0) = 0 \therefore \text{abs. min., } f(1) = \frac{1}{2} \therefore \text{abs. max.}
   \]

2. \( f(t) = \frac{3}{\sqrt[3]{t}} (8-t), \quad [0, 8] \)
   \[
   f(0) = 0 \therefore \text{abs. min., } f(2) = 7.560 \leftarrow \therefore \text{abs. max., } f(8) = 0 \therefore \text{abs. min.}
   \]

3. \( f(z) = ze^{-z}, \quad [0, 2] \)
   \[
   f(0) = 0 \therefore \text{abs. min., } f(1) = \frac{1}{e} \therefore \text{abs. max.}
   \]

4. \( f(x) = \frac{\ln(x)}{x}, \quad [1, 3] \)
   \[
   f(1) = 0 \therefore \text{abs. min., } f(e) = \frac{1}{e} \therefore \text{abs. max.}
   \]

5. \( f(x) = e^{-x} - e^{-2x}, \quad [0, 1] \)
   \[
   f(0) = 0 \therefore \text{abs. min., } f(\ln 2) = .25 \therefore \text{abs. max.}
   \]

6. \( f(x) = x^2 \ln x \)

\[
\begin{align*}
\text{1st } D_x \text{ Test:} \\
\text{Min: } x &= \frac{1}{\sqrt{e}} \\
\text{Max: } x &= 0
\end{align*}
\]

\[
\begin{align*}
\text{2nd } D_x \text{ Test:} \\
\text{Min: } x &= \frac{1}{\sqrt{e^3}} \\
\text{Max: } x &= 0
\end{align*}
\]
7. \( h(f) = f^3 - 12f + 21 \)

\[
\begin{align*}
h'(f) & \quad + \quad 0 \quad - \quad 0 \quad + \\
x & \quad -\sqrt{12} \quad \sqrt{12}
\end{align*}
\]

\[
\begin{align*}
h''(f) & \quad - \quad 0 \quad + \\
x & \quad 0
\end{align*}
\]

1st \( D_3 \) Test:
Min: \( x = \sqrt{12} \)
Max: \( x = -\sqrt{12} \)

2nd \( D_3 \) Test:
Min: \( x = \sqrt{12} \)
Max: \( x = -\sqrt{12} \)

8. \( f(x) = xe^{-x^2} \)

\[
\begin{align*}
f'(x) & \quad - \quad 0 \quad + \quad 0 \quad - \\
x & \quad -\sqrt{\frac{1}{2}} \quad \sqrt{\frac{1}{2}}
\end{align*}
\]

\[
\begin{align*}
f''(x) & \quad - \quad 0 \quad + \quad 0 \quad - \quad 0 \quad + \\
x & \quad -\sqrt{\frac{1}{2}} \quad 0 \quad \sqrt{\frac{1}{2}}
\end{align*}
\]

1st \( D_3 \) Test:
Min: \( x = -\sqrt{\frac{1}{2}} \)
Max: \( x = \sqrt{\frac{1}{2}} \)

2nd \( D_3 \) Test:
Min: \( x = -\sqrt{\frac{1}{2}} \)
Max: \( x = \sqrt{\frac{1}{2}} \)
3.6: Optimization

In the last section, we looked at extrema and the derivative tests. Optimization is a practical application of finding maxima and minima for functions. As I mentioned before, this revolutionized thinking and is a critical component of all industries. You might remember this topic from chapter 2 of book 2 last year; the word problems many of you avoided on the test last year. This year, they form a much more fundamental part of what we need to be able to do, so we can no longer simply skip these problems on tests.

OBJECTIVES

Solve optimization problems.

Every optimization problem looks a bit different, but they all follow a similar progression. You must first identify your variables and any formula you need. Use algebra to eliminate variables, and **take the derivative of the function you are trying to optimize**. This is the most common mistake in optimization problems; taking the derivative of the wrong function.

Ex 1 The owner of the Rancho Grande has 3000 yards of fencing material with which to enclose a rectangular piece of grazing land along the straight portion of a river. If fencing is not required along the river, what are the dimensions of the largest area he can enclose? What is the area?

\[ A = lw \]

River

The problem with maximizing this area formula lies in the fact that we have two independent variables \(l\) and \(w\). We need the fact about perimeter to complete the problem.
\[ P = l + 2w \]
\[ 3000 = l + 2w \]
\[ 3000 - 2w = l \]
\[ A = (3000 - 2w)w \]
\[ A = 3000w - 2w^2 \]

Now, since we have an equation with one independent variable, we can take the derivative easily.

\[ \frac{dA}{dw} = 3000 - 4w \]
\[ \frac{dA}{dw} = 3000 - 4w = 0 \]
\[ w = 750 \]
\[ l = 1500 \]

So we would want a width of 750 yards and a length of 1500 yards. This would give us an area of 1,125,000 yards\(^2\).

Ex 2 A cylindrical cola can has a volume \(32\pi \text{ in}^3\). What is the minimum surface area?

\[ V = \pi r^2 h \]
\[ 32\pi = \pi r^2 h \]
\[ \frac{32}{r^2} = h \]

\[ S = 2\pi r^2 + 2\pi rh \]
\[ S = 2\pi r^2 + \pi r \left( \frac{64}{r^2} \right) \]
\[ S = 2\pi r^2 + 64\pi r^{-1} \]
\[ \frac{dS}{dr} = 4\pi r - 64\pi r^{-2} = 0 \]
\[ 4\pi r \left( 1 - \frac{16\pi}{r^3} \right) = 0 \]
\[ r = 0 \text{ or } 3.691 \]
There is an implied domain here. You cannot have a radius of 0 inches, so 3.691 inches is the radius for the minimum area. The sign pattern verifies this:

\[
\frac{dA}{dr} \quad 0 \quad - \quad 0 \quad + \quad 3.691
\]

So the minimum surface area would be

\[
S = 2\pi(3.691)^2 + 2\pi(3.691)\left(\frac{32}{(3.691)^2}\right) = 140.058 \text{ in}^2
\]

Ex 3 Find the point on the curve \( y = \frac{e^{-x^2}}{2} \) that is closest to the origin.

We want to minimize the distance to the origin, so we will be using the Pythagorean theorem to find the distance.

\[
D = \sqrt{x^2 + y^2} \\
y = \frac{e^{-x^2}}{2} \\
D = \sqrt{x^2 + \left(\frac{e^{-x^2}}{2}\right)^2} \\
\frac{dD}{dx} = \frac{1}{2} \left(x^2 + \frac{e^{-2x^2}}{4}\right)^{-\frac{1}{2}} \left(2x - xe^{-2x^2}\right) = 0
\]

\[x = 0, \pm .841\]

So the minimum distance from the origin is at the point (.377, .434).

Given the diverse nature of optimization problems, it is helpful to remember all the formulas from geometry.
3.6 Homework

Solve these problems algebraically.

1. Find two positive numbers whose product is 110 and whose sum is a minimum.

2. Find a positive number such that the sum of the number and its reciprocal is a minimum.

3. A farmer with 750 feet of fencing material wants to enclose a rectangular area and divide it into four smaller rectangular pens with sides parallel to one side of the rectangle. What is the largest possible total area?
4. If 1200 cm$^2$ of material is available to make a box with an open top and a square base, find the maximum volume the box can contain.

5. Find the point on the line $y = 4x + 7$ that is closest to the origin.

6. Find the points on the curve $y = \frac{1}{x^2 + 1}$ that are closest to the origin.
7. Find the area of the largest rectangle that can be inscribed in the ellipse \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

8. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square, while the other is bent into an equilateral triangle. Find where the wire should be cut to maximize the area enclosed, then find where the wire should be cut to minimize the area enclosed.
9. You need to enclose 500 cm$^3$ of fluid in a cylinder. If the material you are using costs 0.001 dollars per cm$^3$, find the size of the cylinder that minimizes the cost. If the product that you are containing is a sports drink, do you think that the size that minimizes cost is the most efficient size? Explain.

10. The height of a man jumping off of a high dive is given by the function $h(t) = -4.9t^2 + 2t + 10$ on the domain $0 \leq t \leq 1.64715$. Find the absolute maximum and minimum heights modeled by this function.
11. Given that the area of a triangle can be calculated with the formula 
   \[ A = \frac{1}{2}ab \sin \theta, \]
   what value of \( \theta \) will maximize the area of a triangle (given that \( a \) and \( b \) are constants)?

12. You operate a tour service that offers the following rates for the tours: $200 per person if the minimum number of people book the tour (50 people is the minimum) and for each person past 50, up to a maximum of 80 people, the cost per person is decreased by $2.

   It costs you $6000 to operate the tour plus $32 per person.

   a) Write a function that represents cost, \( C(x) \).
   b) Write a function that represents revenue, \( R(x) \).
   c) Given that profit can be represented as \( P(x) = R(x) - C(x) \), write a function that represents profit and state the domain for the function.
   d) Find the number of people that maximizes your profit. What is the maximum profit?
Answers: 3.6 Homework

1. Find two positive numbers whose product is 110 and whose sum is a minimum.
   \[ x = y = \sqrt{110} \]

2. Find a positive number such that the sum of the number and its reciprocal is a minimum.
   \[ x = 1 \]

3. A farmer with 750 feet of fencing material wants to enclose a rectangular area and divide it into four smaller rectangular pens with sides parallel to one side of the rectangle. What is the largest possible total area?
   \[ 18000 \text{ ft}^2 \]

4. If 1200 cm\(^2\) of material is available to make a box with an open top and a square base, find the maximum volume the box can contain.
   \[ 5700 \text{ cm}^3 \]

5. Find the point on the line \( y = 4x + 7 \) that is closest to the origin.
   \[ x = \frac{-28}{17}, \ y = \frac{7}{17} \]

6. Find the points on the curve \( y = \frac{1}{x^2 + 1} \) that are closest to the origin.
   \[ (\pm 0.510, 0.794) \]

7. Find the area of the largest rectangle that can be inscribed in the ellipse
   \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
   \[ \text{Max. Area} = ab \]

8. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square, while the other is bent into an equilateral triangle. Find where the wire should be cut to maximize the area enclosed, then find where the wire should be cut to minimize the area enclosed.
   \[ \text{Maximum is to use all the wire for the square, min is } x = 2.795 \text{ cm for the square.} \]
9. You need to enclose 500 cm$^3$ of fluid in a cylinder. If the material you are using costs 0.001 dollars per cm$^3$, find the size of the cylinder that minimizes the cost. If the product that you are containing is a sports drink, do you think that the size that minimizes cost is the most efficient size? Explain.

$r = 4.301$ cm, $h = 8.603$ cm. This seems to be an inefficient size for the purpose of holding it in one’s hand: it is 8.603 cm tall (almost 3 and a half inches) and the same distance across – not a comfortable fit for most people’s hands.

10. The height of a man jumping off of a high dive is given by the function $h(t) = -4.9t^2 + 2t + 10$ on the domain $0 \leq t \leq 1.64715$. Find the absolute maximum and minimum heights modeled by this function.

Absolute Maximum is at $y = 10.204$ meters
Absolute Minimum is at $y = 0$ meters

11. Given that the area of a triangle can be calculated with the formula $A = \frac{1}{2}ab\sin \theta$, what value of $\theta$ will maximize the area of a triangle (given that $a$ and $b$ are constants)?

Implied domain of $\theta \in (0, \pi)$

$\theta = \frac{\pi}{2}$

12. You operate a tour service that offers the following rates for the tours: $200 per person if the minimum number of people book the tour (50 people is the minimum) and for each person past 50, up to a maximum of 80 people, the cost per person is decreased by $2.

It costs you $6000 to operate the tour plus $32 per person.

a) Write a function that represents cost, $C(x)$.
b) Write a function that represents revenue, $R(x)$.
c) Given that profit can be represented as $P(x) = R(x) - C(x)$, write a function that represents profit and state the domain for the function.
d) Find the number of people that maximizes your profit. What is the maximum profit?

a) $C(x) = 6000 + 32x$
b) $R(x) = (200 - 2(x - 50))x$
c) $P(x) = (200 - 2(x - 50))x - 6000 - 32x$ for $x \in [50, 80]$
d) 80 people, for a maximum profit of $2,640
3.7: Graphing with Derivatives

OBJECTIVE

Sketch the graph of a function using information from its first and/or second derivatives.
Sketch the graph of a first and/or second derivative from the graph of a function.

All last year, we concerned ourselves with sketching graphs based on traits of a function. We tended to look at the one key aspect of the derivative – that is finding extremes – as it applied to a function. Toward the end of the year, we looked at the first and second derivatives as traits of the function. They gave a much wider range of information than specific details.

Remember:

Critical values representing extremes of a function occur when
i. \( f'(x) = 0 \)
ii. \( f'(x) \) does not exist
or
iii. at endpoints of an arbitrary domain.

If \( f'(x) > 0 \), then \( f(x) \) is increasing.
If \( f'(x) < 0 \), then \( f(x) \) is decreasing.

Critical values representing a Point of Inflection (POI of a function occur when
i. \( f''(x) = 0 \)
or
ii. \( f''(x) \) does not exist

If \( f''(x) > 0 \), then \( f(x) \) is concave up.
If \( f''(x) < 0 \), then \( f(x) \) is concave down.
Ex 1 Find the sign patterns of $y$, $y'$, and $y''$ and sketch $y = xe^{2x}$

Zeros: \[ xe^{2x} = 0 \rightarrow x = 0 \]
\[ x = 0 \rightarrow y = 0 \]

\[ y' \]
\[ x \]
\[ - \hspace{1cm} 0 \hspace{1cm} + \]

Extremes: \[ \frac{dy}{dx} = xe^{2x}(2) + e^{2x}(1) \]
\[ xe^{2x}(2) + e^{2x}(1) = 0 \]
\[ e^{2x}(2x + 1) = 0 \]
\[ x = -\frac{1}{2} \]
\[ y = -.184 \]

\[ y'' \]
\[ x \]
\[ - \hspace{1cm} 0 \hspace{1cm} + \]

POI \[ \frac{d^2y}{dx^2} = e^{2x}(2) + (2x + 1)(e^{2x}(2)) \]
\[ = e^{2x}(4x + 4) = 0 \]
\[ x = -1 \rightarrow y = -.135 \]

\[ y'' \]
\[ x \]
\[ - \hspace{1cm} 0 \hspace{1cm} + \]

Putting together the points, increasing/decreasing and concavity that can be determined from these sign patterns, the graph will look something like this:

![Graph of y = xe^{2x}](image-url)
Ex 2 Sketch the function described as follows: decreasing from \((-\infty, -3) \cup (4, 6)\), increasing from \((-3, 4) \cup (6, \infty)\), concave down from \((-\infty, -4)\), concave up from \((-4, 4) \cup (4, \infty)\).

Note that this is only one possible answer. Since no y-values are given, the points could be at any height.

The trait and sign pattern information could be given in table form rather than in a description.
Ex 3 Sketch the graph of the function whose traits are given below.

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; -5 )</td>
<td>Positive</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>( x = -5 )</td>
<td>0</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>(-5 &lt; x &lt; 3 )</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>( x = 3 )</td>
<td>-5</td>
<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>( 3 &lt; x &lt; 9 )</td>
<td>Negative</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>( x = 9 )</td>
<td>0</td>
<td>Positive</td>
<td>0</td>
</tr>
<tr>
<td>( 9 &lt; x )</td>
<td>Positive</td>
<td>Positive</td>
<td>Negative</td>
</tr>
</tbody>
</table>

Notice that the point of inflection just happens to occur at a zero of the function. It is very possible that the traits can occur at the same point; you can conceive of functions that have maximums that are also points of inflection, minimums that are also zeroes, etc.
The derivatives also have graphs and we can discern what they look like from the information gleaned from the original graph.

Ex 4 Sketch a possible graph of the first and second derivative of the function shown below.

Notice we have critical values at $x = -3, -1, \text{ and } 1$. These should be zeroes on the graph of the derivative. We can also see where the graph is increasing, and where it is decreasing; these regions correspond to where the graph of the derivative should be positive and negative, respectively.
Notice that the points of inflection on the graph of $y$ are maximums or minimums on the graph of the derivative. Those points will also be zeroes on the second derivative.
Ex 5 Sketch the possible graphs of a function and its first derivative given the graph of the second derivative below.

Looking at this graph, we can see a zero at \( x = -1 \), and the graph is positive for \( x < -1 \), and negative for \( x > -1 \). This should correspond to the graph of the first derivative increasing, then decreasing, with a maximum at \( x = -1 \).
Notice that this looks like a parabola. Since its derivative was a line this should make sense. However, we don’t actually know the height of the maximum, nor do we actually know where the zeroes of the derivative are, or even if there are any. This is a sketch that is one of many possible functions that could have had the previous graph as its derivative. If you remember the “+C” from integration, we could have an infinite number of graphs that would match – each with a different $C$ value.

Again, the zeroes of the previous graph are extremes of this graph, and the zero on the initial graph is a point of inflection on this graph.
3.8 Homework

Sketch these graphs using the sign patterns of the derivatives.

1. \( y = 3x^4 - 15x^2 + 7 \)

2. \( y = x^3 + 5x^2 + 3x - 4 \)

3. \( y = \frac{x+1}{x^2 - 2x - 3} \)

4. \( y = \frac{x^2 - 1}{x^2 + x - 6} \)
5. \[ y = 5x^{\frac{2}{3}} - x^{\frac{5}{3}} \]

6. \[ y = (x^2)\sqrt{4-x} \]

7. \[ y = x^2e^{-x} \]
For problems 8 through 10, sketch the graph of the first and second derivatives for the function shown in the graphs.

8.
9.
10.
Sketch the possible graph of a function that satisfies the conditions indicated below

11. Increasing from $(-\infty, 5) \cup (7, 10)$, decreasing from $(5, 7) \cup (10, \infty)$

12. Increasing from $(-\infty, -3) \cup (5, \infty)$, decreasing from $(-3, 5)$, concave up from $(-\infty, -2) \cup (2, \infty)$, concave down from $(-2, 2)$

13. Decreasing from $(-\infty, -5) \cup (5, \infty)$, increasing from $(-5, 5)$, concave down from $(-\infty, -7) \cup (-3, 3) \cup (7, \infty)$, concave up from $(-7, -3) \cup (3, 7)$
14. Increasing and concave up from (2,4), decreasing and concave down from (4,7), increasing and concave up from (7,10), with a domain of [2,10).

For problems 15 through 20 sketch the possible graph of a function that has the traits shown.

15.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; 2$</td>
<td>Positive</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>$x = 2$</td>
<td>0</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>$2 &lt; x &lt; 3$</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>$x = 3$</td>
<td>$-5$</td>
<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>$3 &lt; x &lt; 5$</td>
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<td>Positive</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
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<td>Positive</td>
<td>Negative</td>
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</tr>
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<td>Negative</td>
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<td>Positive</td>
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16. 

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<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
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<td>DNE</td>
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<td>Positive</td>
</tr>
<tr>
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</tr>
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<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
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<td>3</td>
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<td>DNE</td>
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<tr>
<td>$9 &lt; x$</td>
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<td>Negative</td>
<td>Positive</td>
</tr>
</tbody>
</table>

17. Absolute minimum at 1, absolute maximum at 3, local minima at 4 and 7, local maximum at 6

18. Absolute minimum at 2, absolute maximum at 7, local minima at 4 and 6, local maxima at 3 and 5
19. Absolute maximum at 1, absolute minimum at 7, no local maxima

20. Absolute minimum at 1, relative minimum at 7, absolute maximum at 4, no local maxima.
Answers: 3.7 Homework

1. \[ y = 3x^4 - 15x^2 + 7 \]

2. \[ y = x^3 + 5x^2 + 3x - 4 \]

3. \[ y = \frac{x+1}{x^2 - 2x - 3} \]
4. \[ y = \frac{x^2 - 1}{x^2 + x - 6} \]

5. \[ y = 5x^{2/3} - x^{5/3} \]

6. \[ y = (x^2)\sqrt{4-x} \]
7. \( y = x^2 e^{-x} \)

8. 

1\(^{\text{st}}\) Derivative

2\(^{\text{nd}}\) Derivative
9.

1\textsuperscript{st} Derivative

2\textsuperscript{nd} Derivative
10.

1st Derivative

2nd Derivative
11. Increasing from \((-\infty, 5) \cup (7, 10)\), decreasing from \((5, 7) \cup (10, \infty)\)

12. Increasing from \((-\infty, -3) \cup (5, \infty)\), decreasing from \((-3, 5)\), concave up from \((-\infty, -2) \cup (2, \infty)\), concave down from \((-2, 2)\)

13. Decreasing from \((-\infty, -5) \cup (5, \infty)\), increasing from \((-5, 5)\), concave down from \((-\infty, -7) \cup (-3, 3) \cup (7, \infty)\), concave up from \((-7, -3) \cup (3, 7)\)
14. Increasing and concave up from (2,4), decreasing and concave down from (4,7), increasing and concave up from (7,10), with a domain of [2,10).

15.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
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<td>Positive</td>
</tr>
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</tr>
<tr>
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<td>Negative</td>
<td>Positive</td>
</tr>
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<td>Positive</td>
<td>Positive</td>
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<td>0</td>
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<td>Negative</td>
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<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>$x = 9$</td>
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<td>Negative</td>
<td>0</td>
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<tr>
<td>$x = 10$</td>
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<td>Positive</td>
</tr>
<tr>
<td>$10 &lt; x$</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
</tbody>
</table>
16.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -1$</td>
<td>Positive</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>$x = -1$</td>
<td>3</td>
<td>DNE</td>
<td>DNE</td>
</tr>
<tr>
<td>$-1 &lt; x &lt; 4$</td>
<td>Positive</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>$x = 4$</td>
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<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>$4 &lt; x &lt; 9$</td>
<td>Positive</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>$x = 9$</td>
<td>3</td>
<td>DNE</td>
<td>DNE</td>
</tr>
<tr>
<td>$9 &lt; x$</td>
<td>Positive</td>
<td>Negative</td>
<td>Positive</td>
</tr>
</tbody>
</table>

17. Absolute minimum at 1, absolute maximum at 3, local minima at 4 and 7, local maximum at 6

18. Absolute minimum at 2, absolute maximum at 7, local minima at 4 and 6, local maxima at 3 and 5
19. Absolute maximum at 1, absolute minimum at 7, no local maxima

20. Absolute minimum at 1, relative minimum at 7, absolute maximum at 4, no local maxima.
3.8: Graphical Analysis with Derivatives

In the last section, we looked at graphing functions and derivatives, but now we will reverse that process.

**OBJECTIVES**

Interpret information in the graph of a derivative in terms of the graph of the “original” function.

As we noted in the last example of the last section, there is a layering and parallelism between the function, its derivative and its second derivative. The zeros and signs of one tell us about increasing, decreasing, and extremes or the concavity and POIs of another. That interconnectedness can be summarized thus:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>-</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$y$ value positive</th>
<th>$y$ value negative</th>
<th>Interval of increasing</th>
<th>Interval of decreasing</th>
<th>Concave up</th>
<th>POI</th>
<th>Concave down</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$y'$ value positive</td>
<td>$y'$ value negative</td>
<td>$y'$ value positive</td>
<td>$y'$ value negative</td>
<td>Interval of increasing</td>
<td>max. or min.</td>
<td>POI</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>$y''$ value positive</td>
<td>$y''$ value negative</td>
<td>$y''$ value positive</td>
<td>$y''$ value negative</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The relationship presented here is upward. The zeros of a function refer to the extremes of the one above it and to the POIs of the one two levels up. If we had a function defined as $\int_0^x F(t)dt$ it would be a level above $F(x)$ and it would relate to $F'(x)$ in the same way that $F(x)$ related to $F''(x)$. We will explore this idea further in a later chapter.
Ex 1 If the curve below is \( y = f'(x) \), what are (a) the critical values for the maximums and minimums, (b) intervals of increasing and decreasing, (c) the POI, and (d) the intervals of concavity of \( y = f(x) \)?

What we have here is a two dimensional representation of the sign pattern of the first derivative.

\[
\begin{array}{cccccccc}
\frac{dy}{dx} & -9 & + & 0 & - & 0 & - & 0 & + \\
x & -9 & -3 & 0 & 10 \\
\end{array}
\]

(a) \(-9\) and 10 must be the critical values of the minimums, because they are zeros of the derivative and the sign of the derivative changes from \(-\) to \(+\) (the 1st Derivative Test). \(-3\) is the critical value of the maximum. 0 is a POI because the sign does not change on either side of it.

(b) \( f(x) \) is increasing on \( x \in (-9, -3) \) and \( x \in (10, \infty) \), and it is decreasing on \( x \in (-\infty, -9) \cup (-3, 10) \).

The intervals of increasing and decreasing on \( f' \) are the intervals of concavity on \( f \). So signs of the slopes of \( f' \) make up the second derivative sign pattern:

305
\[
\frac{d^2y}{dx^2} \leftarrow + 0 - 0 + 0 - 0 + \rightarrow
\]

(c) \(-7, -2, 0, 7\) are Points of Inflection, because that is where the derivative of the derivative equals 0 and the second derivative signs change.

(d) \(f(x)\) is concave up on \(x \in (-\infty, -7) \cup (-2, 0) \cup (7, \infty)\), and it is concave down on \(x \in (-7, -3) \cup (0, 7)\).

Ex 2 Given the same graph of \(y = f'(x)\) in Ex 1, if \(f(0) = 0\), sketch a likely curve for \(f(x)\) on \(x \in [-2, 2]\).

We know that on \(x \in [-2, 2]\) \(f'(x)\) is negative so \(f(x)\) is decreasing on that interval. Since \(f(x)\) is decreasing and \(f(0)\) is the zero, the curve must be above the \(x\)-axis on \(x \in [-2, 0]\) and below the \(x\)-axis on \(x \in (0, 2]\).

On \(x \in (-2, 0)\), the slope of \(f'\) (which is \(f''\)) is positive, so \(f(x)\) is concave up. Similarly, on \(x \in (0, 2)\), \(f''\) is negative since \(f'\) is decreasing, so \(f(x)\) is concave down. So \((0, 0)\) is not only a zero, it is also a Point of Inflection. Putting it all together, this is a likely sketch:

Note that there are not markings for scale on the \(y\)-axis. This is because we cannot know, from the given information, the \(y\)-values of the endpoints of the given domain.
Ex 3 If the following graph is the velocity of a particle in rectilinear motion, what can be deduced about the acceleration and the distance?

The particle is accelerating until \( t = -6 \), it decelerates from \( t = -6 \) to \( t = 6 \), and then it accelerates again. The distance is a relative maximum at \( t = -3 \) and a relative minimum at \( t = -9 \) and \( t = 10 \), but we do not know what the distances from the origin are.
3.8 Homework

Each curve in problems 1 through 6 is \( y = f'(x) \); show the sign patterns of the first and second derivatives. Then find, on the interval \( x \in [-3, 3] \),

(a) the critical values for the maximums and minimums for \( f \),
(b) the \( x \) values for the points of inflection for \( f \),
(c) intervals of increasing and decreasing for \( f \),
(d) the intervals of concavity of \( y = f(x) \), and
(e) sketch a possible curve for \( f(x) \) on \( x \in [-3, 3] \) with \( y \)-intercept (0, 0).

Note that you may need to make decimal approximations for some of the critical values and/or interval endpoints.

1.

2.
Sketch the possible graph of a function whose derivative is shown below.

7.
8. Let \( h \) be a continuous function with \( h(2) = 5 \). The graph of the piecewise linear function below, \( h' \), is shown below on the domain of \(-3 \leq x \leq 7\).

![Graph of h']

a) Find the \( x \)-coordinates of all points of inflection on the graph of \( y = h(x) \) for the interval \(-3 < x < 7\). Justify your answer.

b) Find the absolute maximum value of \( h \) on the interval \(-3 \leq x \leq 7\). Justify your answer.

c) Find the average rate of change of \( h \) on the interval \(-3 \leq x \leq 7\).

d) Find the average rate of change of \( h' \) on the interval \(-3 \leq x \leq 7\). Does the Mean Value Theorem applied on this interval guarantee a value of \( c \), for \(-3 < c < 7\), such that \( h''(c) \) is equal to this average rate of change? Explain why or why not.

AP Handout: BC2003#4, AB2000#3, AB1996 # 1
Answers: 3.8 Homework

1.

(a) CV: Max at \( x = -3.2, 1.6 \)
Min at \( x = -1.6, 3.2 \)
(b) \( x = -2.4, -1, 0, 1, 2.4 \)
(c) Inc: \( x \in (-1.6, 0) \cup (0, 1.6) \)
Dec: \( x \in (-3, -1.6) \cup (1.6, 3) \)
(d) Up: \( x \in (-2.4, -1) \cup (0, 1) \cup (2.4, 3) \)
Down: \( x \in (-3, -2.4) \cup (-1, 0) \cup (1, 2.4) \)

2.

(a) CV: Max at \( x = 0 \)
Min at \( x = -2, 2 \)
(b) \( x = -1.2, 1.2 \)
(c) Inc: \( x \in (-2, 0) \cup (2, 3) \)
Dec: \( x \in (-3, -2) \cup (0, 2) \)
(d) Up: \( x \in (-3, -1.2) \cup (1.2, 3) \)
Down: \( x \in (-1.2, 1.2) \)
(a) CV: Max at $x = -2.5$
    Min at $x = 2.5$
(b) $x = -1.5, 0, 1.5$
(c) Inc: $x \in (-3, -2.5) \cup (2.5, 3)$
   Dec: $x \in (-2.5, 0) \cup (0, 2.5)$
(d) Up: $x \in (-1.5, 0) \cup (1.5, 3)$
   Down: $x \in (-3, -1.5) \cup (0,1.5)$
(e)
5.

(a) CV: Max at $x = -2, 2$
(b) $x = -1, 1$
(c) Inc: $x \in (-3, -2) \cup (0, 2)$
    Dec: $x \in (-2, 0) \cup (2, 3)$
(d) Up: $x \in (-1, 1)$
    Down: $x \in (-3, -1) \cup (1, 3)$

6.

(a) CV: Max at $x = -2$
(b) $x = 0$
(c) Inc: $x \in (-3, -2) \cup (2, 3)$
    Dec: $x \in (-2, 2)$
(d) Up: $x \in (0, 3)$
    Down: $x \in (-3, 0)$

(e)
9. Let $h$ be a continuous function with $h(2) = 5$. The graph of the piecewise linear function below, $h'$, is shown below on the domain of $-3 \leq x \leq 7$. 
a) Find the $x$-coordinates of all points of inflection on the graph of $y = h(x)$ for the interval $-3 < x < 7$. Justify your answer.

POIs occur where $h'' = 0$ or does not exist, and switches signs. This occurs at $x = 1$ and $4$; $h''$ does not exist because $h'$ is not differentiable for those $x$. The sign of $h''$ switches because the slopes of $h'$ switch from positive to negative or negative to positive.

b) Find the absolute maximum value of $h$ on the interval $-3 \leq x \leq 7$. Justify your answer.

Maxima occur at $x = -3, 2, \text{ and } 7$. $h(-3) = 5 + \int_{-2}^{3} h(x) dx = 3.5$, $h(2) = 5$, and $h(7) = 5 + \int_{2}^{7} h(x) dx = 3.25$. Therefore, the absolute maximum is at $x = 2$.

c) Find the average rate of change of $h$ on the interval $-3 \leq x \leq 7$.

Average rate of change of $h$ would be the slope of $h$ on that interval. This would be $\frac{h(7) - h(-3)}{7 - (-3)} = -\frac{1}{40}$

d) Find the average rate of change of $h'$ on the interval $-3 \leq x \leq 7$. Does the Mean Value Theorem applied on this interval guarantee a value of $c$, for $-3 < c < 7$, such that $h''(c)$ is equal to this average rate of change? Explain why or why not.

$\frac{0.5 - (-2)}{7 - (-3)} = \frac{1}{4}$; the Mean Value Theorem does not guarantee this value for $c$ because $h'$ is not differentiable throughout this interval.

AP Handout: BC2003#4, AB2000#3, AB1996 # 1
Chapter 3 Test

1) If the position of a particle is given by \( x(t) = \ln(t^2 + 9) \), find \( v(t) \), \( a(t) \), and find when the particle is stopped. When the particle is stopped, what is the position and acceleration?

2) For the function \( V = \frac{1}{3} \pi r^2 h + 2\pi r^3 \)
   a) find the rate of change of \( V \) with respect to \( r \), assuming \( h \) is constant.
   b) find the rate of change of \( V \) with respect to \( h \), assuming \( r \) is constant.
   c) find the rate of change of \( V \) with respect to \( t \), assuming \( r \) and \( h \) are both variables.
3) Two cars are leaving an intersection; one headed north, the other headed east. The northbound car is traveling at 35 miles per hour, while the eastbound car is traveling at 45 miles per hour. Find the rate at which the direct distance is increasing when the eastbound car is 0.6 miles from the intersection and the northbound car is 0.8 miles from the intersection.

4) You spill some milk on a tablecloth, and you notice that the stain is elliptical and that the major axis (x) is always twice the minor axis (y). Given that the area of an ellipse is \( A = \frac{\pi}{4} xy \), find how fast the area of the stain is increasing when \( x = 6 \text{ mm} \), \( y = 3 \text{ mm} \), and the minor axis is increasing at 0.2 mm per second.
5) Find \( \frac{dy}{dx} \) for \( e^{y^2} + 5xy = \tan(y + 1) + \ln(x + 1) \)

6) Given the function \( x^2 - y^2 = 16 \)
   
a) Use implicit differentiation to show that \( \frac{dy}{dx} = \frac{x}{y} \)
   
b) Use implicit differentiation on \( \frac{dy}{dx} \) to show that \( \frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3} \)
7) Given the position function \( x(t) = 2\tan^{-1}(t) \)
   a) Find the velocity and the acceleration functions for this position function.

b) Find the position, velocity, and acceleration at \( t = -1 \)

c) In which direction is the particle moving at that time?

d) Is the particle speeding up or slowing down at that time?

8) You are filling a conical glass with a total height of 5 inches and a radius of 5 inches at the top. If you pour water into the glass at a rate of 3 cubic inches per second, how fast is the height increasing when the radius of the liquid in the glass is 3 inches.
9) Your friend has an ice cream cone that is melting. He notices that the ice cream is dripping from the bottom of the cone at a rate of 0.4 cm³/sec. He also notices that the level of the ice cream in the cone is at a height of 4 cm, and that the radius of the cone at that point is equal to the height. He poses the following solution to how fast the height of the ice cream is changing:

\[ V_{\text{cone}} = \frac{1}{3} \pi r^2 h \quad r = h \]

\[ V_{\text{cone}} = \frac{1}{3} \pi h^3 \quad h = 4 \]

so \[ V_{\text{cone}} = \frac{1}{3} \pi (4)^3 \]

\[ V_{\text{cone}} = \frac{64\pi}{3} \]

\[ \frac{dV}{dt} = 0 \]

He says, “Obviously, my ice cream cone isn’t really melting.”

Your other friend says, “No, no, no – you have that all wrong. This is what you should do:”

\[ V_{\text{cone}} = \frac{1}{3} \pi r^2 h \quad r = h \]

\[ V_{\text{cone}} = \frac{1}{3} \pi h^3 \]

\[ \frac{dV}{dt} = \pi h^2 \frac{dh}{dt} \quad h = 4, \frac{dV}{dt} = 0.4 \]

\[ 0.4 = 16\pi \frac{dh}{dt} \]

\[ \frac{dh}{dt} = \frac{0.4}{16\pi} = 0.00796 \text{ cm/sec} \]

“So the level of ice cream is rising at 0.00796 cm/sec.”

Is either of your friends’ solution right? Correct the mistakes in their work, if there are any, and explain what went wrong.
Find the critical values and extremes for each of the following functions in problems 10 and 11:

10) \( y = (x+1)e^{-x} \)

11) \( y = 2x - 2\tan x \) on \( x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \)

12) Given the table of values for \( f(x) \) and its first and second derivatives, find the points at which the curve is at a maximum or minimum. Explain why this is the case.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>6</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>18</td>
<td>-27</td>
<td>30</td>
<td>42</td>
<td>67</td>
<td>-11</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>19</td>
<td>0</td>
<td>-22</td>
<td>0</td>
<td>0</td>
<td>83</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>18</td>
<td>-28</td>
<td>23</td>
<td>19</td>
<td>-12</td>
<td>23</td>
</tr>
</tbody>
</table>
For each of the following, list the point and describe what is happening at each point on \( f(x) \):

\[
\begin{align*}
\text{a)} & \quad x = 2 & \quad x = 3 & \quad x = -1 \\
& \quad f(2) = -5 & \quad f(3) = 9 & \quad f(-1) = 0 \\
& \quad f'(2) = -13 & \quad f'(3) = 0 & \quad f'(-1) = 0 \\
& \quad f''(2) = 27 & \quad f''(3) = -29 & \quad f''(-1) = 18 \\
\end{align*}
\]

Below is a chart showing the volume of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

<table>
<thead>
<tr>
<th>( t ) (in minutes)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V(t) ) (in m(^3))</td>
<td>12</td>
<td>14</td>
<td>8</td>
<td>12</td>
<td>19</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>

a) Find the approximate value of \( V'(2) \). Show your work and use units.

b) Find the approximate value of \( V'(4) \). Show your work and use units.

c) Find the approximate value of \( V'(10) \). Show your work and use units.

d) Given that \( V(t) \) is continuous and differentiable on the interval \( 0 < t < 12 \). Must there be a value of \( c \) in that interval such that \( V'(c) \) equals \( \frac{V(12) - V(0)}{12 - 0} \). Explain why or why not.
15) Given the sign patterns below, find the critical values and determine which ones are associated with maximums and which are associated with minimums. Explain why. Assume \( y \) is a continuous function.

a)

\[
\frac{dy}{dx} \begin{array}{ccccc}
- & 0 & + & \text{D.N.E.} & + \\
\hline 
x & -3 & 2 & 7 
\end{array}
\]

Maximums: _______ Minimums: _______

b)

\[
\frac{dy}{dx} \begin{array}{ccccc}
- & 0 & + & \text{D.N.E.} & - \\
\hline 
x & -2 & 0 & 3 
\end{array}
\]

Maximums: _______ Minimums: _______
16. Sketch a graph with the following traits:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; 2$</td>
<td>Negative</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>$x = 2$</td>
<td>0</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>$2 &lt; x &lt; 3$</td>
<td>Positive</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>$x = 3$</td>
<td>5</td>
<td>0</td>
<td>Negative</td>
</tr>
<tr>
<td>$3 &lt; x &lt; 5$</td>
<td>Positive</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>$x = 5$</td>
<td>0</td>
<td>Negative</td>
<td>0</td>
</tr>
<tr>
<td>$5 &lt; x &lt; 7$</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>$x = 7$</td>
<td>$-9$</td>
<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>$7 &lt; x &lt; 9$</td>
<td>Negative</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>$x = 9$</td>
<td>$-1$</td>
<td>Positive</td>
<td>0</td>
</tr>
<tr>
<td>$9 &lt; x &lt; 10$</td>
<td>Negative</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>$x = 10$</td>
<td>0</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>$10 &lt; x$</td>
<td>Positive</td>
<td>Positive</td>
<td>Negative</td>
</tr>
</tbody>
</table>

17. It costs you $c$ dollars each to manufacture and distribute video games. If you sell the games at $x$ dollars each, you will sell $n$ games, where

$$n = \frac{x - c}{a} + b(100 - x),$$

where $a$ and $b$ are positive constants, and $a < b$.

a) Write an equation that represents profit for selling $n$ video games.

b) What price, $x$, will maximize profit?
18. Given the graph of $h'$ illustrated below, find each of the following for $h$.

a) Find all critical values for $h$.

b) Find which critical values are associated with maxima or minima. Justify your answer.

c) Find all points of inflection for $h$. Explain why they are points of inflection.
19. In a certain community, an epidemic spreads in such a way that the percentage $P$ of the population that is infected after $t$ months is modeled by

$$P(t) = \frac{kt^2}{(C + t^2)^2},$$

where $C$ and $k$ are constants. Find $t$, in terms of $C$, such that $P$ is least.
Answers: Chapter 3 Test

1) If the position of a particle is given by \( x(t) = \ln(t^2 + 9) \), find \( v(t) \), \( a(t) \), and find when the particle is stopped. When the particle is stopped, what is the position and acceleration?

\[
\begin{align*}
v(t) &= \frac{2t}{t^2 + 9} \\
a(t) &= \frac{18 - 2t^2}{(t^2 + 9)^2}
\end{align*}
\]

Particle stops at \( t = 0 \); \( x(0) = \ln(9) \), \( a(0) = \frac{2}{9} \)

2) For the function \( V = \frac{1}{3} \pi r^2 h + 2\pi r^3 \)

a) find the rate of change of \( V \) with respect to \( r \), assuming \( h \) is constant.

\[
\frac{dV}{dr} = \frac{2}{3} \pi rh + 6\pi r^2
\]

b) find the rate of change of \( V \) with respect to \( h \), assuming \( r \) is constant.

\[
\frac{dV}{dh} = \frac{1}{3} \pi r^2
\]

c) find the rate of change of \( V \) with respect to \( t \), assuming \( r \) and \( h \) are both variables.

\[
\frac{dV}{dt} = \frac{2}{3} \pi rh \frac{dr}{dt} + \frac{1}{3} \pi r^2 \frac{dh}{dt} + 6\pi r^2 \frac{dr}{dt}
\]

3) Two cars are leaving an intersection; one headed north, the other headed east. The northbound car is traveling at 35 miles per hour, while the eastbound car is traveling at 45 miles per hour. Find the rate at which the direct distance is increasing when the eastbound car is 0.6 miles from the intersection and the northbound car is 0.8 miles from the intersection.

55 miles per hour

4) You spill some milk on a tablecloth, and you notice that the stain is elliptical and that the major axis \((x)\) is always twice the minor axis \((y)\).

Given that the area of an ellipse is \( A = \frac{\pi}{4} \ xy \), find how fast the area of the stain is increasing when \( x = 6 \) mm, \( y = 3 \) mm, and the minor axis is increasing at 0.2 mm per second.

\[
\frac{dA}{dt} = 0.6\pi \ \text{mm}^2/\text{sec}
\]
5) Find $\frac{dy}{dx}$ for $e^{x^2} + 5xy = \tan(y+1) + \ln(x+1)$

\[
\frac{dy}{dx} = \frac{1 - 5xy - 5y}{(x+1)\left(2ye^{x^2} + 5x - \sec^2(y+1)\right)}
\]

6) Given the function $x^2 - y^2 = 16$

a) Use implicit differentiation to show that $\frac{dy}{dx} = \frac{x}{y}$

\[
\frac{d}{dx}[x^2 - y^2 = 16]
\]

\[
2x - 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{x}{y}
\]

b) Use implicit differentiation on $\frac{dy}{dx}$ to show that $\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$

\[
\frac{d}{dx}\left[\frac{dy}{dx} = \frac{x}{y}\right]
\]

\[
\frac{d^2y}{dx^2} = \frac{y - x \frac{dy}{dx}}{y^2}
\]

\[
\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}
\]

7) Given the position function $x(t) = 2\tan^{-1}(t)$

a) Find the velocity and the acceleration functions for this position function.

\[
v(t) = \frac{2}{1+t^2}
\]

\[
a(t) = -4\left(1+t^2\right)^{-2}
\]

b) Find the position, velocity, and acceleration at $t = -1$

\[
x(-1) = -\frac{\pi}{2}
\]

\[
v(-1) = 1
\]

\[
a(-1) = 1
\]
c) In which direction is the particle moving at that time?
   The particle is moving right because the velocity is positive.

d) Is the particle speeding up or slowing down at that time?
   It is speeding up because velocity and acceleration are both positive.

8) You are filling a conical glass with a total height of 5 inches and a radius of 5 inches at the top. If you pour water into the glass at a rate of 3 cubic inches per second, how fast is the height increasing when the radius of the liquid in the glass is 3 inches.

\[ \frac{dh}{dt} = \sqrt{\frac{1}{3\pi}} \text{ cm/sec} \]

9) Your friend has an ice cream cone that is melting. He notices that the ice cream is dripping from the bottom of the cone at a rate of 0.4 cm³/sec. He also notices that the level of the ice cream in the cone is at a height of 4 cm, and that the radius of the cone at that point is equal to the height. He poses the following solution to how fast the height of the ice cream is changing:

\[ V_{cone} = \frac{1}{3} \pi r^2 h \quad r = h \]
\[ V_{cone} = \frac{1}{3} \pi h^3 \quad h = 4 \]

so \[ V_{cone} = \frac{1}{3} \pi (4)^3 \]

\[ V_{cone} = \frac{64\pi}{3} \]

\[ \frac{dV}{dt} = 0 \]

He says, “Obviously, my ice cream cone isn’t really melting.”
Your other friend says, “No, no, no – you have that all wrong. This is what you should do:”

\[ V_{cone} = \frac{1}{3} \pi r^2 h \quad r = h \]

\[ V_{cone} = \frac{1}{3} \pi h^3 \]

\[ \frac{dV}{dt} = \pi h^2 \frac{dh}{dt} \]

\[ 0.4 = 16\pi \frac{dh}{dt} \]

\[ \frac{dh}{dt} = \frac{0.4}{16\pi} = 0.00796 \text{ cm/sec} \]

“So the level of ice cream is rising at 0.00796 cm/sec.”

Is either of your friends’ solution right? Correct the mistakes in their work, if there are any, and explain what went wrong.

Both the solutions are wrong. The first friend assumed that \( h \) was constant (which it was not) leading to a derivative = 0. The second friend missed the fact that the ice cream dripping out was a decreasing volume, making \( \frac{dV}{dt} \) negative.

Find the critical values and extremes for each of the following functions in problems 10 and 11:

10) \( y = (x+1)e^{-x} \)

Critical Values: \( x = 0 \)

Extreme Values: \( y = 1 \)

11) \( y = 2x - 2 \tan x \) on \( x \in \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \)

Critical Values: \( x = -\frac{\pi}{4}, 0, \frac{\pi}{4} \)

Extreme Values: \( x = -\frac{\pi}{2} + 2, 0, \frac{\pi}{2} - 2 \)
12) Given the table of values for \( f(x) \) and its first and second derivatives, find the points at which the curve is at a maximum or minimum. Explain why this is the case.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>6</th>
<th>9</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>18</td>
<td>-27</td>
<td>30</td>
<td>42</td>
<td>67</td>
<td>-11</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>19</td>
<td>0</td>
<td>-22</td>
<td>0</td>
<td>0</td>
<td>83</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>18</td>
<td>-28</td>
<td>23</td>
<td>19</td>
<td>-12</td>
<td>23</td>
</tr>
</tbody>
</table>

(–2, –27) is a maximum because the first derivative = 0 and the curve is concave down. (6, 42) is a minimum because the first derivative = 0 and the curve is concave up. (9, 67) is a maximum because the first derivative = 0 and the curve is concave down.

13) For each of the following, list the point and describe what is happening at each point on \( f(x) \):

a) \( x = 2 \)
   \( f(2) = -5 \)
   \( f'(2) = -13 \)
   \( f''(2) = 27 \)
   (2, –5) decreasing, concave up

b) \( x = 3 \)
   \( f(3) = 9 \)
   \( f'(3) = 0 \)
   \( f''(3) = -29 \)
   (3, 9) Maximum

(1, 0) Minimum

14) Below is a chart showing the volume of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

<table>
<thead>
<tr>
<th>( t ) (in minutes)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V(t) ) (in m³)</td>
<td>12</td>
<td>14</td>
<td>8</td>
<td>12</td>
<td>19</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>

a) Find the approximate value of \( V'(2) \). Show your work and use units.
   \( V'(2) \approx -1 \text{ m}^3/\text{minute} \)

b) Find the approximate value of \( V'(4) \). Show your work and use units.
   \( V'(4) \approx -\frac{1}{2} \text{ m}^3/\text{minute} \)

c) Find the approximate value of \( V'(10) \). Show your work and use units.
   \( V'(10) \approx -\frac{1}{4} \text{ m}^3/\text{minute} \)

d) Given that \( V(t) \) is continuous and differentiable on the interval \( 0 < t < 12 \). Must there be a value of \( c \) in that interval such that \( V'(c) \) equals \( \frac{V(12) - V(0)}{12 - 0} \). Explain why or why not.
   Yes, because of the Mean Value Theorem.
15) Given the sign patterns below, find the critical values and determine which ones are associated with maximums and which are associated with minimums. Explain why. Assume y is a continuous function.

a) 

\[
\frac{dy}{dx} \quad - \quad 0 \quad + \quad \text{D.N.E.} \quad + \quad 0 \quad -
\]

\[
x \quad -3 \quad 2 \quad 7
\]

Maximums: \( x = 7 \)  
\( \frac{dy}{dx} = 0 \) and switches positive to negative.

Minimums: \( x = -3 \)  
\( \frac{dy}{dx} = 0 \) and switches negative to positive.

b) 

\[
\frac{dy}{dx} \quad - \quad 0 \quad + \quad \text{D.N.E.} \quad - \quad 0 \quad +
\]

\[
x \quad -2 \quad 0 \quad 3
\]

Maximums: \( x = 0 \)  
\( \frac{dy}{dx} = \text{D.N.E.} \) and switches positive to negative.

Minimums: \( x = -2 \) and \( 3 \)  
\( \frac{dy}{dx} = 0 \) and switches negative to positive.
16. Sketch a graph with the following traits:

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>f'(x)</th>
<th>f''(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x &lt; 2</td>
<td>Negative</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>x = 2</td>
<td>0</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>2 &lt; x &lt; 3</td>
<td>Positive</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>x = 3</td>
<td>5</td>
<td>0</td>
<td>Negative</td>
</tr>
<tr>
<td>3 &lt; x &lt; 5</td>
<td>Positive</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>x = 5</td>
<td>0</td>
<td>Negative</td>
<td>0</td>
</tr>
<tr>
<td>5 &lt; x &lt; 7</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>x = 7</td>
<td>−9</td>
<td>0</td>
<td>Positive</td>
</tr>
<tr>
<td>7 &lt; x &lt; 9</td>
<td>Negative</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>x = 9</td>
<td>−1</td>
<td>Positive</td>
<td>0</td>
</tr>
<tr>
<td>9 &lt; x &lt; 10</td>
<td>Negative</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>x = 10</td>
<td>0</td>
<td>Positive</td>
<td>Negative</td>
</tr>
<tr>
<td>10 &lt; x</td>
<td>Positive</td>
<td>Positive</td>
<td>Negative</td>
</tr>
</tbody>
</table>

17. It costs you c dollars each to manufacture and distribute video games. If you sell the games at x dollars each, you will sell n games, where

\[ n = \frac{x-c}{a} + b(100-x) \]

where a and b are positive constants, and \( a < b \).

a) Write an equation that represents profit for selling \( n \) video games.

\[ P(x) = \frac{x(x-c)}{a} + bx(100-x) - cx \]

b) What price, \( x \), will maximize profit?

\[ x = \frac{ac - 100ab + c}{2 - 2ab} \]
18. Given the graph of $h'$ illustrated below, find each of the following for $h$.

a) Find all critical values for $h$.
   \[ x = -4 \]

b) Find which critical values are associated with maxima or minima. Justify your answer.
   It is associated with a minimum: $f'$ switches from negative to positive, which means $f$ switches from decreasing to increasing.

c) Find all points of inflection for $h$. Explain why they are points of inflection.
   \[ x = -2, 0, 2, 3, \] because $f'$ goes increasing to decreasing (for $x = -2, 2$) or decreasing to increasing (for $x = 0, 3$) which means that $f''$ switches signs; therefore $f$ switches concavity at those values of $x$.

19. In a certain community, an epidemic spreads in such a way that the percentage $P$ of the population that is infected after $t$ months is modeled by

\[ P(t) = \frac{kt^2}{(t^2 - C)^2}, \quad t \geq 0 \]

where $C$ and $k$ are positive constant. Find $t$, in terms of $C$ and $k$, such that $P$ is least.

$P$ is at a maximum when $t = \sqrt{C}$
3. __: Optional Topic: Logarithmic Differentiation

With implicit differentiation and the chain rule, we learned some powerful tools for differentiating functions and relations. The product and quotient rules also allowed us to take derivatives of certain functions that would otherwise be impossible to differentiate. Sometimes, however, with very complex functions, it becomes easier to manipulate an equation so that it is easier to take the derivative. This is where logarithmic differentiation comes in.

**OBJECTIVES**

Determine when it is appropriate to use logarithmic differentiation.
Use logarithmic differentiation to take the derivatives of complicated functions.

Before we begin, it would be helpful to look at a few rules that we should remember from algebra and precalculus concerning logarithms.

\[
\begin{align*}
    a^x a^y &= a^{x+y} & \log_a x + \log_a y &= \log_a (xy) \\
    \frac{a^x}{a^y} &= a^{x-y} & \log_a x - \log_a y &= \log_a \frac{x}{y} \\
    (a^x)^y &= a^{xy} & \log_a x^n &= n \log_a x
\end{align*}
\]

Since logarithms are exponents expressed in a different form, all of the above rules are derived from the rules for exponents, and you can see the corresponding exponential rule. Because of our algebraic rules, we can do whatever we want to both sides of an equation. In algebra, we usually used this to solve for a variable. In Calculus, we can use this principle to make many derivative problems significantly easier.
Ex 1  Find the derivative of \( y = (x^2 + 7x - 3)(\sin(x)) \)

What we would traditionally use to take the derivative of this function is the product rule.

\[
\frac{dy}{dx} = \left( x^2 + 7x - 3 \right) (\cos(x)) + (2x + 7)(\sin(x))
\]

Obviously, this is a straightforward problem that can be easily done using the product rule. If, however, I took the natural log of both sides of the equation, I can achieve the same results, and never use the product rule. (Remember, we will almost exclusively use the natural log because it works so well within the framework of Calculus)

\[
\ln(y) = \ln\left( x^2 + 7x - 3 \right) (\sin(x))
\]

\[
\ln(y) = \ln(x^2 + 7x - 3) + \ln(\sin(x)) \quad \text{This is simplifying using log rules.}
\]

\[
\frac{d}{dx}\left[ \ln(y) = \ln(x^2 + 7x - 3) + \ln(\sin(x)) \right]
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{2x + 7}{x^2 + 7x - 3} + \frac{\cos(x)}{\sin(x)}
\]

\[
\frac{dy}{dx} = \left( \frac{2x + 7}{x^2 + 7x - 3} + \frac{\cos(x)}{\sin(x)} \right) (y) \quad \text{Now just substitute } y \text{ back in and simplify.}
\]

\[
\frac{dy}{dx} = \left( \frac{2x + 7}{x^2 + 7x - 3} + \frac{\cos(x)}{\sin(x)} \right) \left( x^2 + 7x - 3 \right) (\sin(x))
\]

\[
\frac{dy}{dx} = (2x + 7)(\sin(x)) + (\cos(x))(x^2 + 7x - 3)
\]

Clearly, we got the same answer that we got from the product rule, but with significantly more effort. Logarithmic differentiation is a tool we can use, but we have to use it judiciously, we don’t want to make problems more difficult than they have to be.

Interestingly enough, logarithmic differentiation can be used to easily derive the product and the quotient rules.
Ex 2 If \( y = (u)(v) \), and \( u \) and \( v \) are both functions of \( x \), find \( \frac{dy}{dx} \).

\[
\ln y = \ln [(u)(v)] \quad \text{Take the log of both sides.}
\]

\[
\ln y = \ln(u) + \ln(v) \quad \text{Apply the log rules.}
\]

\[
\frac{d}{dx} \left[ \ln y = \ln(u) + \ln(v) \right] \quad \text{Take the derivative.}
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx}
\]

\[
\frac{dy}{dx} = \left( \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} \right) y \quad \text{Solve for } \frac{dy}{dx}, \text{ substitute for } y, \text{ and simplify.}
\]

\[
\frac{dy}{dx} = \left( \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} \right) (u)(v)
\]

\[
\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}
\]

Notice that we have just proved the product rule. This is significantly easier than some of the first proofs of the product rule (you can look them up if you are interested, they interesting in that they involve the limit definition of the derivative). The proof for the quotient rule is very similar, and you will be doing it yourself in the homework.

Where logarithmic differentiation is really useful is in functions that are excessively painful to work with (or impossible to take the derivative of any other way) because of multiple operations.

Ex 3 Find \( \frac{dy}{dx} \) for \( y = \frac{(x^2 + 5) \sin(3x^3)}{\tan(5x + 2)} \)

We could take the derivative by applying the chain rule, quotient rule, and product rule, but that would be a time-consuming and tedious process. It’s much easier to take the log of both sides, simplify and then take the derivative.
\[ \ln y = \ln \left( \frac{(x^2 + 5) \sin(3x^3)}{\tan(5x + 2)} \right) \]

\[ \ln y = \ln (x^2 + 5) + \ln (\sin(3x^3)) - \ln (\tan(5x + 2)) \]

\[ \frac{d}{dx} \left[ \ln y = \ln (x^2 + 5) + \ln (\sin(3x^3)) - \ln (\tan(5x + 2)) \right] \]

\[ \frac{1}{y} \frac{dy}{dx} = \frac{2x}{x^2 + 5} + \frac{9x^2 \cos(3x^3)}{\sin(3x^3)} - \frac{5 \sec^2(5x + 2)}{\tan(5x + 2)} \]

\[ \frac{dy}{dx} = \left( \frac{2x}{x^2 + 5} + \frac{9x^2 \cos(3x^3)}{\sin(3x^3)} - \frac{5 \sec^2(5x + 2)}{\tan(5x + 2)} \right) y \]

\[ \frac{dy}{dx} = \left( \frac{2x}{x^2 + 5} + \frac{9x^2 \cos(3x^3)}{\sin(3x^3)} - \frac{5 \sec^2(5x + 2)}{\tan(5x + 2)} \right) \left( \frac{x^2 + 5}{\tan(5x + 2)} \right) \]

Now that may seem long and messy, but try it any other way, and you might end up taking a lot more time, with a lot more algebra and a lot more potential spots to make mistakes.

Ex 4 Find \( f'(\pi) \) for \( f(z) = z^{\cos z} \)

\[ \ln[f(z)] = \ln[z^{\cos z}] \]

\[ \frac{d}{dx} \ln[f(z)] = (\cos z)(\ln z) \]

\[ \frac{f'(z)}{f(z)} = \cos z \frac{z}{(\ln z)(\sin z)} \]

\[ f'(z) = \left( \cos z \frac{z}{(\ln z)(\sin z)} \right) f(z) \]

\[ f'(z) = \left( \cos z \frac{z}{(\ln z)(\sin z)} \right) (z^{\cos z}) \]

\[ f'(\pi) = \left( \cos \pi \frac{\pi}{(\ln \pi)(\sin \pi)} \right) (\pi^{\cos \pi}) = -\frac{1}{\pi^2} \]
You could have also done this problem using the change of base property that you learned in Precalculus and you would get the same answer in about the same number of steps, but you would have to remember how to change the base of an exponential function using an old algebra rule.

Ex 4B Find \( f'(\pi) \) for \( f(z) = z^{\cos z} \)

\[
f(z) = z^{\cos z}
\]
\[
f(z) = e^{\cos z \ln z}
\]

Since \( a^x = e^{x \ln a} \)

\[
f'(z) = e^{\cos z \ln z} \left( \cos z \cdot \frac{1}{z} - \sin z \cdot \ln z \right)
\]
\[
f'(z) = e^{\cos z \ln z} \left( \frac{\cos z}{z} - (\ln z)(\sin z) \right)
\]
\[
f'(z) = z^{\cos z} \left( \frac{\cos z}{z} - (\ln z)(\sin z) \right)
\]
\[
f'(\pi) = \left( \frac{\cos \pi}{\pi} - (\ln \pi)(\sin \pi) \right) (\pi^{\cos \pi}) = -\frac{1}{\pi^2}
\]

Again, there are often more than one way to do a specific problem, and part of what we do as mathematicians is decide on the simplest correct method to solving a problem.

The issue many people have when learning more difficult mathematical concepts is that they try to oversimplify a problem and end up getting it wrong as a result.

As Albert Einstein once said, “Make things as simple as possible but not simpler.”
3. **Homework Set A**

Find the derivatives of the following functions. Use logarithmic differentiation when appropriate.

1. \( y = (2x+1)^4(x^3-3)^5 \)

2. \( z = (y^3-3)e^{(2y+1)} \)

3. \( y = \frac{\sin^2 x \tan^4 x}{(x^2+5)^2} \)

4. \( g(t) = t \ln(t) \)

5. \( y = \ln^x(x) \)

6. \( p(v) = v^e \)
7. Use logarithmic differentiation to prove the quotient rule.

8. For the function, \( f(x) = x^{\ln x} \), find an equation for a tangent line at \( x = e \), and use that to approximate the value for \( f(2.7) \). Find the percent difference between this and the actual value of \( f(2.7) \).

10. Explain why you think we use natural logs rather than other bases for logs in Calculus. (Hint: think back to the derivative rules)
3. **Homework Set B**

1. Use logarithmic differentiation to find \( \frac{dq}{dt} \) if \( q = \frac{e^{4-15 \sin^5 3t}}{(\ln t)^{10}} \)

2. Use logarithmic differentiation to find \( \frac{dy}{dx} \) when

\[
y = e^{150x-19 \ln(\sin x)^{100}} \sqrt{x^2 - 1}
\]

3. Use logarithmic differentiation to find \( \frac{dq}{dt} \) if \( q = \frac{e^{4-15 \csc^5 3t}}{\ln t^{10}} \)
4. Find \( \frac{dy}{dx} \) for the function \( y = \left( e^{17x^4} \right) \left( \sin^7 x \right) \left( 5x - 17 \right)^{12} \left( \cot 5x \right) \)
Answers: 3. __ Homewo...
8. For the function, \( f(x) = x^{\ln x} \), find an equation for a tangent line at \( x = e \), and use that to approximate the value for \( f(2.7) \). Find the percent difference between this and the actual value of \( f(2.7) \).

\[
y - e = 1(x - e) \quad \text{and} \quad f(2.7) \approx 2.7
\]

\[
f(2.7) = 2.681963256... \quad \text{which is a 0.673% difference.}
\]

9. Explain why you think we use natural logs rather than other bases for logs in Calculus. (Hint: think back to the derivative rules)

We use natural logs because the derivative rule is significantly easier for problems involving the natural log, rather than the common (or other) logs.

3.__ Homework Set B

1. Use logarithmic differentiation to find \( \frac{dq}{dt} \) if \( q = \frac{e^{t^4 - 15} \sin^5 3t}{(\ln t)^{10}} \)

\[
\frac{dq}{dt} = \left( \frac{e^{t^4 - 15} \sin^5 3t}{(\ln t)^{10}} \right) \left( 4t^3 + 15 \cot (3t) - \frac{10}{t \ln t} \right)
\]

2. Use logarithmic differentiation to find \( \frac{dy}{dx} \) when

\[
\frac{dy}{dx} = \left( e^{150x - 19} \ln(\sin x)^{100} \sqrt{x^2 - 1} \right) \left( -19 + \frac{\cot x}{\ln(\sin x)} + \frac{x}{x^2 - 1} \right)
\]

3. Use logarithmic differentiation to find \( \frac{dq}{dt} \) if \( q = \frac{e^{t^4 - 15} \csc 5t}{\ln t^{10}} \)

\[
\frac{dq}{dt} = \left( \frac{e^{t^4 - 15} \csc 5t}{\ln t^{10}} \right) \left( 4t^3 - 15 \cot (3t) - \frac{1}{t \ln t} \right)
\]

4. Find \( \frac{dy}{dx} \) for the function \( y = (e^{17x^4})(\sin^7 x)(5x - 17)^{12}(\cot 5x) \)

\[
\frac{dy}{dx} = \left[ (e^{17x^4})(\sin^7 x)(5x - 17)^{12}(\cot 5x) \right] \left( 68x^3 + 7\cot x + \frac{60}{5x - 17} - \frac{5\csc^2 5x}{\cot 5x} \right)
\]