Chapter 0: Reminders from Precalculus

Analytic Geometry and Calculus are closely related subjects. Analytic Geometry is the study of functions and relations as to how their graphs on the Cartesian System relate to the algebra of their equations. Traditionally, Calculus has been the study of functions with a particular interest in tangent lines, maximum/minimum points, and area under the curve. (The Reform Calculus movement places an emphasis on how functions change, rather than the static graph, and consider Calculus to be a study of change and of motion.) Consequently, there is a great deal of overlap between the subjects. The advent of graphing calculators has blurred the distinctions between these fields and made subjects that had previously been strictly Calculus topics easily accessible at the lower level. The point of this course is to thoroughly discuss the subjects of Analytic Geometry that directly pertain to entry-level Calculus and to introduce the concepts and algebraic processes of first semester Calculus. ***Because we are considering these topics on the Cartesian Coordinate System, it is assumed that we are using Real Numbers, unless the directions state otherwise.***

Last year's text was designed to study the various families of functions in light of the main characteristics--or TRAITS-- that the graphs of each family possess. Each chapter a different family and a) review what is known about that family from Algebra 2, b) investigate the analytic traits, c) introduce the Calculus Rule that most applies to that family, d) put it all together in full sketches, and e) take one step beyond. In Chapter 0 of this text, we will review the traits that do not involve the derivative before going into the material that is properly part of Calculus. Part of what will make the Calculus much easier is if we have an arsenal of basic facts that we can bring to any problem. Many of these facts concern the graphs that we learned about individually in Precalculus. We need to synthesize all of this material into a cohesive body of information so that we can make the best use of it in calculus.

0.1 Basic Concepts and Definitions

Before we can start, we have to develop a common vocabulary that will be used throughout this course. Much of it comes directly from Algebra 1 and 2.

Domain-- Defn: "The set of values of the independent variable."
Means: the set of *x*-values that can be substituted into the equation to get a Real *y*-value (i.e., *no zero denominator, no negative under an even radical, and no negatives or zero in a logarithm*).

Range-- Defn: "The set of values of the dependent variable."
Means: the set of *y*-values that can come from the equation.

Relation-- Defn: "A set of ordered pairs."
Means: the equation that creates/defines the pairs.

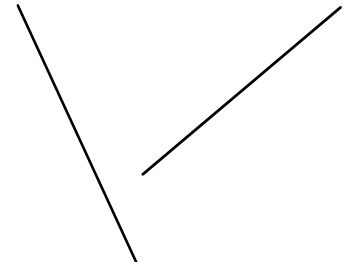
Function-- Defn: "A relation for which there is exactly one value of the dependent variable for each value of the independent variable."
Means: an equation where every *x* gets only one *y*.

Degree-- Defn: "The maximum number of variables that are multiplied together in any one term of the polynomial."
Means: Usually, the highest exponent on a variable.

Families of Functions: Algebraic

I. *Polynomial*s

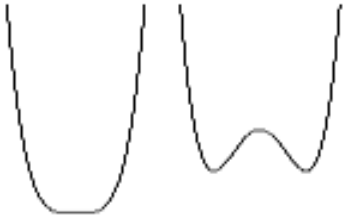
- Defn: "An expression containing no other operations than addition, subtraction, and multiplication performed on the variable."
- Means: any equation of the form $y = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, where *n* is a non-negative integer.
- Most Important Traits: Zeros (*x*-intercepts) and Extreme Points.



Linear polynomials

Quadratic polynomials

Cubic polynomials



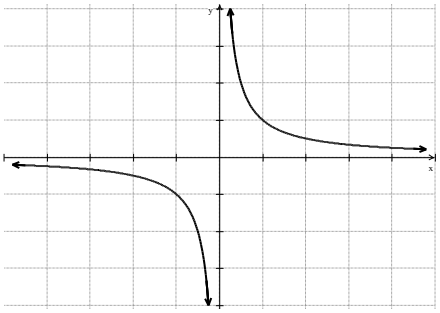
Quartic polynomials

Polynomials are among the most basic of graphs – you can recognize them by the fact that they have no horizontal or vertical asymptotes. They generally look like parabolas or cubics but sometimes with more "turns" or "bumps" (the maxima and mininma.

II. *Rationals*

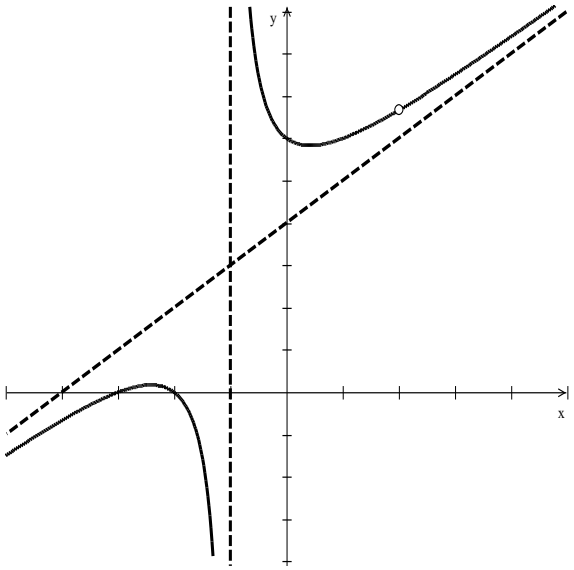
- Defn: "An expression that can be written as the ratio of one polynomial to another."
- Means: an equation with an x in the denominator.
- Most Important Traits: Zeros vs. VAs vs. POEs and End Behavior.

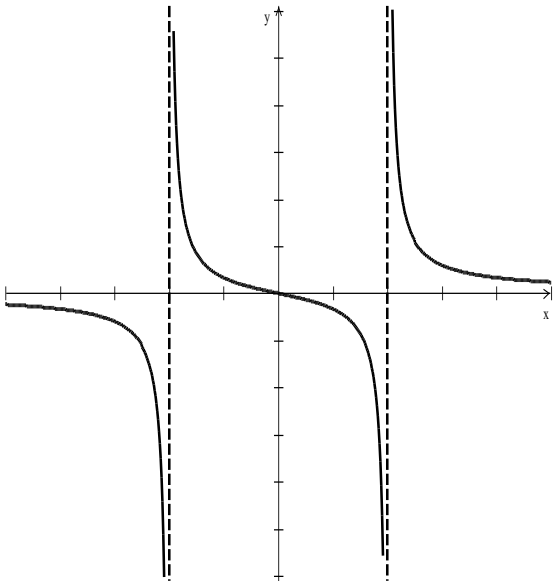
The most basic rational function is $y = \frac{1}{x}$, pictured below.

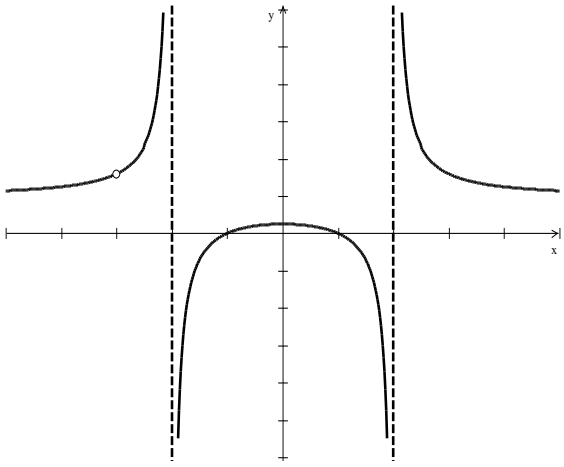


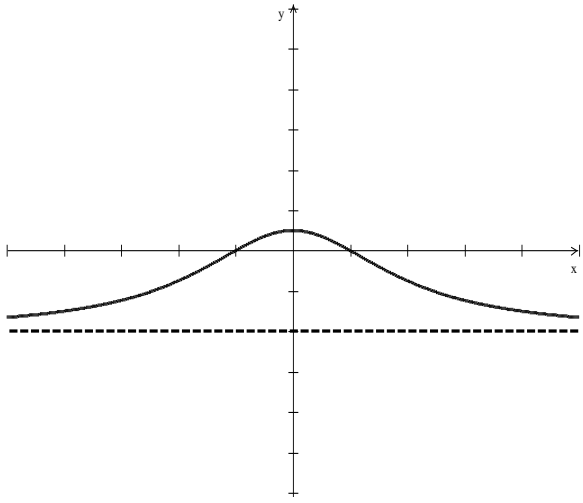
From this graph, you can see the two most common features – the horizontal and the vertical asymptotes. They also generally (though not always) alternate directions around the vertical asymptote. Points of exclusion (POE) also occur on rational function, though not on this particular one.

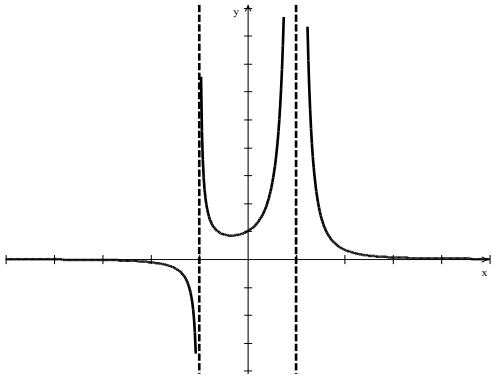
More varied rational functions:









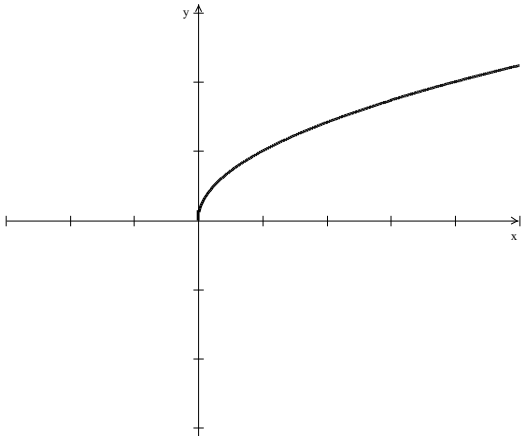


You should be able to see some common features in the VAs, HAs, and POEs. Not all of the graphs have all of these features, but they give you a guide as to what rational functions all generally look like. These are also functions that may have a Slant Asymptote (SA) instead of a HA.

III. *Radicals* (*Irrationals*)

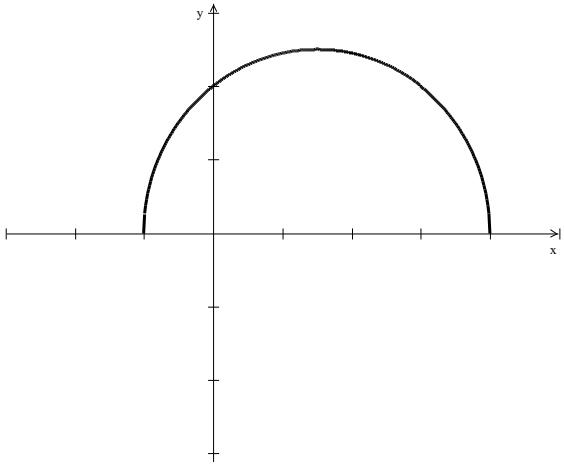
- Defn: "An expression whose general equation contains a root of a variable and possibly addition, subtraction, multiplication, and/or division."
- Means: An equation with an x in a radical.
- Most Important Traits: Domain and Extreme Points.

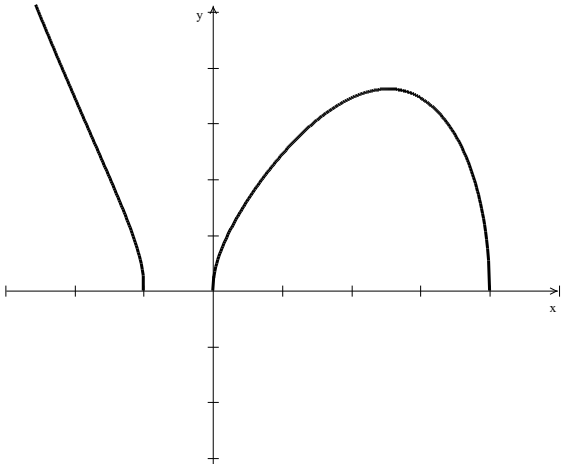
The most basic radical function is $y = \sqrt{x}$, pictured below.

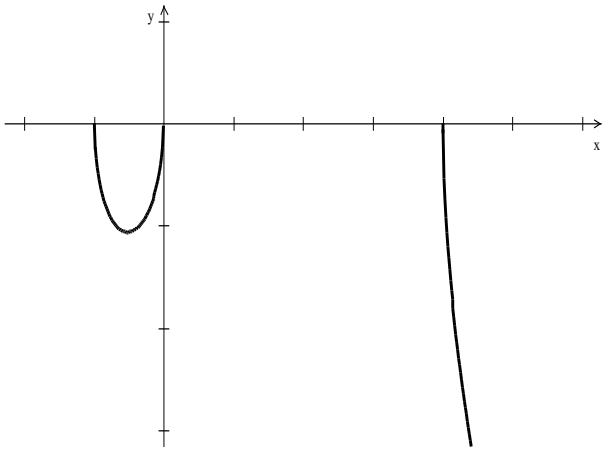


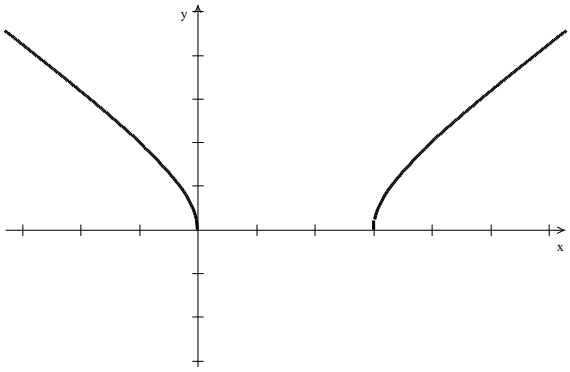
Radicals are easiest to recognize by their profoundly limited domains and ranges. Generally, the range is either $y \in [0, \infty)$ or $y \in (-\infty, 0]$ (entirely above or below the x-axis).

More radical functions:





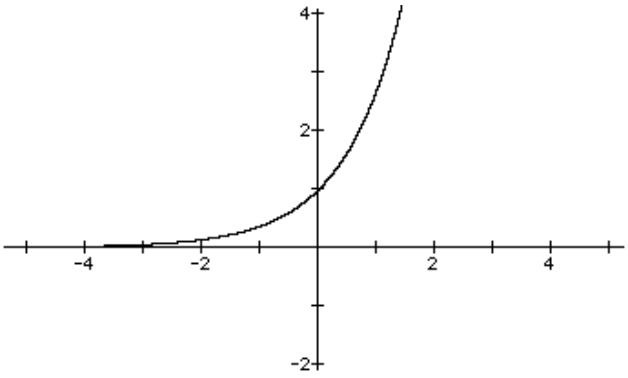


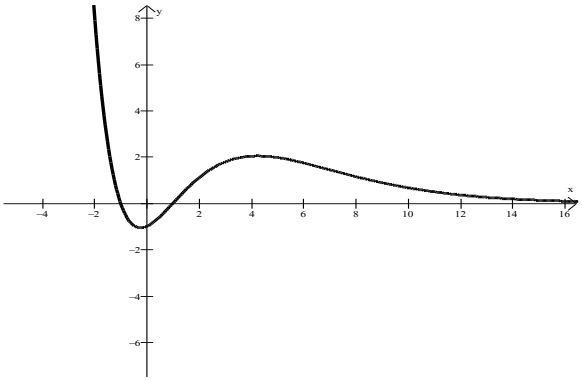


Families of Functions: Transcendental

IV. *Exponentials*

- Defn: "A function whose general equation is of the form $y = a \cdot b^x$."
- Means: there is an x in the exponent.
- Most Important Traits: Extreme Points and End Behavior.

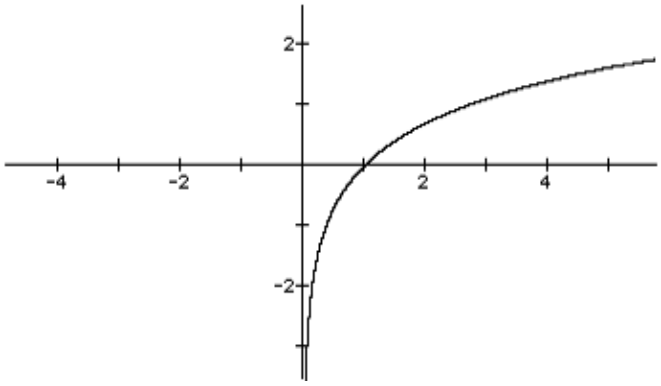




Exponentials tend to flatten on one end and head rapidly up (or down) on the other. There are exceptions to this, but recognizing the basic exponential above is most important. On their own, they do not have zeroes – they must be combined with another function to have zeroes. Their domain is not limited.

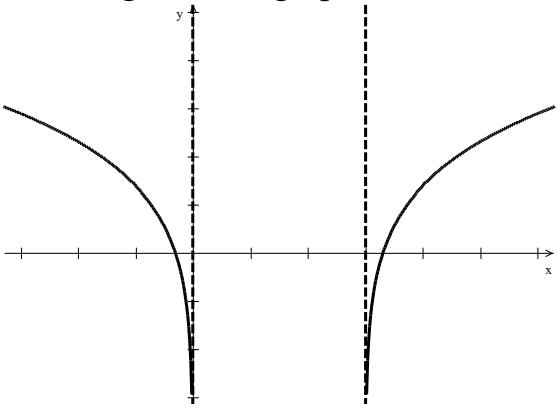
V. *Logarithmic Functions*

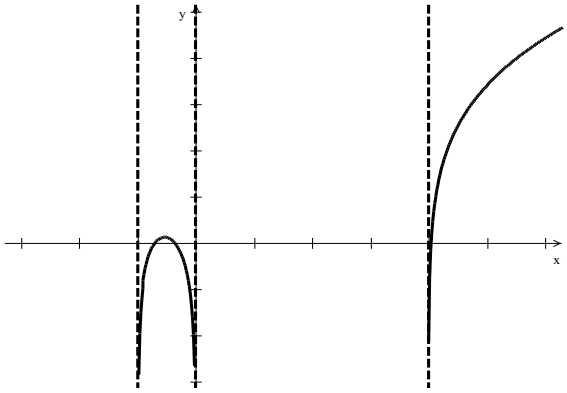
- Defn: "The inverse of an exponential function."
- Means: there is a Log or Ln in the equation.
- Most Important Traits: Domain, VAs, and Extreme Points.



Logarithmic graphs have limited domains. They also have vertical asymptotes and usually only head "downwards" to their vertical asymptotes.

More logarithmic graphs:





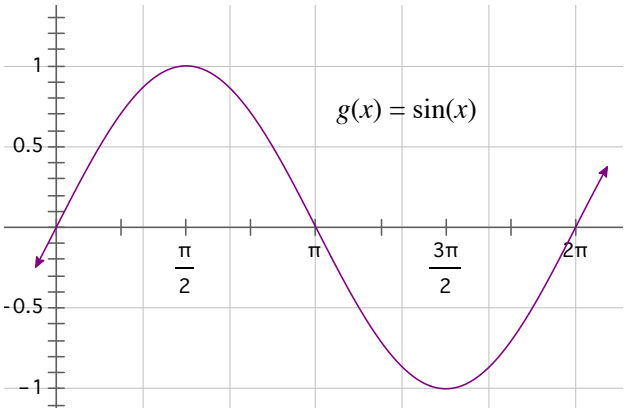
VI. *Trigonometric Functions*

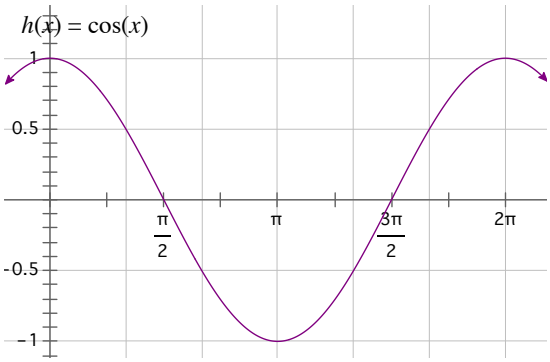
Defn: "A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure."

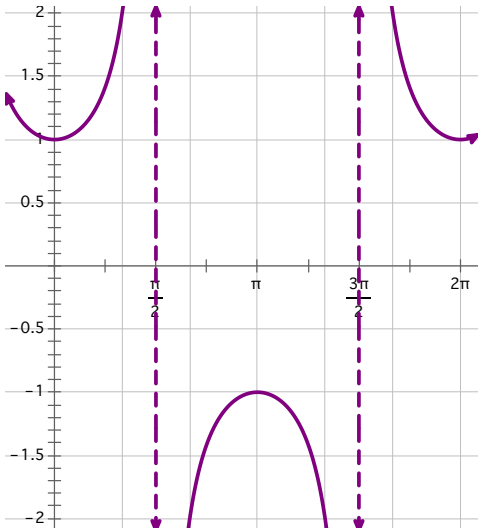
Means: an equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

- Most Important Traits: VAs, Axis Points, and Extreme Points.

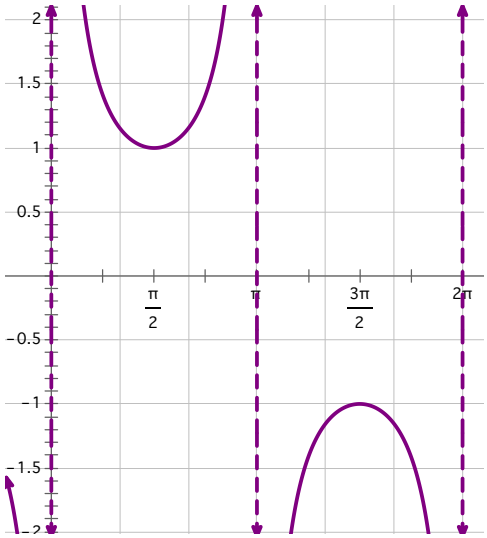
The graphs of trigonometric functions are widely varied – each pair of functions (sine and cosine, secant and cosecant, tangent and cotangent) have very similar graphs. All of them repeat themselves periodically – that is the key to recognizing them.


g(x) = sin(x)
1
0.5
-0.5
-1
π/2
π
3π/2
2π

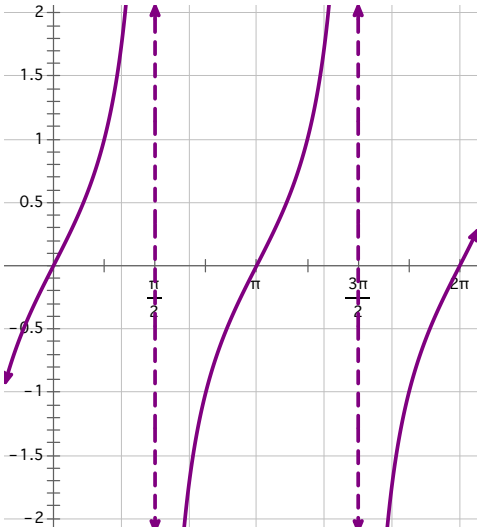

h(x) = cos(x)
1
0.5
-0.5
-1
π/2
π
3π/2
2π


2
1.5
1
0.5
-0.5
-1
-1.5
-2
π/2
π
3π/2
2π

$y = \sec x$

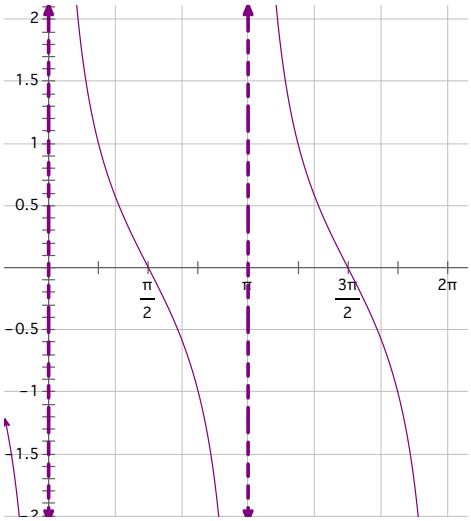

2
1.5
1
0.5
-0.5
-1
-1.5
π/2
π
3π/2
2π

$y = \csc x$



$y = \tan x$

$y = \tan\ x$



$y = \cot x$

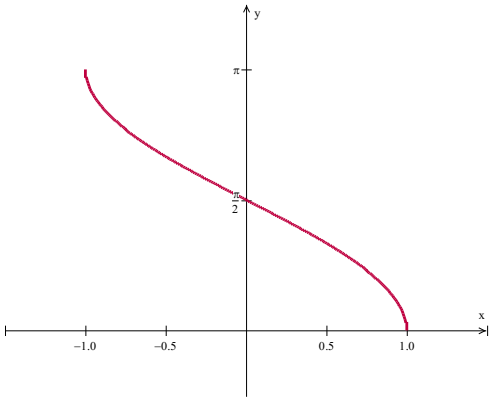
$y = \cot\ x$

VII *Trigonometric Inverse (or ArcTrig)*

Defn: "A function ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\sec^{-1}$, $\csc^{-1}$, or $\cot^{-1}$) whose <u>dependent</u> variable represents an angle measure."

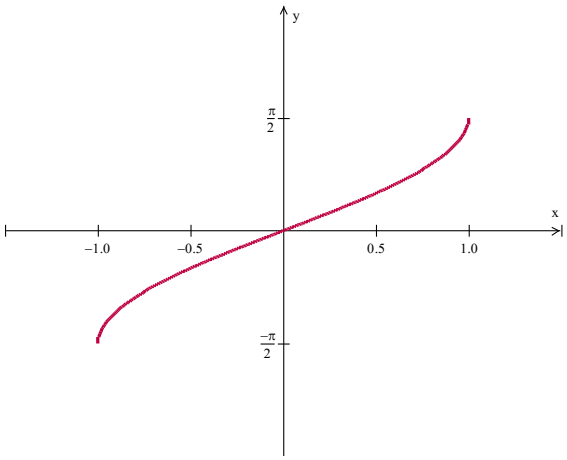
Means: an equation with Arcsine, Arccosine, Arctangent, Arcsecant, Arccosecant, or Arccotangent in it.

- Most Important Traits: Domain and Range.



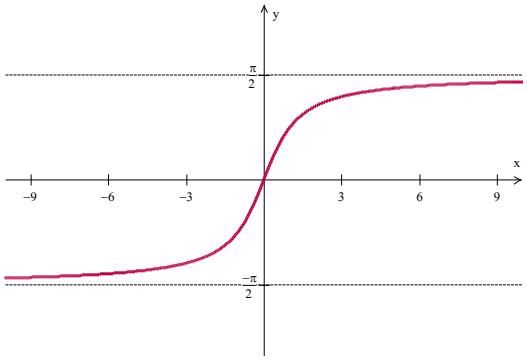
$y = \cos^{-1} x$

$0 \le y \le \pi$

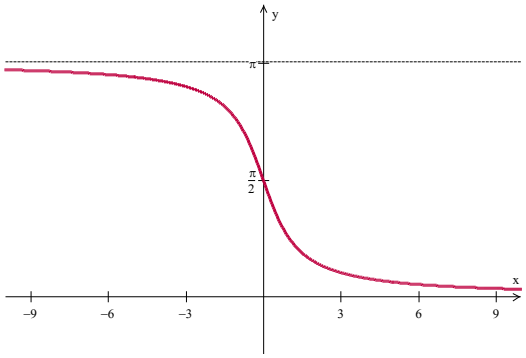


$y = \sin^{-1} x$

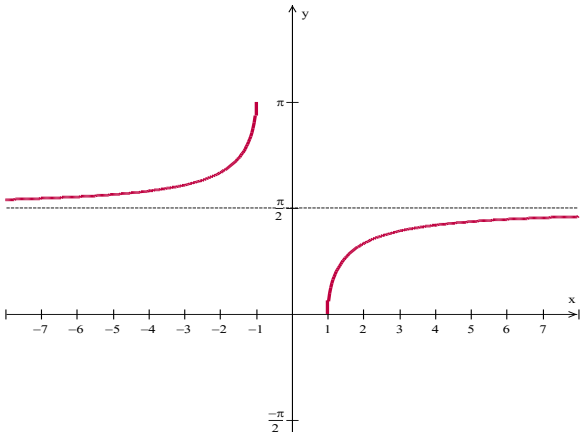
$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



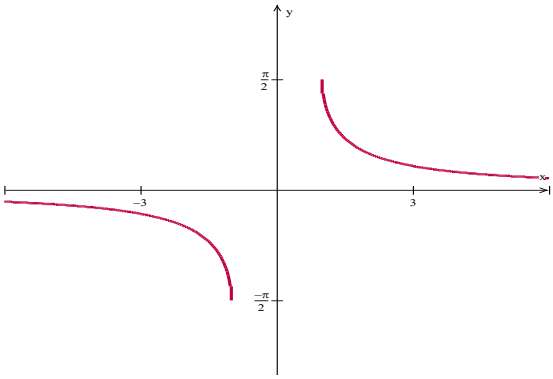
$$y = \tan^{-1} x$$
$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$$



$$y = \cot^{-1} x$$
$$0 \le y \le \pi$$



$$y = \sec^{-1} x$$
$$0 \le y \le \pi$$

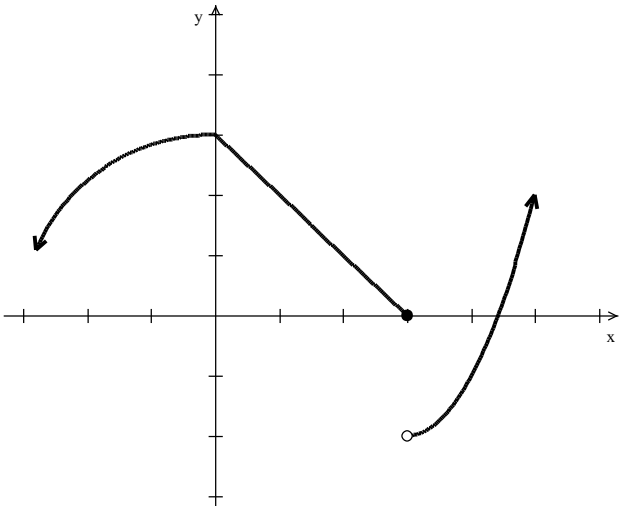


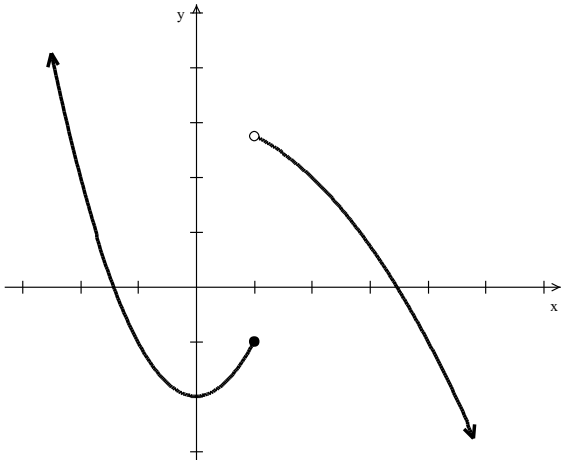
$$y = \csc^{-1} x$$
$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

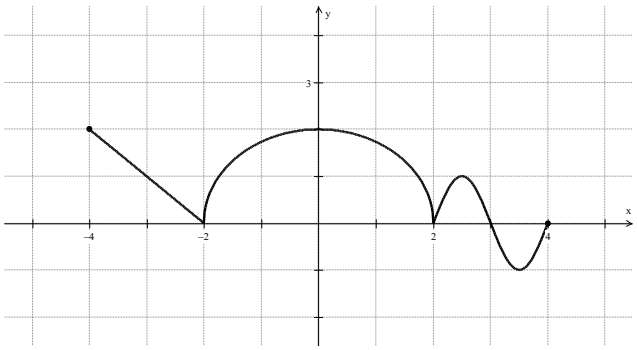
VIII. *Piece-wise Defined*

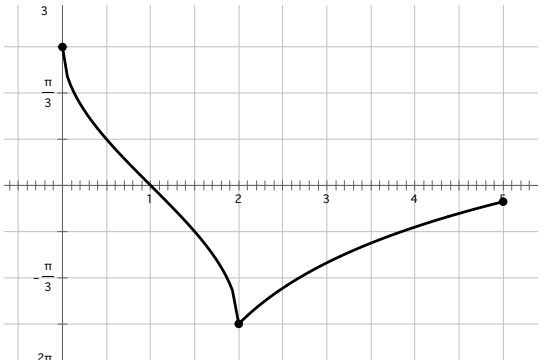
- Defn: A function that is defined by different equations for different parts of its domain.
- Most Important Traits: Depends on the pieces.

They are made up of two or more pieces of other functions. They may or may not be continuous, and combine traits and characteristics of other functions.


y
x


y
x


y
x
3
−4
−2
2
4


3
π/3
1
2
3
4
5
−π/3

0.2 Continuity and Differentiablity

CONTINUITY

One of the main topics early in Calculus is CONTINUITY. It really is a simple concept, which, like the Limit, is made complicated by its mathematical definition. Let us take a look at the formal definition of continuous:

Continuous--Defn: "A function f(x) is continuous at x = a if and only if:

i. $f(a)$ exists*,

ii. $\lim_{x \to a} f(x)$ exists*,

and iii. $\lim_{x \to a} f(x) = f(a)$."

**By "exists," it means "equals a real number."*

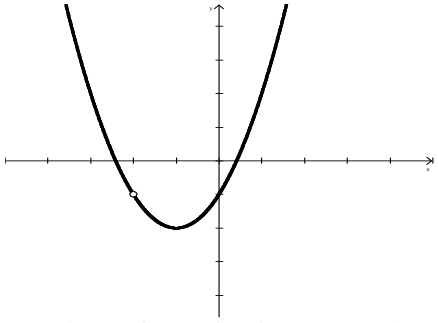
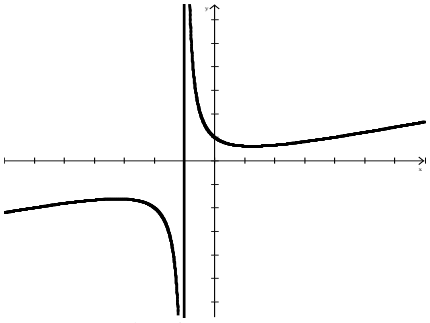
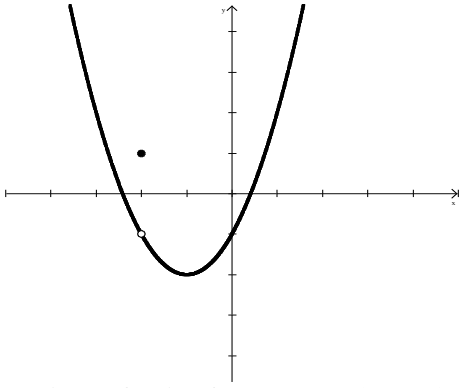
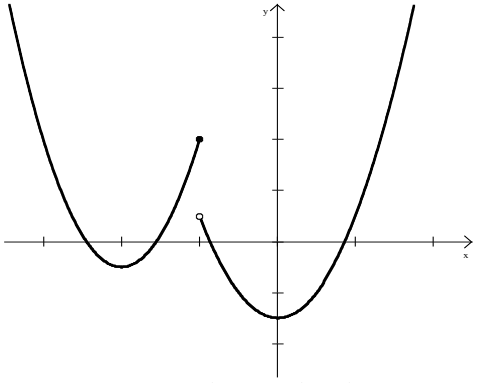
i) "$f(a)$ exists" means a must be in the domain.

ii) "$\lim_{x \to a} f(x)$ exists" means $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

iii) "$\lim_{x \to a} f(x) = f(a)$" should be self explanatory.

NB. All the families of functions which were explored in PreCalculus are continuous in their domain.

Continuity basically means a function's graph has no breaks in it. The formal definition involved limits. Since all the families of functions investigated in PreCalculus are continuous in their domain, it is easier to look at when a curve is discontinuous rather than continuous. There are four kinds of discontinuity:

	Removable Discontinuity ($\lim_{x\to a} f(x)$ does exist)	***Essential Discontinuity*** ($\lim_{x\to a} f(x)$ does not exist)
f(a) does not exist	 Point of Exclusion (POE)	 Vertical asymptotes
f(a) exists	 Point of Displacement (POD)	 Jump Discontinuity

NB. It will be a great advantage to know the shapes of the graphs of various equations. Please review Chapter Appendix A.1.

Differentiability—Defn: the derivative exists. *F(x)* is differentiable at *a* if and only if *F'(a)* exists and is differentiable on *[a, b]* if and only *F(x)* is differentiable at every point in the interval.

F(x) is differentiable at $x=a$ if and only if

i. *F(x)* is continuous at $x=a$,

and ii. $\lim_{x\to a^-} F'(x) = \lim_{x\to a^+} F'(x)$ as a real number.

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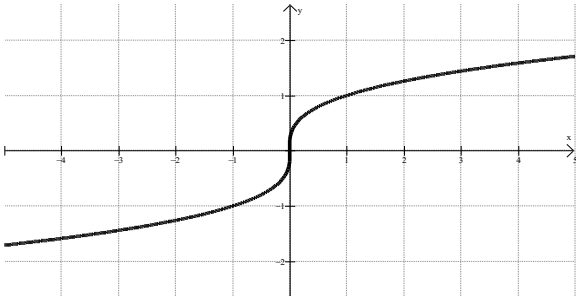
i. *F(x)* is continuous at $x=a$,

and ii. $\lim_{x\to a^-} F'(x) = \lim_{x\to a^+} F'(x)$ as a real number.

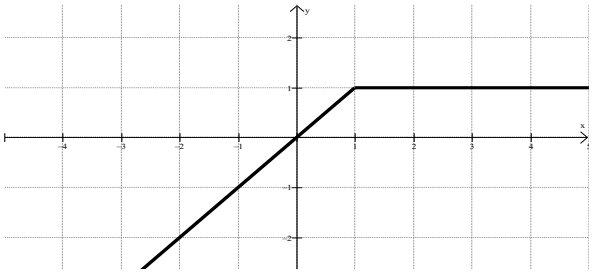
"Not differentiable" simply means the derivative does not exist. This can happen one of three ways.

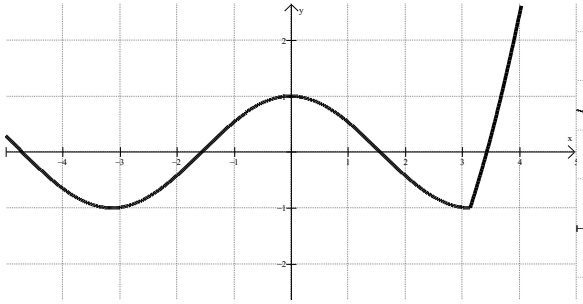
1. A function is not differentiable if it is not continuous.
   For pictures, see above.

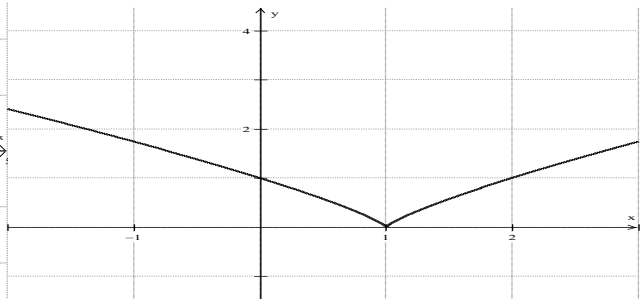
2. A function is not differentiable if the tangent line is vertical.



3. A function is not differentiable if it is not "smooth."







NB. All the families of functions studied in PreCaculus are Continuous and differentiable in their domains.

0.3: Important Facts from PreCalculus for Calculus:

When taking derivatives (or anti-derivatives, for that matter) it is usually the best policy to simplify everything as much as possible before taking the derivative. There are 3 basic categories of simplification: Algebraic, Logarithmic/Exponential, and Trigonometric.

Algebraic:

Simplify any algebraic expression to make it as much like a polynomial as possible. (**Algebra before Calculus!!!**)

- Distribute/FOIL: If there are multiple factors that are simple to multiply, do it.
- Distribute division by a common **monomial**: Eliminate the need for the quotient rule when dividing by a monomial by dividing each term by the monomial
- Write out everything in terms of fractional or negative exponents – you cannot use the power rule without doing this.

Logarithmic/Exponential

- Remember that natural logarithms and base *e* functions cancel one another:
  - $\ln\left(e^{u}\right)=u$
  - $e^{\ln(u)}=u$
- Any power on **the entire argument** of a logarithm can come out as multiplication. This includes fractional exponents
  - Example: $\ln\left(\sqrt{x^2+5x+6}\right)$; since there is a ½ power on the entire argument, it simplifies to $\frac{1}{2}\ln\left(x^2+5x+6\right)$
  - $\ln\left(u^{a}\right)=a\ln\left(u\right)$

More simplification rules:

Rules of exponents

1. $x^a x^b = x^{a+b}$
2. $\dfrac{x^a}{x^b} = x^{a-b}$
3. $\left(x^a\right)^b = x^{ab}$
4. $x^{-n} = \dfrac{1}{x^n}$

Rules of Radicals

$\sqrt[n]{x} = x^{1/n}$

$\sqrt[n]{x^a} = x^{a/n}$

Rules of Logarithms

$\log_a x + \log_a y = \log_a (xy)$

$\log_a x - \log_a y = \log_a \dfrac{x}{y}$

$\log_a x^n = n \log_a x$

Some Facts to Remember:

$\ln 1 = 0$ $\qquad$ $\ln e = 1$

$\ln e^u = u$ $\qquad$ $e^{\ln u} = u$

Change of Base Rules:	$\log_a u = \dfrac{\log u}{\log a} \quad or \quad = \dfrac{\ln u}{\ln a}$ $a^u = e^{u \cdot \ln a}$

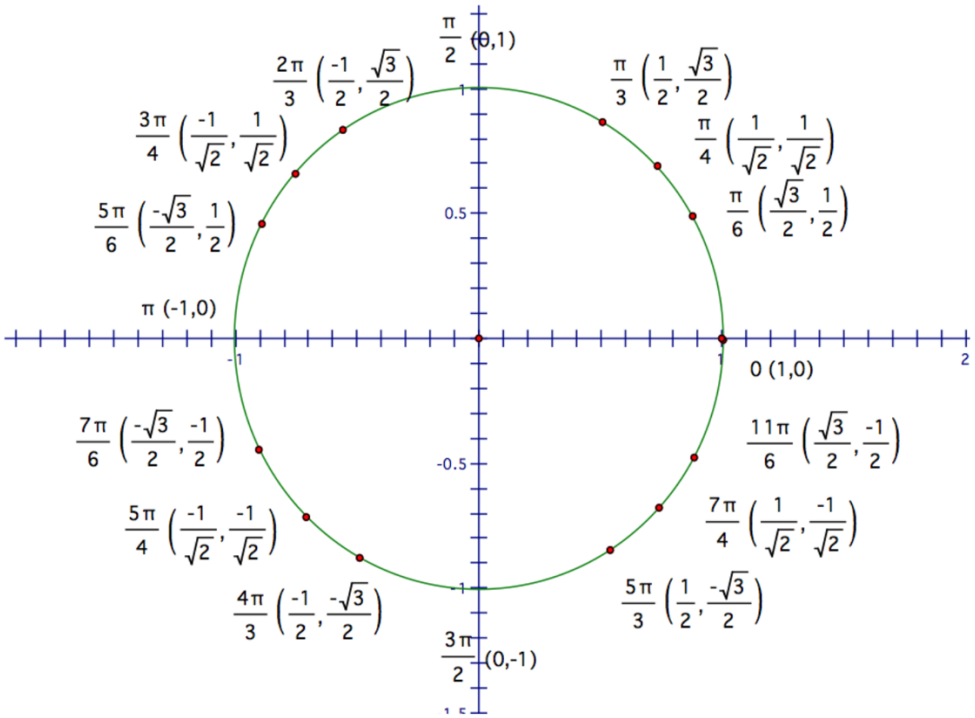
Trigonometric:

- Remember the quotient identities, they can make your life much easier
  - $\dfrac{\sin u}{\cos u} = \tan u$
  - $\dfrac{\cos u}{\sin u} = \cot u$
- The double angle for sine, and the basic Pythagorean identities come up fairly often:
  - $2\sin A\cos A = \sin(2A)$
  - $\cos^2 A + \sin^2 A = 1,\ \tan^2\theta + 1 = \sec^2\theta,\ 1 + \cot^2\theta = \csc^2\theta$

NB. These are simplifications done <u>before</u> you take a derivative – they are not derivative rules. Make sure you finish the problem by taking the derivative after you simplify.

- Calculators only give one of two answers for an inverse. You need to remember how to get the second one:
  - $\cos^{-1}\dfrac{x}{r} = \begin{Bmatrix} \text{calculator} \pm 2\pi n \\ -\text{calculator} \pm 2\pi n \end{Bmatrix}$
  - $\sin^{-1}\dfrac{y}{r} = \begin{Bmatrix} \text{calculator} \pm 2\pi n \\ \pi - \text{calculator} \pm 2\pi n \end{Bmatrix}$
  - $\tan^{-1}\dfrac{y}{r} = \begin{Bmatrix} \text{calculator} \pm 2\pi n \\ \pi + \text{calculator} \pm 2\pi n \end{Bmatrix}$

Special Values:



Some specific values that often arise in Calculus:

$$\cos(x)=0 \rightarrow x=\frac{\pi}{2}\pm\pi n \qquad \sin(x)=0 \rightarrow x=0\pm\pi n$$

$$\cos(x)=1 \rightarrow x=0\pm 2\pi n \qquad \sin(x)=1 \rightarrow x=\frac{\pi}{2}\pm 2\pi n$$

$$\cos(x)=-1 \rightarrow x=\pi\pm 2\pi n \qquad \sin(x)=-1 \rightarrow x=-\frac{\pi}{2}\pm 2\pi n$$

$$\tan(x)=0 \rightarrow x=0\pm\pi n$$

$$\tan(x)=1 \rightarrow x=\frac{\pi}{4}\pm\pi n$$

The Table

Rads	Deg	Cos x	Sin x	Tan x
0	0	1	0	0
$\frac{\pi}{6}$	30	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90	0	1	DNE
$\frac{2\pi}{3}$	120	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	135	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
$\frac{5\pi}{6}$	150	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$
π	180	-1	0	0
$\frac{7\pi}{6}$	210	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{5\pi}{4}$	225	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
$\frac{4\pi}{3}$	240	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$	270	0	-1	DNE
$\frac{5\pi}{3}$	300	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}$	315	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\frac{11\pi}{6}$	330	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$
2π	360	1	0	0

Unless stated otherwise, the problems are always in Radian mode.

A.4: Factoring Review

One of the primary concerns of Precalculus was finding the *x*-intercepts of functions.

Quadratic functions:

1. Factoring by rules

$$a(b\pm c)=ab\pm ac \qquad ab\pm ac=a(b\pm c)$$
$$a^2-b^2=(a+b)(a-b)$$
$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$a^4-b^4=(a^2+b^2)(a^2-b^2)$$

2. Factoring by guess and check
3. Factoring by splitting the middle term
4. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Higher order functions:

1. By Calculator. As was seen two sections ago, zeros can be found by graphing the function and using 2nd Trace 2.

2. Factor by Grouping. This is the same process done after splitting the middle term with quadratics. There is no middle term to split, though.

EX 1: Factor $y=2x^3-x^2-18x+9$

$$\begin{aligned} y&=x^2(2x-1)-9(2x-1)\\ &=(x^2-9)(2x-1)\\ &=(x+3)(x-3)(2x-1)\end{aligned}$$

Zeros: $(\pm 3,0),\left(\tfrac{1}{2},0\right)$

NB. Grouping only works some of the time.

3. Factor like a Quadratic. If the function has only three terms and the powers are common multiples of the powers of a quadratic, the function can be factored in the same manner as a quadratic—that is, by "guess-and-check" or spitting the middle term.

EX 2: Factor $y = x^4 - 10x^2 + 9$

$$\begin{aligned} y &= x^4 - 10x^2 + 9 \\ &= \left(x^2 - 1\right)\left(x^2 - 9\right) \\ &= (x-1)(x+1)(x-3)(x+3) \end{aligned}$$

Zeros: $(\pm 3, 0), (\pm 1, 0)$

NB. Guess-and-Check only works some of the time.

2a/3a Note: There is a Cubic Formula and a Quartic Formula which, like the Quadratic formula, will solve for the zeros of a cubic and quartic function, but they are far to intricate and involved for exploration here.

4. Factoring by synthetic substitution. Factoring by synthetic substitution (a.k.a. Horner's Algorithm) is a quicker form of polynomial division than long division (which will be used and explored later) that tells two things: the other factor if $(x-c)$ is one factor, and the remainder from the division. According to the Remainder Theorem, the remainder when $f(x)$ is divided by $(x-c)$ is $f(c)$. If this remainder is 0, then *c* is a zero of the function. If it is not, then the number *c* cannot be a zero.

Actually, the one most often forgotten is the one that occurs most often in Calculus:

5. Monomial Factoring

- $ab \pm ac = a\left(b \pm c\right)$

Ex 3: Factor $x^2e^{x^2}+5xe^{x^2}$

$x^2e^{x^2}+5xe^{x^2}$ Note that xe^{x^2} is in to both terms, we can take it out.

$=xe^{x^2}(x+5)$ This is as factored as it gets.

Notice that we did not set it equal to zero or solve for x as we were not asked to.

Ex 4 Factor $4x\sec^5(x+1)\tan(5x)-20\sec^4(x+1)$

$4x\sec^5(x+1)\tan(5x)-20\sec^4(x+1)$

$=4\sec^4(x+1)\left[x\tan(5x)-5\sec(x+1)\right]$

Ex 5 Factor $5x^2e^{2x+1}-80e^{2x+1}$

$5x^2e^{2x+1}-80e^{2x+1}$

$=5e^{2x+1}(x^2-16)$

$=5e^{2x+1}(x-4)(x+4)$

Ex 8 Find the zeroes of $y=(x^2-4)(1-\tan(x))$

$0=(x^2-4)(1-\tan(x))$

$0=(x-2)(x+2)(1-\tan(x))$

$(x-2)=0$ $(x+2)=0$ $1-\tan(x)=0$

$x=2$ $x=-2$ $\tan(x)=1$

$x=\tan^{-1}(1)=\frac{\pi}{4}\pm\pi n$

A.5: Sign Patterns

Sign Pattern – an organized way of listing information about a function a function.

Though it looks like it, **a sign pattern is really not a number line**. It is a one-dimensional representation of two coordinated data sets. In many ways, a sign pattern is short-hand representation of a table. Above the line are zeros and plus or minus signs. Below the line are x-values.

$$\begin{array}{cc} y & +\quad 0\quad -\quad 0\quad + \\ x & \longleftrightarrow \\ & -5\qquad 1 \end{array}$$

As a table, the same information would be listed like this:

y	+	0	–	0	+
x	$x<-5$	-5	$-5<x<1$	1	$1<x$

The variables do not have to be x and y. They can represent anything that matches two variables together. Thus, the sign pattern must be labeled in order to know how to interpret the information shown.

Remember: A sign pattern is a tool, not a solution. Context determines the interpretation. The two most common are:

Pattern of:	Interpretation
y	To determine where the curve is above or below the x-axis
$\frac{dy}{dx}$	To determine where the curve is increasing or decreasing
$\frac{d^2y}{dx^2}$	To determine where the curve is concave up or concave down
y^2 or e^y	To determine where the function exists (Domain)
$v(t)$	To determine when the motion is in which direction
$a(t)$	To determine—with $v(t)$—when the motion is speeding up or slowing down

Sign patterns can also interact:

EX 1 The sign patterns associated with $f(x)$ are given.

$$\begin{array}{c} f(x) \\ x \end{array} \quad \begin{array}{ccccccc} + & 0 & - & 0 & + & 0 & + \\ \longleftarrow & & & & & & \longrightarrow \\ & -3 & & -1 & & 1 & \end{array} \qquad \begin{array}{c} f'(x) \\ x \end{array} \quad \begin{array}{ccccccc} - & 0 & + & 0 & - & 0 & + \\ \longleftarrow & & & & & & \longrightarrow \\ & -2 & & 0 & & 1 & \end{array}$$

What can be said about the point where $x=-3$? Why?

Three things can be said about the point where $x=-3$:

1. The most obvious is that $x=-3$ is a zero of $f(x)$. The $f(x)$ sign pattern says so.
2. The $f(x)$ sign pattern also informs us that, at $x=-3$, $f(x)$ switches from above the *x*-axis to below the *x*-axis.
3. The thing most people forget is that, even though $x=-3$ is not one of the numbers on the $f'(x)$ sign pattern, it is still present to the left of $x=-2$. At $x=-3$, $f'(x)$ is negative, therefore, $f(x)$ is decreasing at $x=-3$.

EX 2 The sign patterns associated with $f(x)$ are given.

$$\begin{array}{c} f(x) \\ x \end{array} \quad \begin{array}{ccccccc} + & 0 & - & 0 & + & 0 & + \\ \longleftarrow & & & & & & \longrightarrow \\ & -3 & & -1 & & 1 & \end{array} \qquad \begin{array}{c} f'(x) \\ x \end{array} \quad \begin{array}{ccccccc} - & 0 & + & 0 & - & 0 & + \\ \longleftarrow & & & & & & \longrightarrow \\ & -2 & & 0 & & 1 & \end{array}$$

What can be said about the point where $x=1$? Why?

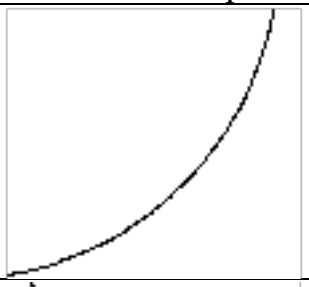
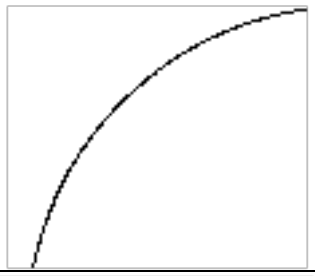
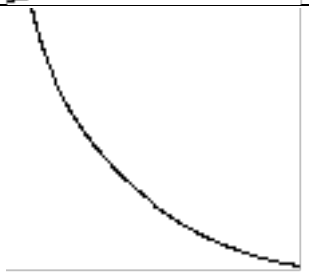
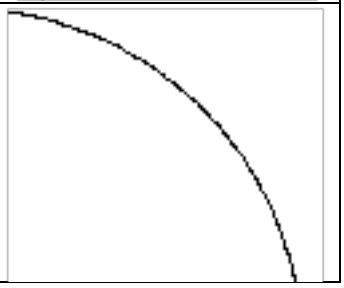
Three things can be said about the point where $x=1$ as well:

1. The $f(x)$ sign pattern informs us that $x=1$ is a zero of $f(x)$.
2. The $f(x)$ sign pattern also informs us that $x=1$ is a bouncer—meaning that $x=1$ hits but does not cross the *x*-axis.
3. The $f'(x)$ sign pattern informs us that $x=1$ is at a minimum of $f(x)$ because the signs of $f'(x)$ switch from negative to positive.

EX 3 Given the $f(x)$and its sign patterns in EX 2, for what values of x is $f(x)$ both below the x-axis and increasing?

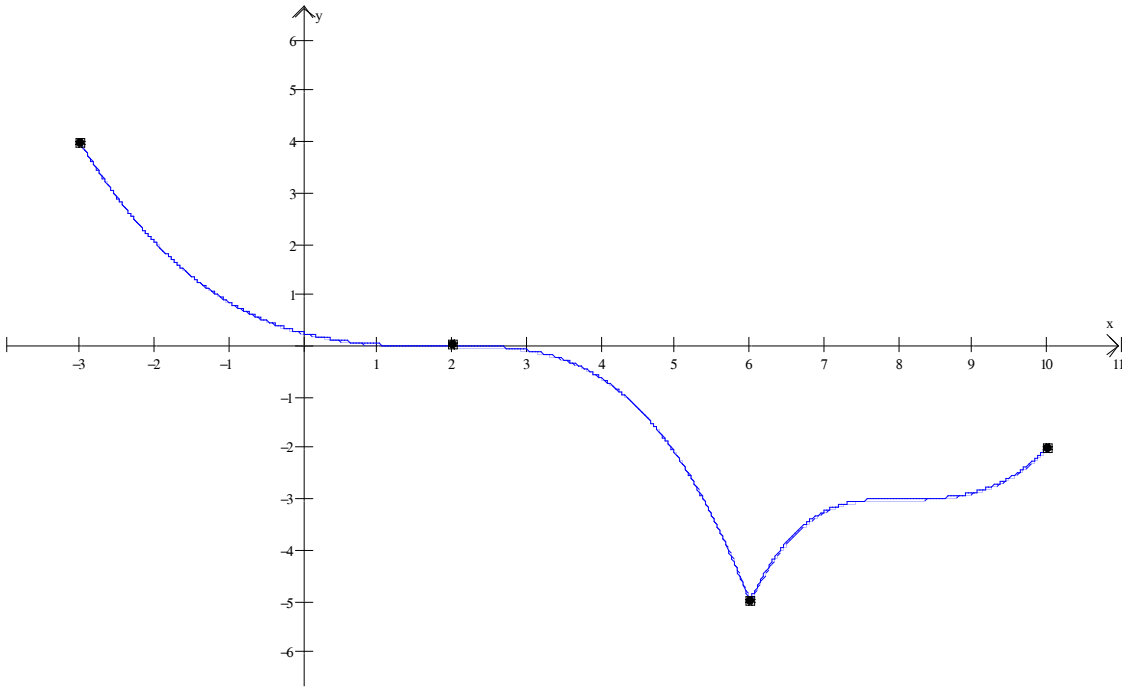
From the $f(x)$ sign pattern, we can see that $f(x)$ is below the x-axis on $x \in (-3,-1)$. From the $f'(x)$ sign pattern, we know that $f(x)$ is decreasing before $x=-2$ and increasing after. Therefore,

$f(x)$ both below the x-axis and increasing on $x \in (-2,-3)$

	Concave Up	Concave Down
Increasing		
Decreasing		

EX 4 Sketch the graph of the function whose traits are given below.

x	–3		2		6		8		10
$f(x)$	4	+	0	–	–5	–	–3	–	–2
$f'(x)$	DNE	–	0	–	DNE	+	+	+	DNE
$f''(x)$	DNE	+	0	–	DNE	–	0	+	DNE



Like the y and the $\frac{dy}{dx}$ equations and sign patterns, the velocity and acceleration equations and sign patterns both hold information about the motion of an object. In particular, questions about whether an object is speeding up or slowing down are answered by a combination of the velocity and acceleration.

An object is speeding up when $v(t)$and $a(t)$ have the same sign.

An object is slowing when $v(t)$and $a(t)$ have opposite signs.

NB. Speeding up and slowing down is not determined by the sign of the acceleration.

EX 5 The position of a particle is described by $x(t)=t^3-3t^2-24t+3$. Is the particle speeding up or slowing down at $t=3$ seconds?

$$v(t)=x'(t)=3t^2-6t-24$$
$$v(3)=3(3)^2-6(3)-24=-15$$

$$a(t)=v'(t)=6t-6$$
$$a(3)=v'(t)=6(3)-6=12$$

Since these have opposite signs, the particle is slowing down.

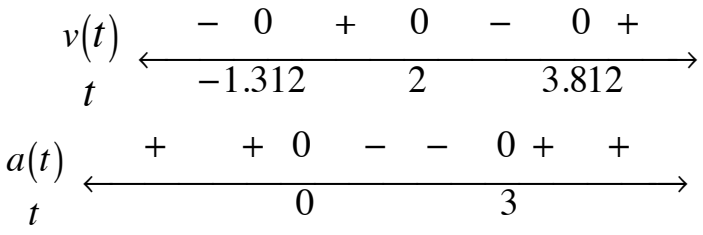
EX 6 The position of a particle is described by $x(t)=\frac{1}{2}t^4-3t^3+20t+3$. When is the particle speeding up?

First, we need the velocity sign pattern:

$$v(t)=2t^3-9t^2+20$$

Synthetic Division shows that $x=2$ is a zero and the velocity eqwuation factors into $(x-1)(2x^2-5x-10)$. The quadratice formuls then shows the other zeros are $x=-1.312$ and 3.812. So

$$\begin{array}{cc} v(t) & - \quad 0 \quad + \quad 0 \quad - \quad 0 \quad + \\ t & \longleftarrow\!\!\!



Since the question was "When is the particle speeding up?" we need to name the intervals of time when $v(t)$ and $a(t)$ have the same sign; that is:

$$t \in (-1.312, 0) \cup (2, 3) \cup (3.812, \infty)$$

A third kind of problem that involves two sign patterns is the parametric motion problems.

In parametric mode, there are two components of position, velocity, and acceleration:

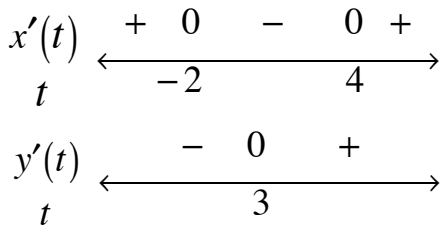
EX 7 A particle's position $\langle x(t), y(t) \rangle$ at time *t* is described by

$$\left\langle \frac{1}{3}t^3 - t^2 - 8t + 1, -t^2 + 6t + 2 \right\rangle.$$

When is the particle moving both left and up?

Direction of motion is determined the velocity so:

$$\langle x'(t), y'(t) \rangle = \langle t^2 - 2t - 8, -2t + 6 \rangle$$



According to these sign patterns, the particle is moving left on $t \in (-2, 4)$ and up on $t \in (3, \infty)$. The question was when is it doing both. So, the particle moving both right and down on

$$t \in (3, 4)$$