Chapter 6 Answer Key

6.1 Free Response Solutions

1. $\lim_{x \to -3} \dfrac{x^2+7x-12}{x^2-9} = \dfrac{0}{0} \overset{LH}{=} \lim_{x \to -3} \dfrac{2x+7}{2x} = -\dfrac{1}{6}$

3. $\lim_{x \to 0} \dfrac{x^2}{\tan x} = \dfrac{0}{0} \overset{LH}{=} \lim_{x \to 0} \dfrac{2x}{\sec^2 x} = 0$

5. $\lim_{x \to 0} x\cot x = \lim_{x \to 0} \dfrac{x}{\tan x} = \dfrac{0}{0} \overset{LH}{=} \lim_{x \to 0} \dfrac{1}{\sec^2 x} = 1$

7. $\lim_{x \to -3} \dfrac{x^3-1}{x^2+3x-4} = \dfrac{0}{0} \overset{LH}{=} \lim_{x \to -3} \dfrac{3x^2}{2x+3} = \dfrac{3}{5}$

9. $\lim_{x \to 1} \dfrac{\ln x}{\tan x} = 0$

11. $\lim_{x \to \sqrt{2}} \dfrac{x^4+4x^2-12}{x^3+x^2-2x-2} = \dfrac{0}{0} \overset{LH}{=} \lim_{x \to \sqrt{2}} \dfrac{4x^3+8x}{3x^2+2x-2} = \dfrac{16\sqrt{2}}{4+2\sqrt{2}} = \dfrac{8\sqrt{2}}{2+\sqrt{2}}$

13. $\lim_{x \to \pi} \dfrac{\cos^2 x-1}{2\cos^2 x-5\cos x-7} = \dfrac{0}{0} \overset{LH}{=} \lim_{x \to \pi} \dfrac{2\cos x \cdot -\sin x}{4\cos x \cdot -\sin x+5\sin x} = \dfrac{0}{0}$

$\overset{LH}{=} \lim_{x \to \pi} \dfrac{2\left[\cos x \cdot -\cos x + -\sin x \cdot -\sin x\right]}{4\left[\cos x \cdot -\cos x + -\sin x \cdot -\sin x\right]+5\cos x} = \dfrac{2(-1)}{4(-1)-5} = \dfrac{2}{9}$

15. $\lim_{x \to -1} \dfrac{1-x^2}{e^{x+1}-1} = \dfrac{0}{0} \overset{LH}{=} \lim_{x \to -1} \dfrac{-2x}{e^{x+1}} = 2$

17. $\lim_{x \to -1} \dfrac{x+1}{e^{x+1}} = \dfrac{0}{1} = 0$

19. $\lim_{x\to 0}\left(1+\frac{x}{2}\right)^{\cot x}$.

$\Rightarrow \ln y = \ln\left(1+\frac{x}{2}\right)^{\cot x} = \cot x \ln\left(1+\frac{x}{2}\right) \Rightarrow \ln y = \ln\left(1+\frac{x}{2}\right)^{\cot x} = \cot x \ln\left(1+\frac{x}{2}\right)$

$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln\left(1+\frac{x}{2}\right)}{\tan x} = \frac{0}{0} \overset{LH}{=} \lim_{x\to 0} \frac{\frac{1}{1+\frac{x}{2}}\cdot\frac{1}{2}}{\sec^2 x} = \frac{1}{2}$

$\therefore \lim_{x\to 0} \ln y = \frac{1}{2} \Rightarrow e^{\lim_{x\to 0} \ln y} = e^{1/2} \Rightarrow \lim_{x\to 0} e^{\ln y} = e^{1/2} \Rightarrow \lim_{x\to 0} y = e^{1/2}$

$\Rightarrow \lim_{x\to 0}\left(1+\frac{x}{2}\right)^{\cot x} = e^{1/2}$

21. $\lim_{x\to 5} \csc(x-5)\cdot\ln(x-4) = \lim_{x\to 5} \frac{\ln(x-4)}{\sin(x-5)} = \frac{0}{0} \overset{LH}{=} \lim_{x\to 5} \frac{\frac{1}{x-4}}{\cos(x-5)} = \frac{1}{1} = 1$

23. $\lim_{h\to 0} \frac{(3+h)^3 - 27}{h} = 27$

We are being asked to find $f'(3)$ where $f(x) = x^3$. $f'(x) = 3x^2 \Rightarrow f'(3) = 27$

25. $\lim_{h\to 0} \frac{e^{3+h} - e^3}{h} = e^3$

We are being asked to find $f'(3)$ where $f(x) = e^x$. $f'(x) = e^x \Rightarrow f'(3) = e^3$

27. $\lim_{h\to 0} \frac{\cos(\pi + h) + 1}{h} = 0$

We are being asked to find $f'(\pi)$ where $f(x) = \cos x$. $f'(x) = \sin x \Rightarrow f'(\pi) = 0$

29. $\lim_{x\to 4} \frac{\sin x - \sin 4}{x-3} = 0$. This is NOT a derivative limit. Substitute $x = 4$.

31. $\lim_{x\to a} \frac{x^3 - a^3}{x^2 - a^2} = \frac{3}{2}a$

Like #29, this is NOT a derivative limit.

$$\lim_{x\to a}\frac{x^3-a^3}{x^2-a^2}=\frac{0}{0}\overset{LH}{=}\lim_{x\to a}\frac{3x^2}{2x}=\frac{3a^2}{2a}=\frac{3}{2}a$$

6.1 Multiple Choice Solutions

1. $$\lim_{x\to a}\frac{x^4-a^4}{x-a}\overset{L'H}{=}\lim_{x\to a}\frac{4x^3}{1}=4a^3$$

The correct answer is C

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3. $$\lim_{x\to 2}\frac{x^2+x-2}{x^2-4}\overset{L'H}{=}\lim_{x\to 2}\frac{2x+1}{2x}=\frac{5}{4}$$

The correct answer is A

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5. $$\lim_{x\to 1}\frac{\ln x}{e^x-1}=\frac{\ln 1}{e^1-1}=\frac{0}{e^1-1}=0$$

The correct answer is A

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7. $$\lim_{x\to 1}\frac{\ln x}{x^2-1}\overset{L'H}{=}\lim_{x\to 1}\frac{1/x}{2x}=\lim_{x\to 1}\frac{1}{2x^2}=\frac{1}{2}$$

The correct answer is C

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9. C and D are mutually exclusive. Since $f'(3)=5$, C is true and D must be false. E might be false, but D must be.

The correct answer is D

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11. $$\lim_{x\to 2}\frac{\int_{-2}^{x} t^3dt}{x^2-4} \overset{L'H}{=} \lim_{x\to 2}\frac{\frac{d}{dx}\int_{-2}^{x} t^3dt}{\frac{d}{dx}\left(x^2-4\right)} = \lim_{x\to 2}\frac{\left(x^3\right)}{(2x)} = \lim_{x\to 2}\frac{x^2}{2} = 2$$

The correct answer is B.

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13. $$\lim_{x\to 0}\frac{\int_{0}^{x^3}\cos t^2dt}{x^3} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{d}{dx}\int_{0}^{x^3}\cos t^2dt}{\frac{d}{dx}x^3} = \lim_{x\to 0}\frac{\cos x^6\left(3x^2\right)}{3x^2} = \lim_{x\to 0}\left(\cos x^6\right) = 1$$

The correct answer is B.

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15. $$\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h} = \left.\frac{d}{dx}(\sin x)\right|_{x=\frac{\pi}{2}} = \cos\frac{\pi}{2} = 0$$

The correct answer is C

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17. $$\lim_{h\to 0}\frac{\sin^2\left(\frac{\pi}{3}+h\right)-\frac{3}{4}}{h} \overset{L'H}{=} \lim_{h\to 0}\frac{2\sin\left(\frac{\pi}{3}+h\right)\cos\left(\frac{\pi}{3}+h\right)}{1}$$
$$= 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

The correct answer is C

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19. I. is true: $f'(3) = \lim_{x \to 3} \dfrac{\sqrt{x^2-1} - \sqrt{8}}{x-3}$

II is true: $f'(3) = \lim_{h \to 0} \dfrac{\sqrt{(x+h)^2-1} - \sqrt{8}}{h}$

III is true: $f'(3) = \lim_{h \to 0} \dfrac{\sqrt{(3+h)^2-1} - \sqrt{3^2-1}}{h}$

The correct answer is AC

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21. I. $\lim_{x \to 2} \dfrac{f(x)-3}{x-2} = f'(2) = 0$. False

II. $f(2) = 3$. True

III. $\lim_{x \to 2} f(x) = 3$. True

The correct answer is C) II and III

6.2 Free Response Solutions

1(a) $\lim_{x \to -4^-} f(x) = 4$ (b) $\lim_{x \to -4^+} f(x) = 1$ (c) $\lim_{x \to 3} f(x) = 2$

(d) $\lim_{x \to -4} f(x) = dne$ (e) $\lim_{x \to 5^+} f(x) = -\infty$ (f) $\lim_{x \to 5^-} f(x) = \infty$

(g) $f(-4) = 4$ (h) $\lim_{x \to 0} f(x) = -3$ (i) $f(0) = dne$

(j) $f(3) = 5$ (k) $\lim_{x \to 0^+} f(x) = -3$ (l) $\lim_{x \to 0^-} f(x) = -3$

(m) $\lim_{x \to 3} f(x) = 2$ (n) $f(5) = dne$

3. $\lim_{x \to 1^-} \dfrac{2x}{x^2 - 1} = \dfrac{2}{0^-} = -\infty$

5. $\lim_{x \to 2^-} \dfrac{1-2x}{x^2 - 4x + 4} = \lim_{x \to 2^-} \dfrac{1-2x}{(x-2)^2} = \dfrac{-3}{0^+} = -\infty$

7. $\lim_{x \to 0^+} \dfrac{\ln x}{\sin x} = -\infty$; because of the vertical asymptote on $\ln x$

9. $\lim_{x \to 0^-} \dfrac{\cot(x^2 - 5x)}{\ln x^2} = \dfrac{\infty}{-\infty}$

$\overset{LH}{=} \lim_{x \to 0^-} \dfrac{-\csc^2(x^2 - 5x) \cdot (2x - 5)}{1/x^2 \cdot 2x}$

$\overset{LH}{=} \lim_{x \to 0^-} \dfrac{-x \cdot (2x - 5)}{2\sin^2(x^2 - 5x)} \overset{LH}{=} \lim_{x \to 0^-} \dfrac{-4x - 5}{4\sin(x^2 - 5x)\cos(x^2 - 5x)} = -\dfrac{-5}{4(0^+)(1)} = -\infty$

6.2 Multiple Choice Solutions

1. "The derivative of $f(x)$ is positive for all $x < -2$ and negative for all $-2 < x$" means the curve goes up from left to right before $x = -2$ and down from left to right after $x = -2$. Therefore, $\lim_{x \to -2^-} f(x) = +\infty$ and $\lim_{x \to -2^+} f(x) = +\infty$.

The correct answer is C

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3. I. False : $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

   II. $f(2) = 2$: True

   III. $\lim_{x \to 2} f(x)$ does not exist: True

Therefore, the correct answer is E) II and III only

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5. I. f *is continuous at* $x = 3$: True

   II. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$: True

   III. $\lim_{x \to 2} f(x)$ does not exist: False

Therefore, the correct answer is D) I and II

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7. I. $\lim_{x \to 2} f(x) = 1 = f(2) \therefore$ True

   II. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$: True

   III. $\lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = f'(2) = dne \therefore$ True

Therefore, the correct answer is C) I, II, and III

6.3 Free Response Solutions

1. $f(x)$ is discontinuous at $x = -4$, because $\lim_{x \to -4^-} f(x) = 4$, but $\lim_{x \to -4^+} f(x) = 1$. Therefore, $\lim_{x \to -4} f(x) = dne$

$f(x)$ is discontinuous at $x = 0$, because $f(0) = dne$

$f(x)$ is discontinuous at $x = 3$, because $\lim_{x \to 3} f(x) = 2$, but $f(3) = 5$. Therefore, $\lim_{x \to 3} f(x) \neq f(3)$

$f(x)$ is discontinuous at $x = 5$, because $f(5) = dne$.

3. $f(x) = \begin{cases} x^2 - 1, if x > -1 \\ 0, if x = -1 \\ 4 - x, if x < -1 \end{cases}$; a = -1; Discontinuous because $\lim_{x \to -1^+} f(x) \neq \lim_{x \to -1^-} f(x)$, therefore $\lim_{x \to -1} f(x)$ does not exist.

5. $h(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, if x \neq 2 \\ 4, if x = 2 \end{cases}$; $a = 2$; $\lim_{x \to 2} h(x) = 4$ and $h(2) = 4$, therefore, continuous.

7. $k(x) = \begin{cases} 2x + 1, if x \leq -1 \\ 3 - x^2, if x > -1 \end{cases}$; a = -1; Discontinuous because $k(-1)$ does not exist.

9. $H(x) = \begin{cases} 2x + 1, \ if \ x > -1 \\ 0, \ if \ x = -1 \\ -x^2, \ if \ x < -1 \end{cases}$; a = -1; $\lim_{x \to -1^+} H(x) = \lim_{x \to -1^-} H(x) = -1$, but $H(x) = 0$, therefore, $\lim_{x \to -1} H(x) \neq H(0)$. So $H(x)$ is discontinuous

11. $f(x)=\begin{cases}\tan^{-1}(x-3)\text{if}x\le 3\\ 1-\cos(x-3)\text{if}x>3\end{cases}$; $a=3$; Continuous because
$\lim_{x\to 3} f(x)=\tan^{-1}0=0=1-\cos 0=\lim_{x\to 3} f(x)$

15. $f(x)=\begin{cases}secx, & if\ x<0\\ 1, & if\ x=0\\ 1-x^3, & if\ x>0\end{cases}$ is continuous at $x=0$ because $f(0)=1$ and

$\lim_{x\to 0^-} f(x)=Sec\ 0=1=1-0^3=\lim_{x\to 0^+} f(x).$

17. $f(x)=\begin{cases}\sqrt{9-x^2}, & if\ -3<x<3\\ 0, & if\ x=3\\ (x-3)^2, & if\ x>3\end{cases}$ is continuous at $x=3$ because $f(3)=0$

and $\lim_{x\to 3^-} f(x)=\sqrt{9-3^2}=0=\lim_{x\to 03} f(x)=(3-3)^2=0$

6.3 Multiple Choice Solutions

1. A If f is continuous then $\lim_{x\to b} f(x)$ must exist

A through D are parts of the definition of continuous, so they must all be true. E might be false, but does not have to be.

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3. I. False: There is a POE at $x=1$
   II. False: There is a POE at $x=1$
   III. True: The graph of f has a horizontal asymptote at $y=\frac{1}{2}$

The correct answer is C

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5. $\lim_{x \to 1} \dfrac{\sqrt{x+3} - \sqrt{3x+1}}{x-1} = k$

$$\lim_{x \to 1} \frac{(x+3)^{1/2} - (3x+1)^{1/2}}{x-1} \overset{L'H}{=} \lim_{x \to 1} \frac{\frac{1}{2}(x+3)^{-1/2} - \frac{1}{2}(3x+1)^{-1/2}(3)}{1} = \frac{\frac{1}{4} - \frac{3}{4}}{1} = \frac{1}{2} = k$$

The correct answer is C

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7. "The derivative of $f(x)$ is negative for all $x < -2$ and positive for all $-2 < x$" means the curve is decreasing before the VA and increasing after. So the curve goes down on both sides of $x = 2$. I and II are true and III is false.

The correct answer is D

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9. I. False: $\lim_{x \to 2} f(x) = 3$

   II. True: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

   III. True: $\lim_{x \to 2} f(x) = 3$

The correct answer is C

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11. I. $f'(2) > \lim_{x \to 2} f(x) \to 0 > 2 \therefore$ False

    II. $f'(1) < 0$ and $\lim_{x \to 2} f(x) = 2 \therefore$ True

    III. $\lim_{x \to 2} f(x)$ does not exist: False

Therefore, the correct answer is B) II only

6.4 Free Response Solutions

1. $f(x)=\sqrt[3]{x^2}$ is not differentiable at $x=0$, because $\frac{dy}{dx}$ *dne*.

3. $f(x)=\begin{cases} x^2-2x+1, \text{if} x\le 1 \\ \ln x, \text{if} x>1 \end{cases}$ is not differentiable because, $f'(1^-)\neq f'(1^+)$.

5. $f(x)=\begin{cases} mx+2, \text{ if } x\le 1 \\ k\ln x, \text{ if } x>1 \end{cases} \rightarrow f'(x)=\begin{cases} m, \text{ if } x<1 \\ \frac{k}{x}, \text{ if } x>1 \end{cases}$

Continuous: $m(1)+2=k\ln(1)\rightarrow m+2=0\rightarrow m=-2$

Differentiable: $m=\frac{k}{1}\rightarrow m=k\rightarrow m=-2=k$

$m=-2, k=-2$

7. $f(x)=\begin{cases} ke^{2x}, \text{ if } x\le 0 \\ 3-mx, \text{ if } x>0 \end{cases} \rightarrow f(x)=\begin{cases} 2ke^{2x}, \text{ if } x<0 \\ -m, \text{ if } x>0 \end{cases}$

Continuous: $ke^0=3-m(0)\rightarrow k=3$

Differentiable: $2ke^0=-m\rightarrow m=2(3)=6$

$-2m-6=-m\rightarrow -6=m$

$m=-6, k=3$

9. $f(x)$ is not differentiable at $x=-4$ because $f(x)$ is not continuous and at $x=-1$ because it is not smooth.

11. $f(x)$ is not differentiable at $x=\pm 2$ because $f(x)$ is not smooth there.

6.4 Multiple Choice Solutions

1. A is not false because the endpoints are extremes. C is false because the curve is not smooth. D is false because the curve goes from increasing to increasing at 2. E is false because line segments have 0 concavity.

The correct answer is B

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3. All the statements must be true. A thru C are the parts of the definition of continuity and the curve must be continuous in order to be differentiable. D is part of the definition of differentiable.

The correct answer is E

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5. Differentiable: $f'(x) = \begin{cases} 2x + k & \text{for} x < 1 \\ 3 & \text{for} x > 1 \end{cases}$

$2(1) + k = 3 \rightarrow k = 1$

Continuous: $(1)^2 + (2)(1) - 3 = 3(1) + b \rightarrow b = -3$

The correct answer is A

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7. There is a point of displacement at b.

The correct answer is B

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9. I. is True: $\lim_{x \to 3^-} f(x) = 3 + 2 = 5$ and $\lim_{x \to 3^-} f(x) = 4(3) - 7 = 5$

II. is True: The three parts of the definition of continuity hold.

III. is False: $\lim_{x \to 3^-} f'(x) = 1$ but $\lim_{x \to 3^+} f'(x) = 4$

The correct answer is D

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11. The denominator degree is higher, therefore $\lim_{x \to \infty} \frac{P(x)}{Q(x)} = 0$

The correct answer is C

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13. I. is true by the numerical definition of $f'(2)$
II. is true because if $f'(2)$ exists, $f(x)$ must be continuous at $x = 2$
III. is true because $f'(2) = 0$

The correct answer is E

6.5 Free Response Solutions

1. $\lim_{x\to\infty}\dfrac{1-15x+12x^2}{16x^2-1}=\dfrac{12}{16}=\dfrac{3}{4}$ -OR-

$\lim_{x\to\infty}\dfrac{1-15x+12x^2}{16x^2-1}=\dfrac{\infty}{\infty}\overset{LH}{=}\lim_{x\to\infty}\dfrac{-15+24x}{32x}=\dfrac{\infty}{\infty}\overset{LH}{=}\lim_{x\to\infty}\dfrac{24}{32}=\dfrac{3}{4}$

-OR-

$$\lim_{x\to\infty}\frac{1-15x+12x^2}{16x^2-1}=\lim_{x\to\infty}\frac{{}^{1}\!/_{x^2}-{}^{15x}\!/_{x^2}+{}^{12x^2}\!/_{x^2}}{{}^{16x^2}\!/_{x^2}-{}^{1}\!/_{x^2}}=\lim_{x\to\infty}\frac{{}^{1}\!/_{x^2}-{}^{15}\!/_{x}+12}{16-{}^{1}\!/_{x^2}}=\frac{12}{16}=\frac{3}{4}$$

3. $\lim_{x\to-\infty}\left(2+e^x\right)\cdot\tan^{-1}\left(x^2+1\right)=2\cdot\dfrac{\pi}{2}=\pi$

5. $\lim_{x\to\infty}\dfrac{\cos x}{18-4x+x^2}=0$ because the denominator goes to infinity while the numerator stays between 1 and -1.

7, $\lim_{x\to-\infty}\dfrac{\sin x}{\ln(-x)}=0$ because the denominator goes to infinity while the numerator stays between 1 and -1.

9. $\lim_{x\to-\infty}\tan^{-1}\left(\dfrac{x^3+7x^2+10x}{x+5}\right)=\tan^{-1}(\infty)=\dfrac{\pi}{2}$

11. $\lim_{x\to\infty}\dfrac{\cot^{-1}x}{e^{-x}+1}=\lim_{x\to\infty}\dfrac{\tan^{-1}({}^{1}\!/_{x})}{e^{-x}+1}=\dfrac{0}{1}=0$

13. $f(x)=\dfrac{\cos x}{x^2}$

Left end: $\lim_{x\to-\infty}\dfrac{\cos x}{x^2}=0\therefore$horizontal asymptote at $y=0$

Right end: $\lim_{x\to\infty}\dfrac{\cos x}{x^2}=0\therefore$horizontal asymptote at $y=0$

15. $f(x)=\dfrac{\ln x^2}{\sqrt{4-x}}=\dfrac{2\ln x}{\sqrt{4-x}}$

Left end: $\lim\limits_{x\to-\infty}\dfrac{2\ln x}{\sqrt{x-4}}=dne\therefore$ none

Right end: $\lim\limits_{x\to\infty}\dfrac{2\ln x}{\sqrt{x-4}}\overset{L'H}{=}\lim\limits_{x\to\infty}\dfrac{{}^{2}\!/_{x}}{{}^{1}\!/_{2(x-4)^{1}\!/_{2}}}\lim\limits_{x\to\infty}\dfrac{(x-4)^{1}\!/_{2}}{x}=0\therefore$ horizontal asymptote at $y=0$

6.5 Multiple Choice Solutions

1. The denominator degree is higher than the numerator degree.

The correct answer is A

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3. The numerator degree is higher than the denominator degree.

The correct answer is E

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5. HA: $\dfrac{ax}{x}=2\to a=2$; VA: $-3+c=0\to c=3$; then $a+c=2+3=5$

The correct answer is E

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7. $\lim\limits_{x\to\infty}\left(4x-x^2\right)e^{-x}\approx\lim\limits_{x\to\infty}e^{-x}=0$

The correct answer is A

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9. I. True: $f'(x) < 0$ means $f(x)$ is decreasing, which it is.

II. True: $f(x)$ approaches a horiizantal Asymptote, so $\lim_{x \to +\infty} f'(x) = 0$

III. False: $\lim_{x \to -\infty} f(x) = 3$, but $\lim_{x \to -\infty} f'(x) = 0$

The correct answer is D

6.6 Free Response Solutions

1. $$\int_1^\infty \frac{1}{(3x+1)^2}dx = \lim_{b\to\infty}\int_1^b (3x+1)^{-2}dx \qquad \boxed{\begin{array}{l} u = 3x+1 \\ du = dx \end{array}}$$

$$= \lim_{b\to\infty} \frac{1}{3}\int_1^b 3(3x+1)^{-2}dx$$

$$= \lim_{b\to\infty} \frac{1}{3}\int_4^{3b+1} u^{-2}du = \lim_{b\to\infty} \frac{1}{3}\left[-u^{-1}\right]_4^{3b+1} = \lim_{b\to\infty} -\frac{1}{3}\left[\frac{1}{3b+1} - \frac{1}{4}\right]$$

$$= \frac{1}{12}$$

3. $$\int_0^\infty e^{-x}dx = \lim_{b\to\infty}\int_0^b e^{-x}dx \qquad \boxed{\begin{array}{l} u = -x \\ du = -dx \end{array}}$$

$$= \lim_{b\to\infty} -\int_0^b -e^{-x}dx = \lim_{b\to\infty} -\int_0^{-b} e^u = \lim_{b\to\infty} -\left[e^{-b} - e^0\right] = 1$$

5. $$\int_{-\infty}^\infty xe^{-x^2}dx = \int_{-\infty}^0 xe^{-x^2}dx + \int_0^\infty xe^{-x^2}dx$$

$$\boxed{\begin{array}{l} u = -x^2 \\ du = -2xdx \end{array}}$$

$$= \lim_{a\to-\infty}\int_a^0 xe^{-x^2}dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}dx$$

$$= \lim_{a\to-\infty} -\frac{1}{2}\int_a^0 -2xe^{-x^2}dx + \lim_{b\to\infty} -\frac{1}{2}\int_0^b -2xe^{-x^2}dx$$

$$= \lim_{a\to-\infty} -\frac{1}{2}e^u\Big|_{-a^2}^0 + \lim_{b\to\infty} -\frac{1}{2}\,e^u\Big|_0^{-b^2}$$

$$= \lim_{a\to-\infty} -\frac{1}{2}\left(1 - e^{-a^2}\right) + \lim_{b\to\infty} -\frac{1}{2}\left(e^{-b^2} - 1\right) = -1$$

7. $\displaystyle\int_1^{\infty}\frac{\ln x}{x}dx=\lim_{b\to\infty}\int_1^{b}\frac{\ln x}{x}dx\left|\begin{array}{l}u=\ln x\\ du=\frac{1}{x}dx\end{array}\right.$

$\displaystyle=\lim_{b\to\infty}\int_0^{\ln b}u\,du=\lim_{b\to\infty}\frac{u^2}{2}\Big|_0^{\ln b}=\lim_{b\to\infty}\frac{1}{2}(\ln b)^2-0=\infty$

Diverges

9. $\displaystyle\int_{-\infty}^{0}x3^{-x^2}dx=-\frac{1}{2}\lim_{a\to-\infty}\int_a^{0}(-2x)3^{-x^2}dx$

$\displaystyle=-\frac{1}{2}\lim_{a\to-\infty}3^{-x^2}\ln 3\Big|_a^0=-\frac{\ln 3}{2}\left[\lim_{a\to-\infty}3^{-a^2}-1\right]=\frac{\ln 3}{2}$

11. $\displaystyle\int_e^{\infty}\frac{1}{x\ln^2 x}dx=\lim_{b\to\infty}\int_e^{b}u^{-2}dx=\lim_{b\to\infty}\left(-\frac{1}{u}\right)\Big|_e^b=0-\left(-\frac{1}{e}\right)=\frac{1}{e}$

6.6 Multiple Choice Solutions

1. I. $\displaystyle\int_0^{\infty}\frac{1}{1+x^2}\,dx=\lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b=\frac{\pi}{2}\therefore\text{convergent}$

II. $\displaystyle\int_2^{\infty}\frac{1}{x\ln x}\,dx=\lim_{b\to\infty}\left[\ln(\ln x)\right]_2^b=\infty\therefore\text{divergent}$

III. $\displaystyle\int_1^{\infty}\frac{1}{\sqrt{x}}\,dx=\lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b=\infty\therefore\text{divergent}$

The correct answer is A

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3. $$\int_1^{\infty} \frac{t^3}{e^{t^4}}dt = \lim_{b\to 0}\left(-\frac{1}{4}\right)\int_1^b \left(-4t^3e^{-t^4}dt\right) = \lim_{b\to 0}\left(-\frac{1}{4}e^{-t^4}\right)_1^b$$

$$= 0-\left(-\frac{1}{4}e^{-1}\right) = \frac{1}{4e}$$

The correct answer is C

---

5. $$\int_1^{\infty} \frac{xdx}{1+x^2} = \frac{1}{2}\int_1^{\infty} \frac{2xdx}{1+x^2} = \lim_{b\to\infty}\frac{1}{2}\left[\ln\left(1+x^2\right)\right]_1^b = divergent$$

The correct answer is E

---

7. $$area = \int_1^{\infty} \frac{1}{\sqrt[3]{x^2}}dx = \lim_{b\to 0}\int_1^b x^{-2/3}dx = \lim_{b\to 0}\left[3x^{1/3}\right]_1^b = divergent$$

The correct answer is D

---

9. $$area = \int_0^{\infty} 4xe^{-x^2}dx = -2\lim_{b\to\infty}\int_0^b -2xe^{-x^2}dx = \lim_{b\to\infty}\left[-2e^{-x^2}\right]_0^b$$

$$= 0-(-2) = 2$$

The picture above is the graph of $f(x) = 4xe^{-x^2}$. The area is

The correct answer is C

6.7 Free Response Solutions

1. $\int_0^5 \frac{1}{x^5}dx = \lim_{a\to 0^+}\int_a^5 x^{-5}dx = \lim_{a\to 0^+} \left.\frac{(x+1)^{-4}}{-4}\right|_a^5$

$= -\frac{1}{4(6)^4} - \lim_{a\to 0^+}\frac{(a+1)^{-4}}{-4} = \text{diverges}$

3. $\int_0^{\pi/4}\csc^2 t dt = \lim_{a\to 0^+}\int_a^{\pi/4}\csc^2 t dt = \lim_{a\to 0^+} -\cot t\Big|_a^{\pi/4}$

$= -\cot\frac{\pi}{4} + \lim_{a\to 0^+}\cot a = dne$. Therefore, $\int_0^{\pi/4}\csc^2 t dt$ is divergent.

5. $\int_0^{\pi}\sec x dx = \int_0^{\pi/2}\sec x dx + \int_{\pi/2}^{\pi}\sec x dx$

$= \lim_{b\to \pi/2^-}\int_0^b \sec x dx + \lim_{a\to \pi/2^+}\int_a^{\pi}\sec x dx$

$= \lim_{b\to \pi/2^-}\ln\left|\sec x + \tan x\right|\Big|_0^b + \lim_{a\to \pi/2^+}\ln\left|\sec x + \tan x\right|\Big|_a^{\pi}$

$= \lim_{b\to \pi/2^-}\ln\left|\sec b + \tan b\right| - \ln\left|\sec 0 + \tan 0\right| + \lim_{a\to \pi/2^+}\ln\left|\sec\pi + \tan\pi\right| - \ln\left|\sec a + \tan a\right|$

$= \lim_{b\to \pi/2^-}\ln\left|\sec b + \tan b\right| + \lim_{a\to \pi/2^+} -\ln\left|\sec a + \tan a\right| = dne$

$\int_0^{\pi}\sec x dx$ *is divergent* Divergent

6.7 Multiple Choice Solutions

1. I. $\int_0^{-\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to-\infty}\left[\tan^{-1}x\right]_0^b = -\frac{\pi}{2} \therefore$ convergent

II. $\int_1^e \frac{1}{x\ln x}\,dx = \lim_{a\to 1^+}\left[\ln(\ln x)\right]_a^e = -\infty \therefore$ divergent

III. $\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0}\left[2\sqrt{x}\right]_0^4 = 4-0 = 4 \therefore$ convergent

The correct answer is D

---

3. I. $\int_0^e \frac{2x}{1-x^2}\,dx = \int_0^1 \frac{2x}{1-x^2}\,dx + \int_1^e \frac{2x}{1-x^2}\,dx$

$= \lim_{b\to 1^-}\left[-\ln\left|1-x^2\right|_0^b\right] + \lim_{a\to 1^+}\left[-\ln\left|1-x^2\right|_a^e\right]$

$\lim_{b\to 1^-}\left[-\ln\left|1-x^2\right|_0^b\right] dne \therefore \int_0^e \frac{2x}{1-x^2}\,dx$ diverges

II.

$\int_{-1}^{\infty} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to 0}\left[-\frac{1}{x}\right]_{-1}^b + \lim_{a\to 0}\left[\lim_{b\to\infty}\left[-\frac{1}{x}\right]_a^b\right]$

$\lim_{b\to 0}\left[-\frac{1}{b}\right]_{-1}^b - 1 = dne \therefore$ divergent

III. $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0}\left[\ln x\right]_0^1 = -\infty \therefore$ divergent

The correct answer is E

---

5. I. $\int_0^{16} u^{-\frac{5}{4}}\,du = \left[-4u^{-\frac{1}{4}}\right]_0^{16} \rightarrow divergent\ \ at\ \ x=0$

II. $\int_0^1 \frac{1}{x^2}dx = \left[\frac{x^{-1}}{-1}\right]_0^1 \rightarrow divergent\ \ at\ \ x=0$

III. $\int_0^1 \frac{1}{\sqrt{x}}dx = \left[2x^{1/2}\right]_0^1 = 2 \rightarrow convergent$

The correct answer is B

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7. $\int_0^{\infty} f(x)\,dx = \lim_{a\to 0^+}\lim_{b\to\infty}\int_a^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left(2\sqrt{b}\right) - \lim_{a\to 0^+}\left(2\sqrt{a}\right)$

$\lim_{b\to\infty}\frac{-1}{b} = 0$

$\lim_{b\to\infty}\left(2\sqrt{b}\right) = diverges,\ but\ \lim_{a^+\to 0}\left(2\sqrt{a}\right) = 0$

The correct answer is C

Chapter 6 Practice Test (Part 1) Solutions

1. $\lim_{h\to 0}\dfrac{sec^2\left(\frac{\pi}{4}+h\right)-sec^2\left(\frac{\pi}{4}\right)}{h}=$

$\overset{L'H}{=}\lim_{h\to 0}\dfrac{2sec\left(\frac{\pi}{4}+h\right)sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)}{1}=2\sqrt{2}\sqrt{2}(1)=4$

The correct answer is D

---

3. Since $9-(-3)^2=0=6(-3)+18,\ f(x)$ is continuous.

$f(x)=\begin{cases}-2x, & if\ x<-3\\ 6, & if\ -3<x\end{cases}$. Since $-2(-3)=6,\ f(x)$ is differentiable.

The correct answer is E

---

5. $\int_0^{\infty}e^{-x}\,dx=\lim_{b\to\infty}\int_0^{b}e^{-x}dx=\lim_{b\to\infty}\ 3-e^{-b}-\left(-e^{0}\right)=1$

$\int_0^{1}\dfrac{1}{\sqrt{x^3}}\,dx=\lim_{a\to 0^+}\int_a^{1}x^{-3/2}\,dx=\lim_{a\to 0^+}\left[-2x^{-1/2}\right]=dne$

$\int_0^{\infty}\dfrac{x}{4+x^2}\,dx=\dfrac{1}{2}\lim_{b\to\infty}\int_0^{b}\dfrac{2x}{4+x^2}\,dx=\dfrac{1}{2}\lim_{b\to\infty}\left[\ln\left(4+x^2\right)\right]_0^{b}=dne$

Two of these diverge

The correct answer is D

---

7. $G'(x) = \dfrac{d}{dx}\displaystyle\int_8^{x^3} \sqrt[3]{t}\,dt = \sqrt[3]{x^3}\cdot\left(3x^2\right) = 3x^3$

The correct answer is D

---

9. $f(x) = \ln\left(x - \dfrac{\pi}{2}\right)$ has a vertical asymptote at $x = \dfrac{\pi}{2}$.

The correct answer is C

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11. $\displaystyle\lim_{x \to \pi^-} \frac{x - \pi}{\sec x} = \frac{0}{-1} = 0$

The correct answer is C

Chapter 6 Practice Test (Part 2) Solutions

1(a) $\displaystyle\lim_{x \to -3^-} f(x) = 0$ because the left side of the curve approaches $(-3, 0)$

1(b) $\displaystyle\lim_{x \to 4^-} f(x) = -\infty$ because the curve goes down along the left side of the vertical asymptote.

1(c) $\displaystyle\lim_{x \to -3} f(x) = 0$ because the two parts of the curve come together at $(-3, 0)$

1(d) $\displaystyle\lim_{x \to 4^+} f(x) = +\infty$ because the curve goes up along the right side of the vertical asymptote.

1(e) $\displaystyle\lim_{x \to 3^+} f(x) =$ -1 because the right side of the curve approaches $(3, -1)$

1(f) $\lim_{x \to -1} f(x) = 2.8$ because the curve is continuous at $x = -1$, so

$\lim_{x \to -1} f(x) = f(-1) = \sqrt{9-(-1)^2} = \sqrt{8} = 2.828$

1(g) $f(4)$=DNE because there is a vertical asymptote at x=4

1(h) $\lim_{x \to 3^-} f(x) = 0$ because the left side of the curve approaches $(3, 0)$

1(i) $f(-3) = 1$ because the actual point is at $(-3, 1)$

1(j) $f(3)$= -1 because the actual point is at $(-3, -1)$

1(k) $\lim_{x \to -3^+} f(x) = 0$ because the right side of the curve approaches $(-3, 0)$

1(l) $\lim_{x \to 3} f(x) =$ DNE because the two parts of the curve do not come together at as x approaches 3

Solutions to The Funnel Problem II

3(a) $r_{ave} = \frac{1}{10}\int_0^{10} \frac{1}{30}(h^2+4)dh =$

$r_{ave} = \frac{1}{300}\left[\frac{h^3}{3} + 4h\right]_0^{10} = \frac{1}{300}\left[\frac{1000}{3} + 400\right] = \frac{22}{9}$

3(b) $V = \pi\int_0^{10}\left[\frac{1}{30}(h^2+4)\right]^2 dh = \frac{\pi}{900}\int_0^{10}(h^4+8h^2+4)dh$

$V = = \frac{\pi}{900}\left(\frac{h^5}{5} + \frac{8}{3}h^3 + 4h\right)_0^{10} = \frac{\pi}{900}\left(\frac{10^5}{5} + \frac{8}{3}10^3 + 4(10)\right) = \frac{3406\pi}{135}$

3(c) $V = \pi \int_{-\infty}^{10} \left[\frac{1}{30}\left(h^2+4\right) \right]^2 dh = \frac{\pi}{900} \lim_{a \to -\infty} \int_{a}^{10} \left(h^4+8h^2+4\right) dh =$

$= \frac{\pi}{900} \lim_{a \to -\infty} \left(\frac{h^5}{5} + \frac{8}{3}h^3 + 4h \right)_{a}^{10} = \frac{\pi}{900} \left(\frac{10^5}{5} + \frac{8}{3}(10)^3 + 4(10h) \right) - dne$

Therefore, the volume diverges.

3(d) $\frac{d}{dt}\left[V = \pi \int_0^h \left[\frac{1}{30}\left(t^2+4\right) \right]^2 dt \right] = \frac{\pi}{900} \left(h^4+8h^2+4\right) \frac{dh}{dt}$

$\left. \frac{dV}{dt} \right|_{h=3} = \frac{\pi}{900} \left(3^4+8(3)^2+4\right) \left(\frac{1}{4} \right) = \frac{157\pi}{3600}$