Chapter 13:

Integrals
Chapter 13 Overview: The Integral

Calculus is essentially comprised of two operations. Interspersed throughout the chapters of this book has been the first of these operations—the derivative. The surface of this field has barely been scratched and only those areas that are most related to Analytic Geometry were viewed. College Calculus will begin with the derivative and look much further into it, as well as looking at subjects other than Analytic Geometry where the derivative plays a major role. But some basics of the other operation—the integral—should be considered.

There are two kinds of integrals—the indefinite integral (or anti-derivative) and the definite integral. The indefinite integral is referred to as the anti-derivative, because, as an operation, it and the derivative are inverses (just as squares and square roots, or exponential and logarithmic functions). As it pertains to Analytic Geometry, the definite integral is an operation that gives the area between a curve and the x-axis. It has nothing to do with traits or sketching a graph. This area is purely Calculus and is an appropriate place to finish our introduction to the subject.
13-1: Anti-Derivatives: The Power Rule

As seen in a previous chapter, certain traits of a function can be deduced if its derivative is known. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called anti-differentiation, or integration.

Symbol: \( \int (f(x)) \, dx = \text{“the integral of } f \text{ of } x \, dx \)

The \( dx \) is called the differential. For now, just treat it as part of the integral symbol. It tells the independent variable of the function (usually, but not always, \( x \)) and, in a sense, is where the increase in the exponent comes from. It does have meaning on its own which will be explored in a different course.

Looking at the integral as an anti-derivative, that is, as an operation that reverses the derivative, a basic process for integration emerges.

Remember:

\[
\frac{d}{dx} \left[ x^n \right] = nx^{n-1} \quad \text{and} \quad D_x \left[ \text{constant} \right] = 0
\]

(or, multiply the power in front and subtract one from the power). In reversing the process, the power must increase by one and divide by the new power. The derivative does not allude to what constant, if any, may have been attached to the original function.
The Anti-Power Rule:

\[ \int (x^n) \, dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1 \]

The “+C” is to account for any constant that might have been there before the derivative was taken. NB. This rule will not work if \( n = -1 \), because it would require division by zero. Recall from the derivative rules what yields \( x^{-1} \) (or \( \frac{1}{x} \)) as the derivative—\( \ln x \). The complete the Anti-Power Rule is:

Since \( D_x [f(x) + g(x)] = D_x [f(x)] + D_x [g(x)] \) and \( D_x [cx^n] = cD_x [x^n] \) then

\[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \]

\[ \int c(f(x)) \, dx = c \int f(x) \, dx \]

These allow integration of a polynomial by integrating each term separately.
LEARNING OUTCOMES

Find the indefinite integral of a polynomial.
Use integration to solve rectilinear motion problems.

EX 1 \[\int (3x^2 + 4x + 5) \, dx\]

\[\int (3x^2 + 4x + 5) \, dx = 3 \frac{x^{2+1}}{2+1} + 4 \frac{x^{1+1}}{1+1} + 5 \frac{x^{0+1}}{0+1} + C\]
\[= \frac{3x^3}{3} + \frac{4x^2}{2} + \frac{5x^1}{1} + C\]
\[= x^3 + 2x^2 + 5x + C\]

EX 2 \[\int \left(x^4 + 4x^2 + 5 + \frac{1}{x}\right) \, dx\]

\[\int \left(x^4 + 4x^2 + 5 + \frac{1}{x}\right) \, dx = \frac{x^{4+1}}{4+1} + \frac{4x^{2+1}}{2+1} + \frac{5x^{0+1}}{0+1} + \ln x + C\]
\[= \frac{1}{5} x^5 + \frac{4}{3} x^3 + 5x + \ln|x| + C\]

EX 3 \[\int \left(x^2 + \frac{3}{x} - \frac{4}{x}\right) \, dx\]

\[\int \left(x^2 + \frac{3}{x} - \frac{4}{x}\right) \, dx = \int \left(x^2 + x^{\frac{1}{2}} - \frac{4}{x}\right) \, dx\]
\[= \frac{x^{2+1}}{2+1} + \frac{x^{1/3+1}}{1/3+1} - 4 \ln|x| + C\]
\[= \frac{1}{3} x^3 + \frac{3}{4} x^{4/3} - 4 \ln|x| + C\]
Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

EX 4  \[ \int \left( x^2 + \sqrt[3]{x} \right) \left( 2x + 1 \right) \, dx \]

\[
\int \left( x^2 + \sqrt[3]{x} \right) \left( 2x + 1 \right) \, dx = \int \left( 2x^3 + 2x^{4/3} + x^2 + x^{1/3} \right) \, dx
\]

\[
= \frac{2x^4}{4} + \frac{2x^{7/3}}{\frac{7}{3}} + \frac{x^3}{3} + \frac{x^{4/3}}{\frac{4}{3}} + C
\]

\[
= \frac{1}{2} x^4 + \frac{6}{7} x^{7/3} + \frac{1}{3} x^3 + \frac{3}{4} x^{4/3} + C
\]

Example 5 is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for \( C \).

EX 5  \( f'(x) = 4x^3 - 6x + 3 \). Find \( f(x) \) if \( f(0) = 13 \).

\[
f(x) = \int \left( 4x^3 - 6x + 3 \right) \, dx
\]

\[
= x^4 - 3x^2 + 3x + C
\]

\[
f(0) = 0^4 - 3(0)^2 + 3(0) + C = 13
\]

\[
\therefore \ C = 13
\]

\[
f(x) = x^4 - 3x^2 + 3x + 13
\]
EX 6 The acceleration of a particle is described by $a(t) = 3t^2 + 8t + 1$. Find the distance equation for $x(t)$ if $v(0) = 3$ and $x(0) = 1$.

$$v(t) = \int (a(t)) \, dt = \int (3t^2 + 8t + 1) \, dt$$
$$= t^3 + 4t^2 + t + C_1$$
$$3 = (0)^3 + 4(0)^2 + (0) + C_1$$
$$3 = C_1$$
$$v(t) = t^3 + 4t^2 + t + 3$$

$$x(t) = \int (v(t)) \, dt = \int (t^3 + 4t^2 + t + 3) \, dt$$
$$= \frac{1}{4} t^4 + \frac{4}{3} t^3 + \frac{1}{2} t^2 + 3t + C_2$$
$$1 = \frac{1}{4} (0)^4 + \frac{4}{3} (0)^3 + \frac{1}{2} (0)^2 + 3(0) + C_2$$
$$1 = C_2$$
$$x(t) = \frac{1}{4} t^4 + \frac{4}{3} t^3 + \frac{1}{2} t^2 + 3t + 1$$
EX 7  The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for $x(t)$ if $v(1) = 0$ and $x(1) = 3$.

$$v(t) = \int (a(t)) \, dt = \int (12t^2 - 6t + 4) \, dt$$
$$= 4t^3 - 3t^2 + 4t + C_1$$
$$0 = 4(1)^3 - 3(1)^2 + 4(1) + C_1$$
$$-5 = C_1$$
$$v(t) = 4t^3 - 3t^2 + 4t - 5$$

$$x(t) = \int (v(t)) \, dt = \int (4t^3 - 3t^2 + 4t - 5) \, dt$$
$$= t^4 - t^3 + 2t^2 - 5t + C_2$$
$$3 = (1)^4 - (1)^3 + 2(1)^2 - 5(1) + C_2$$
$$6 = C_2$$

$$x(t) = t^4 - t^3 + 2t^2 - 5t + 6$$
13-1 Free Response Homework

Perform the anti-differentiation.

1. \( \int (6x^2 - 2x + 3) \, dx \)  
2. \( \int (x^3 + 3x^2 - 2x + 4) \, dx \)

3. \( \int \frac{2}{\sqrt{x}} \, dx \)  
4. \( \int (8x^4 - 4x^3 + 9x^2 + 2x + 1) \, dx \)

5. \( \int x^3(4x^2 + 5) \, dx \)  
6. \( \int (4x - 1)(3x + 8) \, dx \)

7. \( \int \left( \sqrt{x} - \frac{6}{\sqrt{x}} \right) \, dx \)  
8. \( \int \left( \frac{x^2 + \sqrt{x} + 3}{x} \right) \, dx \)

9. \( \int (x + 1)^3 \, dx \)  
10. \( \int (4x - 3)^2 \, dx \)

11. \( \int \left( \sqrt{x} + 3 \sqrt[3]{x^3} - \frac{6}{\sqrt{x}} \right) \, dx \)  
12. \( \int \left( \frac{4x^3 + \sqrt{x} + 3}{x^2} \right) \, dx \)

Solve the initial value problems.

13. \( f'(x) = 3x^2 - 6x + 3 \). Find \( f(x) \) if \( f(0) = 2 \).

14. \( f'(x) = x^3 + x^2 - x + 3 \). Find \( f(x) \) if \( f(1) = 0 \).

15. \( f'(x) = (\sqrt{x} - 2)(3\sqrt{x} + 1) \). Find \( f(x) \) if \( f(4) = 1 \).

16. The acceleration of a particle is described by \( a(t) = 36t^2 - 12t + 8 \). Find the distance equation for \( x(t) \) if \( v(1) = 1 \) and \( x(1) = 3 \).

17. The acceleration of a particle is described by \( a(t) = t^2 - 2t + 4 \). Find the distance equation for \( x(t) \) if \( v(0) = 2 \) and \( x(0) = 4 \).
13-1 Multiple Choice Homework

1. \( \int \frac{1}{x^2} \, dx = \)
   a) \( \ln x^2 + C \)    b) \( -\ln x^2 + C \)    c) \( x^{-1} + C \)
   d) \( -x^{-1} + C \)    e) \( -2x^{-3} + C \)

2. \( \int x(10 + 8x^4) \, dx = \)
   a) \( 5x^2 + \frac{4}{3} x^6 + C \)    b) \( 5x^2 + \frac{8}{5} x^5 + C \)    c) \( 10x + \frac{4}{3} x^6 + C \)
   d) \( 5x^2 + 8x^6 + C \)    e) \( 5x^2 + \frac{8}{7} x^6 + C \)

3. \( \int x\sqrt{3x} \, dx = \)
   a) \( \frac{2\sqrt{3}}{5} x^{5/2} + C \)    b) \( \frac{5\sqrt{3}}{2} x^{5/2} + C \)    c) \( \frac{\sqrt{3}}{2} x^{3/2} + C \)
   d) \( 2\sqrt{3}x + C \)    e) \( \frac{5\sqrt{3}}{2} x^{3/2} + C \)
Integration by Substitution: The Chain Rule

The other three derivative rules—the Product, Quotient and Chain Rules—are a little more complicated to reverse than the Power Rule. This is because they yield a more complicated function as a derivative, one which usually has several algebraic simplifications. The integral of a rational function is particularly difficult to unravel because, as seen previously, a rational derivative can be obtained by differentiating a composite function with a logarithm or a radical, or by differentiating another rational function. Reversing the Product Rule is as complicated, though for other reasons. Both these subjects can be left for a traditional Calculus class. The Chain Rule is another matter.

Composite functions are among the most pervasive situations in math. Though not as simple in reverse as the Power Rule, the overwhelming importance of this rule makes it imperative that it be addressed here.

Remember:

The Chain Rule: \[ \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \]

The derivative of a composite function turns into a product of a composite and a non-composite. So if there is a product to integrate, it might be that the product came from the Chain Rule. The integration is not done by a formula so much as a process that might or might not work. One makes an educated guess and hopes it works out. There are other processes in Calculus for when it does not work.

Integration by Substitution:

1. Identify the inside function of the composite and call it \( u \).
2. Find \( du \) from \( u \).
3. If necessary, multiply a constant inside the integral to create \( du \), and balance it by multiplying the reciprocal of that constant outside the integral. (See EX 2)
4. Substitute \( u \) and \( du \) into the equation.
5. Perform the integration using the Anti-Power Rule (or transcendental rules, in next section.)
6. Re-substitute for \( u \).
This is one of those mathematical processes that makes little sense when first seen. But after seeing several examples, the meaning suddenly becomes clear. **BE PATIENT!**

**LEARNING OUTCOME**

Use the Unchain Rule to integrate composite, product expressions.

**EX 1**

\[
\int \left( 3x^2 \left( x^3 + 5 \right)^{10} \right) \, dx
\]

\( \left( x^3 + 5 \right)^{10} \) is the composite function. So \( u = x^3 + 5 \)

\[ du = 3x^2 \, dx \]

\[
\int \left( 3x^2 \left( x^3 + 5 \right)^{10} \right) \, dx = \int \left( u^{10} \right) \, du
\]

\[ = \frac{u^{11}}{11} + C \]

\[ = \frac{1}{11} \left( x^3 + 5 \right)^{11} + C \]
EX 2 \[ \int \left[ x(x^2+5)^3 \right] dx \]

\((x^2+5)^3\) is the composite function. So \( u = x^2 + 5 \)
\[ du = 2x \, dx \]

\[ \int \left[ x(x^2+5)^3 \right] dx = \frac{1}{2} \int \left( x^2 + 5 \right)^3 (2x \, dx) \]
\[ = \frac{1}{2} \int (u^3) \, du \]
\[ = \frac{1}{2} \cdot \frac{u^4}{4} + C \]
\[ = \frac{1}{8}(x^2+5)^4 + C \]

EX 3 \[ \int \left[ \left( \sqrt[4]{x^4 + 2x^2 - 5} \right) \right] dx \]

\(\sqrt[4]{x^4 + 2x^2 - 5}\) is the composite function.

So \( u = x^4 + 2x^2 - 5 \)
\[ du = (4x^3 + 4x) \, dx = 4(x^3 + x) \, dx \]

\[ \int \left[ \left( \sqrt[4]{x^4 + 2x^2 - 5} \right) \right] dx = \frac{1}{4} \int \left( \sqrt[4]{u} \right)^4 (4x^3 + x) \, dx \]
\[ = \frac{1}{4} \int (\sqrt[4]{u}) \, du \]
\[ = \frac{1}{4} \int \left( u^{\frac{1}{5}} \right) \, du \]
\[ = \frac{1}{4} \cdot \frac{u^{\frac{6}{5}}}{\frac{6}{5}} + C \]
\[ = \frac{5}{24} (x^4 + 2x^2 - 5)^{\frac{6}{5}} + C \]
EX 4
\[
\int \left( \frac{3x^2 + 4x - 5}{(x^3 + 2x^2 - 5x + 2)^3} \right) \, dx
\]

\[u = x^3 + 2x^2 - 5x + 2\]
\[du = (3x^2 + 4x - 5) \, dx\]

\[
\int \left( \frac{3x^2 + 4x - 5}{(x^3 + 2x^2 - 5x + 2)^3} \right) \, dx = \int (x^3 + 2x^2 - 5x + 2)^{-3} (3x^2 + 4x - 5) \, dx
\]
\[= \int u^{-3} \, du\]
\[= \frac{u^{-2}}{-2} + C\]
\[= \frac{-1}{2(x^3 + 2x^2 - 5x + 2)^2} + C\]
Sometimes, the other factor is not the $du$, or there is an extra $x$ that must be replaced with some form of $u$.

EX 5 $\int (x+1)\sqrt{x-1} \, dx$

\begin{align*}
u &= x - 1 \\
x &= u + 1 \\
du &= dx
\end{align*}

$$\int (x+1)\sqrt{x-1} \, dx = \int ((u+1)+1)\sqrt{u} \, du$$

$$= \int (u+2)u^{1/2} \, du$$

$$= \int \left( u^{3/2} + 2u^{1/2} \right) \, du$$

$$= \frac{u^{5/2}}{5/2} + \frac{2u^{3/2}}{3/2} + C$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + C$$
EX 6 \( \int \left( x^3 \left( x^2 + 4 \right)^{3/2} \right) \, dx \)

\[ u = x^2 + 4 \]
\[ x^2 = u - 4 \]
\[ du = 2x \, dx \]

\[
\int \left( x^3 \left( x^2 + 4 \right)^{3/2} \right) \, dx = \frac{1}{2} \int \left( x^2 \left( x^2 + 4 \right)^{3/2} \right) (2x \, dx)
\]
\[ = \frac{1}{2} \int (u - 4) u^{3/2} \, du \]
\[ = \frac{1}{2} \int \left( u^{5/2} - 4u^{3/2} \right) \, du \]
\[ = \frac{1}{2} \left( \frac{u^{7/2}}{7/2} - \frac{4u^{5/2}}{5/2} \right) + C \]
\[ = \frac{1}{7} \left( x^2 + 4 \right)^{7/2} - \frac{4}{5} \left( x^2 + 4 \right)^{5/2} + C \]
13-2 Free Response Homework

Perform the anti-differentiation.

1. \[ \int (5x + 3)^3 \, dx \]
2. \[ \int \left( x^3 \left( x^4 + 5 \right)^{24} \right) \, dx \]
3. \[ \int (1 + x^3)^2 \, dx \]
4. \[ \int (2 - x)^{2/3} \, dx \]
5. \[ \int \left( x \sqrt{2x^2 + 3} \right) \, dx \]
6. \[ \int \frac{dx}{5x + 2} \]
7. \[ \int \frac{x^3}{\sqrt{1 + x^4}} \, dx \]
8. \[ \int \frac{x + 1}{\sqrt[3]{x^2 + 2x + 3}} \, dx \]
9. \[ \int \left( x^5 \left( x^2 + 4 \right)^{12} \right) \, dx \]
10. \[ \int \sqrt{x + 3} (x + 1)^2 \, dx \]

13-2 Multiple Choice Homework

1. \[ \int \frac{x}{x^2 - 4} \, dx = \]
   a) \[ \frac{-1}{4(x^2 - 4)^2} + C \]
   b) \[ \frac{1}{2(x^2 - 4)} + C \]
   c) \[ \frac{1}{2} \ln|x^2 - 4| + C \]
   d) \[ 2 \ln|x^2 - 4| + C \]
   e) \[ \frac{1}{2} \arctan\left( \frac{x}{2} \right) + C \]
2. \[ \int \frac{e^{\sqrt{x}}}{2\sqrt{x}} \, dx = \]
   a) \( \ln \sqrt{x} + C \)  
   b) \( x + C \)  
   c) \( e^x + C \)  
   d) \( \frac{1}{2} e^{2\sqrt{x}} + C \)  
   e) \( e^{\sqrt{x}} + C \)

3. When using the substitution \( u = \sqrt{1+x} \), an anti-derivative of \( \int 60x\sqrt{1+x} \, dx \) is
   a) \( 20u^3 - 60u + C \)  
   b) \( 15u^4 - 30u^2 + C \)  
   c) \( 30u^4 - 60u^2 + C \)  
   d) \( 24u^5 - 40u^3 + C \)  
   e) \( 12u^6 - 20u^4 + C \)

4. \[ \int \frac{4x}{1+x^2} \, dx = \]
   a) \( 4 \arctan x + C \)  
   b) \( \frac{4}{x} \arctan x + C \)  
   c) \( \frac{1}{2} \ln(1 + x^2) + C \)  
   d) \( 2 \ln(1 + x^2) + C \)  
   e) \( 2x^2 + 4 \ln|x| + C \)

5. \[ \int x(x^2-1)^4 \, dx = \]
   a) \( \frac{1}{10} x^2(x^2-1)^5 + C \)  
   b) \( \frac{1}{10} (x^2-1)^5 + C \)  
   c) \( \frac{1}{5} (x^3-x)^5 + C \)  
   d) \( \frac{1}{5} (x^2-1)^5 + C \)  
   e) \( \frac{1}{5} (x^2-x)^5 + C \)
13-3: Anti-Derivatives: The Transcendental Rules

The proof of the transcendental integral rules can be left to a more formal Calculus course. But since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

**Derivative Rules:**

\[
\frac{d}{dx} \sin u = (\cos u) \frac{du}{dx} \quad \frac{d}{dx} \csc u = (-\csc u \cot u) \frac{du}{dx}
\]

\[
\frac{d}{dx} \cos u = (-\sin u) \frac{du}{dx} \quad \frac{d}{dx} \sec u = (\sec u \tan u) \frac{du}{dx}
\]

\[
\frac{d}{dx} \tan u = (\sec^2 u) \frac{du}{dx} \quad \frac{d}{dx} \cot u = (-\csc^2 u) \frac{du}{dx}
\]

\[
\frac{d}{dx} e^u = (e^u) \frac{du}{dx} \quad \frac{d}{dx} \ln u = \left( \frac{1}{u} \right) \frac{du}{dx}
\]

\[
\frac{d}{dx} a^u = (a^u \cdot \ln a) \frac{du}{dx} \quad \frac{d}{dx} \log_a u = \left( \frac{1}{u \cdot \ln a} \right) \frac{du}{dx}
\]

**Integral Rules:**

\[
\int (\cos u) \, du = \sin u + C \quad \int (\csc u \cot u) \, du = -\csc u + C
\]

\[
\int (\sin u) \, du = -\cos u + C \quad \int (\sec u \tan u) \, du = \sec u + C
\]

\[
\int (\sec^2 u) \, du = \tan u + C \quad \int (\csc^2 u) \, du = -\cot u + C
\]

\[
\int (e^u) \, du = e^u + C \quad \int \left( \frac{1}{u} \right) \, du = \ln |u| + C
\]

\[
\int (a^u) \, du = \frac{a^u}{\ln a} + C
\]

**LEARNING OUTCOME**

Integrate functions involving transcendental operations.
EX 1  \[ \int (\sin x + 3\cos x) \, dx \]

\[ \int (\sin x + 3\cos x) \, dx = \int (\sin x) \, dx + 3\int (\cos x) \, dx \]

\[ = -\cos x + 3\sin x + C \]

EX 2  \[ \int \left( e^x + 4 + 3\csc^2 x \right) \, dx \]

\[ \int \left( e^x + 4 + 3\csc^2 x \right) \, dx = \int (e^x) \, dx + 4\int dx + 3\int (\csc^2 x) \, dx \]

\[ = e^x + 4x - 3\cot x + C \]

EX 3  \[ \int (\sec x (\sec x + \tan x)) \, dx \]

\[ \int (\sec x (\sec x + \tan x)) \, dx = \int (\sec^2 x) \, dx + \int (\sec x \tan x) \, dx \]

\[ = \tan x + \sec x + C \]

The Unchain Rule will apply to the transcendental functions quite well.

EX 4  \[ \int (\sin 5x) \, dx \]

\[ u = 5x \]
\[ du = 5 \, dx \]

\[ \int (\sin 5x) \, dx = \frac{1}{5} \int (\sin 5x) 5 \, dx \]

\[ = \frac{1}{5} \int (\sin u) \, du \]

\[ = \frac{1}{5} (-\cos u) + C \]

\[ = -\frac{1}{5} \cos 5x + C \]
EX 5 \[
\int (\sin^6 x \cos x) \, dx
\]

\[
u = \sin x
\]
\[
du = \cos x \, dx
\]

\[
\int (\sin^6 x \cos x) \, dx = \int (u^6) \, du
\]

\[
= \frac{1}{7} u^7 + C
\]

\[
= \frac{1}{7} \sin^7 x + C
\]

EX 6 \[
\int (x^5 \sin x^6) \, dx
\]

\[
u = x^6
\]
\[
du = 6x^5 \, dx
\]

\[
\int (x^5 \sin x^6) \, dx = \frac{1}{6} \int (\sin x^6)(6x^5 \, dx)
\]

\[
= \frac{1}{6} \int (\sin u) \, du
\]

\[
= -\frac{1}{6} \cos u + C
\]

\[
= -\frac{1}{6} \cos x^6 + C
\]
EX 7  \[ \int \left( \cot^3 x \csc^2 x \right) \, dx \]

\[ u = \cot x \]
\[ du = -\csc^2 x \, dx \]

\[ \int \left( \cot^3 x \csc^2 x \right) \, dx = \int \left( \cot^3 x \right) \left( -\csc^2 x \, dx \right) \]
\[ = -\int \left( u^3 \right) \, du \]
\[ = -\frac{1}{4} u^4 + C \]
\[ = -\frac{1}{4} \cot^4 x + C \]

EX 8  \[ \int \left( \cos \frac{\sqrt{x}}{\sqrt{x}} \right) \, dx \]

\[ u = \sqrt{x} = x^{\frac{1}{2}} \]
\[ du = \frac{1}{2} x^{-\frac{1}{2}} \, dx = \frac{1}{2x^{\frac{1}{2}}} \, dx \]

\[ \int \left( \cos \frac{\sqrt{x}}{\sqrt{x}} \right) \, dx = 2 \int \left( \cos \frac{1}{\sqrt{x}} \right) \left( \frac{1}{2\sqrt{x}} \, dx \right) \]
\[ = 2\int (\cos u) \, du \]
\[ = 2\sin u + C \]
\[ = 2\sin \sqrt{x} + C \]
EX 9 \[ \int (xe^{x^2+1}) \, dx \]

\[
\begin{align*}
\int (xe^{x^2+1}) \, dx &= \frac{1}{2} \int (e^{x^2+1})(2x \, dx) \\
&= \frac{1}{2} \int (e^{u}) \, du \\
&= \frac{1}{2} e^{u} + C \\
&= \frac{1}{2} e^{x^2+1} + C
\end{align*}
\]

EX 10 \[ \int \left( \frac{\ln^3 x}{x} \right) \, dx \]

\[
\begin{align*}
\int \left( \frac{\ln^3 x}{x} \right) \, dx &= \int u^3 \, du \\
&= \frac{1}{4} u^4 + C \\
&= \frac{1}{4} \ln^4 x + C
\end{align*}
\]
13-3 Free Response Homework

Perform the anti-differentiation.

1. $\int (x^4 \cos x^5) \, dx$
2. $\int (\sin (7x+1)) \, dx$

3. $\int (\sec^2 (3x-1)) \, dx$
4. $\int \left( \frac{\sin \sqrt{x}}{\sqrt{x}} \right) \, dx$

5. $\int (\tan^4 x \sec^2 x) \, dx$
6. $\int \frac{\ln x}{x} \, dx$

7. $\int (e^{6x}) \, dx$
8. $\int \frac{\cos 2x}{\sin^3 2x} \, dx$

9. $\int \frac{x \ln(x^2+1)}{x^2+1} \, dx$
10. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

11. $\int (\sqrt{\cot x} \csc^2 x) \, dx$
12. $\int \frac{1}{x^2} \left( \sin \frac{1}{x} \right) \left( \cos \frac{1}{x} \right) \, dx$

13-3 Multiple Choice Homework

1. $\int \left( x^3 + 2 + \frac{1}{x^2+1} \right) \, dx =$

   a) $\frac{x^4}{4} + 2x + \tan^{-1} x + C$
   b) $x^4 + 2 + \tan^{-1} x + C$
   c) $\frac{x^4}{4} + 2x + \frac{3}{x^3+3} + C$
   d) $\frac{x^4}{4} + 2x + \tan^{-1} 2x^2 + C$
   e) $4 + 2x + \tan^{-1} x + C$
2. \[ \int \cos(3 - 2x) \, dx = \]
   a) \( \sin(3 - 2x) + C \)
   b) \( -\sin(3 - 2x) + C \)
   c) \( \frac{1}{2} \sin(3 - 2x) + C \)
   d) \( -\frac{1}{2} \sin(3 - 2x) + C \)
   e) \( -\frac{1}{5} \sin(3 - 2x) + C \)

3. \[ \int \frac{x - 2}{x - 1} \, dx = \]
   a) \( -\ln|x - 1| + C \)
   b) \( x + \ln|x - 1| + C \)
   c) \( x - \ln|x - 1| + C \)
   d) \( x + \sqrt{x - 1} + C \)
   e) \( x - \sqrt{x - 1} + C \)

4. \[ \int \frac{e^{x^2} - 2x}{e^{x^2}} \, dx = \]
   a) \( x - e^{-x^2} + C \)
   b) \( x - e^{-x^2} + C \)
   c) \( x + e^{-x^2} + C \)
   d) \( -e^{x^2} + C \)
   e) \( -e^{-x^2} + C \)
5. \[ \int 6 \sin x \cos^2 x \, dx = \]

a) \[2 \sin^3 x + C\]  
b) \[-2 \sin^3 x + C\]  
c) \[2 \cos^3 x + C\]  
d) \[-2 \cos^3 x + C\]  
e) \[3 \sin^2 x \cos^2 x + C\]
13-4: Definite Integrals

As noted in the overview, anti-derivatives are known as indefinite integrals because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an Analytic Geometry context of area. Though it is not necessary to know the theory behind this to be able to do it, the theory is a major subject of Integral Calculus, so it will be explored briefly here.

Geometry presented how to find the exact area of various polygons, but it never considered figures where one side is not made of a line segment. Here, consider a figure where one side is the curve \( y = f(x) \) and the other sides are the \( x \)-axis and the lines \( x = a \) and \( x = b \).

As seen above, the area can be approximated by rectangles whose height is the \( y \) value of the equation and whose width is \( \Delta x \). The more rectangles made, the better the approximation. The area of each rectangle would be \( f(x) \cdot \Delta x \) and the total area of \( n \) rectangles would be \( \sum_{i=1}^{n} f(x_i) \cdot \Delta x \). If an infinite number of rectangles (which would be infinitely thin) could be made, the exact area would be the resulting sum. The rectangles can be drawn several ways—with the left side at the height of the curve (as drawn above), with the right side at the curve, with the rectangle straddling the curve, or even with rectangles of different widths. But
once they become infinitely thin, it will not matter how they were drawn—they will have no width and a height equal to the y-value of the curve.

An infinite number of rectangles can be made mathematically by taking the limit as \( n \) approaches infinity, or

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x.
\]

This limit is rewritten as the Definite Integral:

\[
\int_{a}^{b} f(x) \, dx
\]

\( b \) is the “upper bound” and \( a \) is the “lower bound,” and would not mean much if it were not for the following rule:

---

**The Fundamental Theorem of Calculus:**

1. \( \frac{d}{dx} \int_{c}^{x} f(t) \, dt = f(x) \)

2. If \( F'(x) = f(x) \), then \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \).

---

The First Fundamental Theorem of Calculus simply says what is already known—that an integral is an anti-derivative. The Second Fundamental Theorem says the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

This idea of the integral meaning the area may not make sense initially, mainly because Geometry was used, where area is always measured in square units. But that is only because the length and width are always measured in the same kind of units, so multiplying length and width must yield square units. Consider a graph
where the \( x \)-axis is time in seconds and the \( y \)-axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet/sec—that is, feet. So the area under the curve equals the distance traveled in feet. In other words, the integral of velocity is distance.

### LEARNING OUTCOMES

- Evaluate definite integrals.
- Interpret definite integrals as area under a curve.

#### EX 1

Find \( \int_{1}^{4} x^2 \, dx \)

\[
\int_{1}^{4} x^2 \, dx = \left[ \frac{x^3}{3} + C \right]_{1}^{4}
\]

\[
= \left( \frac{4^3}{3} + C \right) - \left( \frac{1^3}{3} + C \right)
\]

\[
= 21
\]

Notice that the arbitrary constant \( C \) does not effect the definite integral. It cancels itself out when we substitute \( a \) and \( b \) and subtract. So, with a definite integral, **the \( + C \) can be ignored**.

#### EX 2

Find \( \int_{\pi/4}^{\pi/2} \sin x \, dx \)

\[
\int_{\pi/4}^{\pi/2} \sin x \, dx = \left[ -\cos x \right]_{\pi/4}^{\pi/2}
\]

\[
= \left[ -\cos \frac{\pi}{2} \right] - \left[ -\cos \frac{\pi}{4} \right]
\]

\[
= 0 - \left( -\frac{1}{\sqrt{2}} \right)
\]
EX 3  \[ \int_0^3 (3x^2 + 4x + 5) \, dx \]

\[ \int_0^3 (3x^2 + 4x + 5) \, dx = \left[ x^3 + 2x^2 + 5x \right]_0^3 \]

\[ = \left[ 3^3 + 2(3)^2 + 5(3) \right] - \left[ 0^3 + 2(0)^2 + 5(0) \right] \]

\[ = 60 - 0 \]

\[ = 60 \]

EX 4  \[ \int_1^3 \left( \frac{3}{x} \right) \, dx \]

\[ \int_1^3 \left( \frac{3}{x} \right) \, dx = 3[\ln x]_1^3 \]

\[ = 3[\ln 3] - 3[\ln 1] \]

\[ = 3\ln 3 = 3.295 \]

If a composite function requires integration by substitution, substitute for the boundaries as well. \( a \) and \( b \) are normally what \( x \) would equal. If switching to \( u \), the boundaries must equal what \( u \) would equal when \( x = a \) and \( x = b \).
EX 5 \[ \int_{0}^{2} \left( x(x^2 + 5)^3 \right) \, dx \]

\[ u = x^2 + 5, \quad du = 2x \, dx, \quad u(0) = 5, \quad u(2) = 9 \]

\[ \int_{0}^{2} \left( x(x^2 + 5)^3 \right) \, dx = \frac{1}{2} \int_{0}^{2} (x^2 + 5)^3 (2x \, dx) \]

\[ = \frac{1}{2} \int_{5}^{9} u^3 \, du \]

\[ = \frac{1}{2} \cdot \left[ \frac{u^4}{4} \right]_{5}^{9} \]

\[ = \frac{1}{2} \cdot \left[ \frac{9^4}{4} - \frac{5^4}{4} \right] \]

\[ = 742 \]

EX 6 \[ \int_{0}^{\pi/2} \left( \sin^6 x \cos x \right) \, dx \]

\[ u = \sin x, \quad du = \cos x \, dx, \quad u\left(\frac{\pi}{2}\right) = 1, \quad u(0) = 0 \]

\[ \int_{0}^{\pi/2} \left( \sin^6 x \cos x \right) \, dx = \int_{0}^{1} (u^6) \, du \]

\[ = \left[ \frac{1}{7}u^7 \right]_{0}^{1} \]

\[ = \frac{1}{7} \]
13-4 Free Response Homework

Evaluate the definite integrals.

1. \( \int_0^3 (x^2 + 5) \, dx \)  
2. \( \int_2^6 \frac{1}{\sqrt{2x - 3}} \, dx \)

3. \( \int_{-1}^1 (x\sqrt{1-x^2}) \, dx \)  
4. \( \int_0^1 (x+2)^3 \, dx \)

5. \( \int_{-2}^2 (x+5)(x^2-3) \, dx \)  
6. \( \int_{-4}^9 \frac{1-\sqrt{x}}{\sqrt{x}} \, dx \)

7. \( \int_{\pi/6}^{\pi/2} \cos^5 x \sin x \, dx \)  
8. \( \int_1^3 \frac{x^2 + 1}{x^2} \, dx \)

9. \( \int_{-1}^2 \frac{dx}{2x+5} \)  
10. \( \int_0^\pi \frac{\sin x}{2 - \cos x} \, dx \)

11. \( \int_{\ln 2}^4 \frac{dx}{x \ln x} \)  
12. \( \int_0^{\ln 2} \frac{e^x}{1+e^{2x}} \, dx \)

13-4 Multiple Choice Homework

1. \( \int_2^6 \left( \frac{1}{x} + 2x \right) \, dx = \)

   a) \( \ln 4 + 32 \)  
   b) \( \ln 3 + 40 \)  
   c) \( \ln 3 + 32 \)  
   d) \( \ln 4 + 40 \)  
   e) \( \ln 12 + 32 \)
2. \( \int_{0}^{1} \sin \pi x \, dx = \)

a) \( \frac{2}{\pi} \)  

b) \( \frac{1}{\pi} \)  

c) 0  

d) \( -\frac{2}{\pi} \)  

e) \( -\frac{1}{\pi} \)

3. \( \int_{\pi/4}^{\pi/3} \sec^2 x \tan x \, dx = \)

a) \( \ln \sqrt{3} \)  

b) \( -\ln \sqrt{3} \)  

c) \( \ln \sqrt{2} \)  

d) \( \sqrt{3} - 1 \)  

e) \( \ln \frac{\pi}{3} - \ln \frac{\pi}{4} \)

4. \( \int_{0}^{2} \sqrt{x^2 - 4x + 4} \, dx = \)

a) 1  

b) \( -1 \)  

c) \( -2 \)  

d) 2  

e) None of these

5. If \( \int_{2}^{4} f(x) \, dx = 6 \), then \( \int_{2}^{4} (f(x) + 3) \, dx = \)

a) 3  

b) 6  

c) 9  

d) 12  

e) 15
6. Let \( f \) be the function defined by \( f(x) = \begin{cases} 
\frac{x+1}{x+1} & \text{for } x < 0 \\
1 + \sin \pi x & \text{for } x \geq 0 
\end{cases} \). Then

\[
\int_{-1}^{1} f(x) \, dx =
\]

a) \( \frac{3}{2} \)  

b) \( \frac{3}{2} - \frac{2}{\pi} \)  

c) \( \frac{1}{2} - \frac{2}{\pi} \)  

d) \( \frac{3}{2} + \frac{2}{\pi} \)  

e) \( \frac{1}{2} + \frac{2}{\pi} \)
13-5: Definite Integrals and Area

Since the definite integrals were originally defined in terms of “area under a curve,” what this context of “area” really means needs to be considered in relation to the definite integral.

Consider this function, \( y = (x \cos(x^2)) \) on \( x \in [0, \sqrt{\pi}] \). The graph looks like this:

![Graph of y = x \cos(x^2) with x in [0, \sqrt{\pi}]](image)

In the last section, \( \int_{0}^{\sqrt{\pi}} (x \cos(x^2)) \, dx = 0 \). But it can be seen there is area under the curve, so how can the integral equal the area and equal 0? Remember that the integral was created from rectangles with width \( dx \) and height \( f(x) \). So the area below the \( x \)-axis would be negative, because the \( f(x) \) values are negative.

**EX 1** What is the area under \( y = (x \cos(x^2)) \) on \( x \in [0, \sqrt{\pi}] \)?

It is already known that \( \int_{0}^{\sqrt{\pi}} (x \cos(x^2)) \, dx = 0 \), so this integral cannot represent the area. The question is to find the positive number that represents the area (total distance), not the difference between the positive and negative “areas” (displacement). The commonly accepted context for area is a positive value. So,

\[
\text{Area} = \int_{0}^{\sqrt{\pi}} |x \cos(x^2)| \, dx \\
= \int_{0}^{\sqrt{\pi}} x \cos(x^2) \, dx - \int_{\sqrt{\pi}}^{1.244} x \cos(x^2) \, dx \\
= 1
\]
The calculator could find this answer: \( \text{fnInt(abs(Xcos(X^2)),X,0,\pi)} \approx 1.00000159 \)

When the phrase “area under the curve” is used, what is really meant is the area between the curve and the \( x \)-axis. CONTEXT IS EVERYTHING. The area under the curve is only equal to the definite integral when the curve is completely above the \( x \)-axis. When the curve goes below the \( x \)-axis, the definite integral is negative, but the area, by definition, is positive.

**LEARNING OUTCOMES**

Relate definite integrals to area under a curve.
Understand the difference between displacement and total distance.
Extend that idea to understanding the difference between the two concepts in other contexts.

**Properties of Definite Integrals:**

1. \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \)
2. \( \int_a^a f(x) \, dx = 0 \)
3. \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \), where \( a < c < b \)
EX 2 Find the area under \( y = x^3 - 2x^2 - 5x - 6 \) on \( x \in [-1, 2] \).

A quick look at the graph reveals that the curve crosses the \( x \)-axis at \( x = 2 \).

Integrate \( y \) on \( x \in [-1, 2] \) to get the **difference** between the areas, not the sum. To get the total area, set up two integrals:

\[
\text{Area} = \int_{-1}^{1} (x^3 - 2x^2 - 5x - 6) \, dx + \left[ -\int_{1}^{2} (x^3 - 2x^2 - 5x - 6) \, dx \right]
\]

\[
= \left[ \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} - 6x \right]_{-1}^{1} + \left[ \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} - 6x \right]_{1}^{2}
\]

\[
= \frac{32}{3} + \frac{29}{12}
\]

\[
= \frac{157}{12}
\]
EX 3 Find the area between \( y = \sin x \) and the \( x \)-axis on \( x \in [0, 2\pi] \).

Evaluating the integral \( \int_0^{2\pi} \sin x \, dx \):

\[
\int_0^{2\pi} \sin x \, dx = [-\cos x]_0^{2\pi} \\
= (-(-1)) - 1 \\
= 0
\]

Obviously, the area cannot be zero. Look at the graph:

![Graph of \( y = \sin x \) from \( x = 0 \) to \( x = 2\pi \)]

What has happened with our integral here is that on \( x \in [0, \pi] \) the curve is above the \( x \)-axis, so the “area” represented by the integral is positive. But on \( x \in [\pi, 2\pi] \), the “area” is negative. Since these areas are the same size, the integrals added to zero.

There are several ways to account for this to find the area directly. The most common is to split the integral into two integrals where the boundaries are the zeros, and multiply the “negative area” by another negative to make it positive.

\[
\text{Area} = \int_0^{\pi} \sin x \, dx - \int_\pi^{2\pi} \sin x \, dx \\
= [-\cos x]_0^{\pi} - [-\cos x]_\pi^{2\pi} \\
= \left[\left((-1)) - (-1)\right] - \left[(-1) - (-(-1))\right]\right] \\
= 4
\]
Here’s a fun scenario—imagine leaving your house to go to school, and that school is 6 miles away. You leave your house and halfway to school you realize you have forgotten your Calculus homework (gasp!). You head back home, pick up your assignment, and then head to school.

There are two different questions that can be asked here. How far are you from where you started? And how far have you actually traveled? You are six miles from where you started but you have traveled 12 miles. These are the two different ideas behind displacement and total distance.

**Vocabulary:**
1. **Displacement** – how far apart the starting position and ending position of an object are (it can be positive or negative)
2. **Total Distance** – how far an object travels in total (this can only be positive)

```
Displacement = \int_a^b v \ dt

Total Distance = \int_a^b |v| \ dt
```

EX 4 A particle moves along a line so that its velocity at any time \( t \) is \( v(t) = t^2 - t - 6 \) (measured in meters per second).

\begin{align*}
\text{a) } & \quad \text{Find the displacement of the particle during the time period } 1 \leq t \leq 4. \\
\text{b) } & \quad \text{Find the distance traveled during the time period } 1 \leq t \leq 4.
\end{align*}

\begin{align*}
\text{a) } & \quad \int_1^4 v \ dt = \int_1^4 (t^2 - t - 6) \ dt \\
& \quad = \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4 \\
& \quad = -4.5
\end{align*}
b) \( \int_{a}^{b} |v| \ dt = \int_{1}^{4} |t^2 - t - 6| \ dt \)
\[= -\int_{1}^{2} (t^2 - t - 6) \ dt + \int_{2}^{4} (t^2 - t - 6) \ dt \]
\[= -\left[ \frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_{1}^{2} + \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_{2}^{4} \]
\[= 10 \frac{1}{6} \]

Note that the properties of integrals are used to split the integral into two integrals that represent the separate positive and negative distance traveled. Putting a “–” in front of the first integral turns the negative value from the integral into a positive value. Split the integral at \( t = 2 \) because that would be where \( v(t) = 0 \).
13-5 Free Response Homework

1. Find the area between $y = x^2 + 1$ and the x-axis on $x \in [-1, 1]$.

2. Find the area between $y = x\sqrt{9-4x^2}$ and the x-axis on $x \in \left[0, \frac{3}{2}\right]$.

3. Find the area between $y = \cos x$ and the x-axis on $x \in [0, 2\pi]$.

4. Find the area between $y = xe^{-x^2}$ and the x-axis on $x \in [-2, 2]$.

5. Find the area between $y = x\sqrt{9-4x^2}$ and the x-axis on $x \in \left[-\frac{3}{2}, \frac{3}{2}\right]$.

6. Find the area between $y = x\sqrt{2x^2-18}$ and the x-axis on $x \in [-2, 1]$.

7. Find the area between $y = 3\sin x\sqrt{1-\cos x}$ and the x-axis on $x \in [-\pi, 0]$.

8. Find the area between $y = x^2e^{x^3}$ and the x-axis on $x \in [0, 1.5]$.

9. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$ for $0 \leq t \leq 3$. Find a) the displacement and b) the distance traveled by the particle during the given time interval.

10. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$ for $1 \leq t \leq 6$. Find a) the displacement and b) the distance traveled by the particle during the given time interval.

11. Find the distance traveled by a particle in rectilinear motion whose velocity, in feet/sec, is described $v(t) = 4t + 1$ from $t = 1$ second to $t = 5$ seconds.

12. Find the distance traveled by a particle in rectilinear motion whose velocity, in feet/sec, is described $v(t) = 3t^2 - 4t + 1$ from $t = 0$ second to $t = 4$ seconds.

[Be Careful!!!]
13-5 Multiple Choice Homework

1. The area under the graph of \( y = 4x^3 + 6x - \frac{1}{x} \) on the interval \( 1 \leq x \leq 2 \) is

   a) \( 32 - \ln 2 \text{ units}^2 \)  b) \( 30 - \ln 2 \text{ units}^2 \)  c) \( 24 - \ln 2 \text{ units}^2 \)
   
   d) \( \frac{99}{4} \text{ units}^2 \)  e) \( 21 \text{ units}^2 \)

2. A particle starts at \((5, 0)\) when \( t = 0 \) and moves along the \(x\)-axis in such a way that at time \( t > 0 \) its velocity is given by \( v(t) = \frac{1}{1 + t} \). Determine the position of the particle at \( t = 3 \).

   a) \( \frac{97}{16} \)  b) \( \frac{95}{16} \)  c) \( \frac{79}{16} \)  d) \( 5 + \ln 4 \)  e) \( 1 + \ln 4 \)

3. At \( t = 0 \), a particle starts at the origin with a velocity of 6 feet per second and moves along the \(x\)-axis in such a way that at time \( t \) its acceleration is \( 12t^2 \) feet per second per second. Through how many feet does the particle move during the first 2 seconds?

   a) 16 ft  b) 20 ft  c) 24 ft  d) 28 ft  e) 32 ft

4. \( \int_{1}^{2} \frac{x^2 - x}{x^3} \, dx = \)

   a) \( \ln 2 - \frac{1}{2} \)  b) \( \ln 2 + \frac{1}{2} \)  c) \( \frac{1}{2} \)  d) 0  e) \( \frac{1}{4} \)
5. \[ \int_{0}^{\sqrt{3}} \frac{x}{\sqrt{1 + x^2}} \, dx = \]

a) \( \frac{1}{2} \)  

b) 1  

c) 2  

d) \ln 2  

e) \arctan 2 - \frac{\pi}{4}
Integrals Practice Test
Part 1: CALCULATOR REQUIRED

Round to 3 decimal places. Show all work.

**Multiple Choice** (3 pts. each)

1. A particle moves along the $x$-axis with the velocity given by $v(t) = 3t^2 + 6t$ for time $t \geq 0$. If the particle is at position $x = 2$ at time $t = 0$, what is the position of the particle at time $t = 1$?

   (a) 4
   (b) 6
   (c) 9
   (d) 11
   (e) 12

2. If $\int_{-5}^{2} f(x) \, dx = -17$ and $\int_{5}^{2} f(x) \, dx = -4$, what is the value of $\int_{-5}^{5} f(x) \, dx$?

   (a) $-21$
   (b) $-13$
   (c) 0
   (d) 13
   (e) 21

3. $\int \frac{x^2}{e^{x^3}} \, dx =$

   (a) $-\frac{1}{3} \ln e^{x^3} + C$
   (b) $-\frac{e^{x^3}}{3} + C$
   (c) $-\frac{1}{3e^{x^3}} + C$
   (d) $\frac{1}{3} \ln e^{x^3} + C$
   (e) $\frac{x^3}{3e^{x^3}} + C$
4. \[ \int_0^x \sin t \, dt = \]
   (a) \( \sin x \)
   (b) \( -\cos x \)
   (c) \( \cos x \)
   (d) \( \cos x - 1 \)
   (e) \( 1 - \cos x \)

5. \[ \int_1^e \frac{x^2 + 1}{x} \, dx = \]
   (a) \( \frac{e^2 - 1}{2} \)
   (b) \( \frac{e^2 + 1}{2} \)
   (c) \( \frac{e^2 + 2}{2} \)
   (d) \( \frac{e^2 - 1}{e^2} \)
   (e) \( \frac{2e^2 - 8e + 6}{3e} \)

6. \[ \int x\sqrt{4 - x^2} \, dx = \]
   (a) \( \frac{(4 - x^2)^{3/2}}{3} + C \)
   (b) \( -(4 - x^2)^{3/2} + C \)
   (c) \( \frac{x^2(4 - x^2)^{3/2}}{3} + C \)
   (d) \( -\frac{x^2(4 - x^2)^{3/2}}{3} + C \)
(e) \(-\frac{\left(4 - x^2\right)^{3/2}}{3} + C\)

7. If \(\int_{-2}^{2} \left(x^7 + k\right) \, dx = 16\), then \(k =\)

(a) \(-12\)
(b) 12
(c) \(-4\)
(d) 4
(e) 0

8. \(\int \sin(2x + 3) \, dx =\)

(a) \(\frac{1}{2} \cos(2x + 3) + C\)
(b) \(\cos(2x + 3) + C\)
(c) \(-\cos(2x + 3) + C\)
(d) \(-\frac{1}{2} \cos(2x + 3) + C\)
(e) \(-\frac{1}{5} \cos(2x + 3) + C\)
Free Response (10 pts. each)

1. Find the area between \( y = \frac{1}{x^2} \) and the \( x \)-axis on the interval \( x \in [1, 2] \).

2. Find the area between \( y = \frac{x}{1 + x^2} \) and the \( x \)-axis from \( x = -2 \) to \( x = 2 \).

3. A particle moves along a line with velocity function \( v(t) = t^2 - t \), where \( v \) is measured in meters per second. Set up, but do not solve, and integral expression for a) the displacement and b) the distance traveled by the particle during the time interval \([0, 5]\).
Integrals Homework Answer Key

13-1 Free Response Homework

1. \(2x^3 - x^2 + 3x + C\)  
2. \(\frac{1}{4}x^4 + x^3 - x^2 + 4x + C\)

3. \(3x^{\frac{2}{3}} + C\)  
4. \(\frac{8}{5}x^5 - x^4 + 3x^3 + x^2 + x + C\)

5. \(\frac{2}{3}x^6 + \frac{5}{4}x^4 + C\)  
6. \(4x^3 + \frac{29}{2}x^2 - 8x + C\)

7. \(\frac{2}{3}x^{\frac{3}{2}} - 12x^{\frac{1}{2}} + C\)  
8. \(\frac{1}{2}x^2 + 2x^{\frac{1}{2}} + 3\ln x + C\)

9. \(\frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + C\)  
10. \(\frac{16}{3}x^3 - 12x^2 + 9x + C\)

11. \(\frac{2}{3}x^{\frac{3}{2}} + \frac{6}{5}x^\frac{5}{2} - 12x^{\frac{1}{2}} + C\)  
12. \(2x^2 - 2x^{-\frac{1}{2}} - \frac{1}{x} + C\)

13. \(f(x) = x^3 - 3x^2 + 3x + 2\)  
14. \(f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x - \frac{37}{12}\)

15. \(f(x) = \frac{3}{2}x^2 - \frac{10}{3}x^{\frac{3}{2}} - 2x + \frac{32}{3}\)  
16. \(x(t) = 3t^4 - 2t^3 + 4t^2 - 13t + 11\)

17. \(x(t) = \frac{1}{12}t^4 - \frac{1}{3}t^3 + 2t^2 + 2t + 4\)

13-1 Multiple Choice Homework

1. D  
2. A  
3. A

13-2 Free Response Homework

1. \(\frac{1}{20}(5x + 3)^4 + C\)  
2. \(\frac{1}{100}(x^4 + 5)^{25} + C\)
3. \( \frac{1}{7}x^7 + \frac{1}{2}x^4 + x + C \)  
4. \( -\frac{3}{5}(2 - x)^{5/3} + C \)  
5. \( \frac{1}{6}(2x^2 + 3)^{3/2} + C \)  
6. \( -\frac{1}{10}(5x + 2)^2 + C \)  
7. \( \frac{1}{2}(1 + x^4)^{1/2} + C \)  
8. \( \frac{3}{4}(x^2 + 2x + 3)^{2/3} + C \)  
9. \( \frac{1}{10}(x^2 + 4)^5 - (x^2 + 4)^4 + \frac{8}{3}(x^2 + 4)^3 + C \)  
10. \( \frac{2}{7}(x + 3)^{7/2} - \frac{8}{5}(x + 3)^{5/2} + \frac{8}{3}(x + 3)^{3/2} + C \)

13-2 Multiple Choice Homework

13-3 Free Response Homework
1. \( \frac{1}{5}\sin x^5 + C \)  
2. \( -\frac{1}{7}\cos (7x + 1) + C \)  
3. \( \frac{1}{3}\tan (3x - 1) + C \)  
4. \( -2\cos \sqrt{x} + C \)  
5. \( \frac{1}{5}\tan^5 x + C \)  
6. \( \frac{1}{2}(\ln x)^2 + C \)  
7. \( \frac{1}{6}e^{6x} + C \)  
8. \( -\frac{1}{4}\sin^{-2} 2x + C = -\frac{1}{4}\csc^2 2x + C \)  
9. \( \frac{1}{4}(\ln(x^2 + 1))^2 + C \)  
10. \( 2e^{\sqrt{2}x} + C \)  
11. \( -\frac{2}{3}\cot^{3/2} x + C \)  
12. \( -\frac{1}{2}\left(\sin\frac{1}{x}\right)^2 + C \), or \( \frac{1}{2}\left(\cos\frac{1}{x}\right)^2 + C \)
13-3 Multiple Choice Homework

13-4 Free Response Homework
1. 24  2. 2  3. 0  4. 16.25
5. $-33 \frac{1}{3}$  6. $-3$  7. 0.070  8. $\frac{8}{3}$
9. 0.549  10. 0.693  11. 0.693  12. 0.322

13-4 Multiple Choice Homework

13-5 Free Response Homework
1. $\frac{8}{3}$  2. 2.25  3. 4  4. 0.982
5. 4.5  6. $2 \cos \sqrt{\pi} + 2$  7. $-\sqrt{32}$
8. $\frac{1}{3}(e^{3.375} - 1)$  9a. $-\frac{3}{2}$ m  b. $\frac{41}{6}$ m
10a. $-\frac{10}{3}$ m  b. $\frac{98}{3}$ m  11. 52 feet  12. 36.296 feet

13-5 Multiple Choice Homework
Integrals Practice Test Answer Key

Multiple Choice

Free Response
1. Area = \frac{1}{2}  2. Area = 2\ln 5

3a. Displacement = \int_{0}^{5} (t^2 - t) \, dt
b. Total Distance Traveled = -\int_{0}^{1} (t^2 - t) \, dt + \int_{1}^{5} (t^2 - t) \, dt